IB Math AI HL — Topic 5 [Set 1]

Differentiation

Derivatives · Tangents & Normals · Increasing/Decreasing · Stationary Points · Optimisation

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Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Marks each	3	4	5
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless otherwise stated. Calculator permitted throughout.

Key Formulae

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ **Constant:** $\frac{d}{dx}(c) = 0$ **Sum Rule:** $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Tangent at $(a, f(a))$**:** $y - f(a) = f'(a)(x-a)$ **Normal gradient:** $m_{\text{normal}} = -\frac{1}{f'(a)}$ (if $f'(a) \neq 0$)

Stationary point: $f'(x) = 0$ **Min/Max test:** $f''(x) > 0 \Rightarrow$ local min; $f''(x) < 0 \Rightarrow$ local max

Increasing: $f'(x) > 0$ **Decreasing:** $f'(x) < 0$ **Rate of change:** $f'(a) =$ instantaneous rate at $x = a$

GDC Tips

- **Graphing:** Enter $y_1 = f(x)$ and use `TRACE` or `CALC` > `dy/dx` to find the gradient at any x-value.
- **Stationary points:** Use `CALC` > `MINIMUM` or `MAXIMUM` to locate turning points; confirm with $f'(x) = 0$.
- **Intersection:** Enter $y_2 = 0$ (or the line equation) and use `CALC` > `INTERSECT`.
- **Verify derivatives:** Use `nDeriv(f(x),x,a)` to evaluate $f'(a)$ numerically as a check.
- **Solving** $f'(x) = 0$: Store $f'(x)$ as y_2 and find intersections with the x-axis.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. **EASY** **Calculator** **[3 marks]**

The height of a ball above the ground (in metres) is modelled by

$$h(t) = 20 + 18t - 5t^2, \quad t \geq 0,$$

where t is the time in seconds after the ball is thrown.

(a) Write down $h'(t)$. [1]

(b) Find the velocity of the ball at $t = 2$ seconds. [1]

(c) Write down the initial height of the ball. [1]

2. **EASY** **No Calculator** **[3 marks]**

Let $f(x) = 3x^4 - 8x^2 + 5$.

(a) Find $f'(x)$. [2]

(b) Write down the value of $f'(0)$. [1]

3. EASY Calculator **[3 marks]**

The number of bacteria N in a culture is modelled by

$$N(t) = 400 + 6t^2 - t^3, \quad 0 \leq t \leq 5,$$

where t is time in hours.

(a) Find the rate of change of N with respect to t. [2]

(b) Find the rate of change at $t = 3$ hours. [1]

4. EASY Calculator **[3 marks]**

A curve has equation $y = x^3 - 6x + 2$. The point $P(2, -2)$ lies on this curve.

(a) Find $\dfrac{dy}{dx}$. [1]

(b) Find the gradient of the tangent to the curve at P. [1]

(c) Write down the equation of the tangent at P. [1]

5. EASY Calculator **[3 marks]**

The profit (in thousands of dollars) earned by a small company after x years of operation is modelled by

$$P(x) = 2x^3 - 9x^2 + 12x, \quad x \geq 0.$$

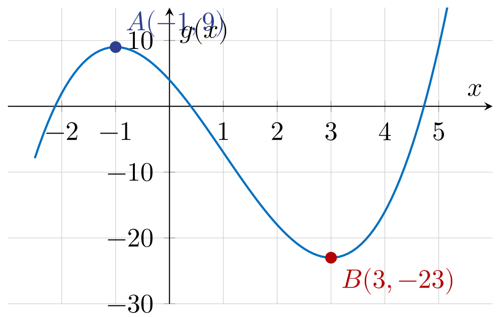
(a) Find $P'(x)$. [2]

(b) Determine whether profit is increasing or decreasing at $x = 1$. [1]

6. EASY Calculator **[3 marks]**

Let $g(x) = x^3 - 3x^2 - 9x + 4$.

The diagram below shows part of the graph of g.



The graph has a local maximum at $A(-1,\ 9)$ and a local minimum at $B(3,\ -23)$.

(a) Write down the interval on which g is decreasing. [1]

(b) Find $g'(x)$. [1]

(c) Verify that $g'(-1) = 0$. [1]

7. EASY No Calculator **[3 marks]**

Let $f(x) = 2x^3 - 3x^2 - 12x + 7$.

(a) Find $f'(x)$. [1]

(b) Set $f'(x) = 0$ and hence find the x-coordinates of any stationary points. [2]

8. **EASY** **Calculator** **[3 marks]**

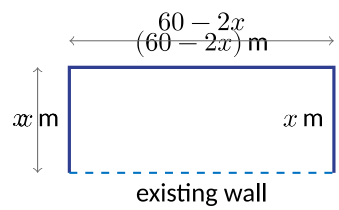
A curve has equation $f(x) = x^3 - 6x^2 + 9x + 1$.

(a) Show that $x = 1$ is a stationary point of f. [2]

(b) Find $f''(x)$ and hence determine the nature of the stationary point at $x = 1$. [1]

9. **EASY** **Calculator** **[3 marks]**

A rectangular enclosure is to be built using 60 m of fencing on three sides; the fourth side is an existing wall.



Let x m be the length of each side perpendicular to the wall.

(a) Write down an expression for the area A of the enclosure in terms of x. [1]

(b) Find $\dfrac{dA}{dx}$. [1]

(c) Write down the value of x that maximises the area. [1]

10. EASY Calculator **[3 marks]**

The curve C has equation $y = x^2 + 3x - 1$. The point $Q(1, 3)$ lies on C.

(a) Find the gradient of the tangent to C at Q. [1]

(b) Write down the gradient of the normal to C at Q. [1]

(c) Find the equation of the normal to C at Q. [1]

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. **MEDIUM** **Calculator** **[4 marks]**

A drainage channel has a cross-section whose depth (in metres) at horizontal distance x m from one bank is modelled by

$$d(x) = -x^2 + 6x - 5, \quad 1 \leq x \leq 5.$$

(a) Find the value of x at which the depth is greatest. [2]

(b) Find the maximum depth of the channel. [1]

(c) Determine whether the depth is increasing or decreasing at $x = 4$. Give a reason. [1]

12. **MEDIUM** **Calculator** **[4 marks]**

The power output P (in watts) of a solar panel depends on the angle of inclination θ (in degrees from horizontal) according to the model

$$P(\theta) = 120\theta - 2\theta^2, \quad 0 < \theta < 60.$$

(a) Find $P'(\theta)$. [1]

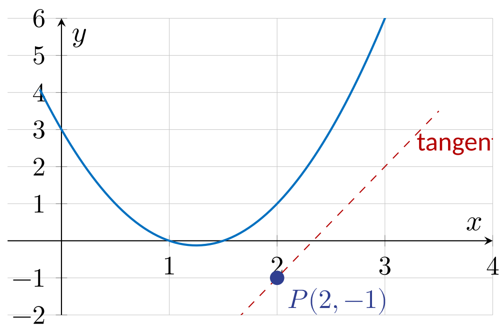
(b) Find the angle θ that maximises P. [2]

(c) Interpret the meaning of $P'(\theta)$ in context. [1]

13. **MEDIUM** **Calculator** **[4 marks]**

A curve has equation $y = 2x^2 - 5x + 3$.

The diagram below shows part of the curve with point P marked.



The tangent to the curve at $P(2, -1)$ is shown.

(a) Find $\dfrac{dy}{dx}$. [1]

(b) Show that the equation of the tangent at P is $y = 3x - 7$. [2]

(c) Find the x-intercept of the tangent. [1]

14. **MEDIUM** **Calculator** **[4 marks]**

The temperature T (in °C) of a liquid cooling in a room is modelled by

$$T(t) = 80 - 12t + t^2, \quad 0 \le t \le 10,$$

where t is time in minutes.

(a) Find the rate of change of temperature at $t = 4$ minutes. [2]

(b) At what time is the temperature decreasing most rapidly? [1]

(c) Find the minimum temperature in the interval $0 \le t \le 10$. [1]

15. MEDIUM No Calculator **[4 marks]**

Let $h(x) = x^4 - 4x^3 + 3$.

(a) Find $h'(x)$ and $h''(x)$. [2]

(b) Find the coordinates of all stationary points of h. [1]

(c) Determine the nature of each stationary point. [1]

16. MEDIUM Calculator **[4 marks]**

Let $f(x) = x^3 - \frac{3}{2}x^2 - 18x + 5$.

(a) Find $f'(x)$. [1]

(b) Find the values of x for which f is increasing. [2]

(c) Find the y-coordinate of the local maximum of f. [1]

17. MEDIUM Calculator **[4 marks]**

A scientist measures the volume V cm^3 of a melting ice block as a function of time t seconds:

$$V(t) = 500 - 30t + \tfrac{3}{4}t^2, \quad 0 \leq t \leq 25.$$

(a) Find $V'(t)$. [1]

(b) Find the time at which the volume is decreasing most rapidly. [1]

(c) Find the minimum volume of the ice block on the given interval. [2]

18. MEDIUM Calculator **[4 marks]**

The curve C has equation $y = x^3 - 3x^2 + 4$.

(a) Find the coordinates of both stationary points of C. [2]

(b) Determine the nature of each stationary point, justifying your answer. [2]

19. MEDIUM Calculator **[4 marks]**

A curve has equation $y = x^3 - 2x^2 + 4$. The normal to the curve at the point $R(3,\ 13)$ meets the y-axis at point S.

(a) Find $\frac{dy}{dx}$. [1]

(b) Find the gradient of the normal at R. [1]

(c) Find the coordinates of S. [2]

20. MEDIUM Calculator **[4 marks]**

The table below shows the values of a function f and its derivative f' at selected values of x.

x	-2	-1	0	1	2
$f(x)$	-11	0	3	2	3
$f'(x)$	9	3	0	-1	3

(a) Write down the value of x in the table where f has a stationary point. [1]

(b) Using the table, determine on which interval(s) f is increasing. [1]

(c) The tangent to $y = f(x)$ at $x = -1$ meets the x-axis. Find the x-coordinate of this meeting point. [2]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **Calculator** **[5 marks]**

An open-topped box is made from a rectangular sheet of cardboard measuring 30 cm by 20 cm by cutting equal squares of side x cm from each corner and folding up the sides.

(a) Show that the volume of the box is $V(x) = 4x^3 - 100x^2 + 600x$. [2]

(b) Find the value of x that maximises the volume. Give your answer correct to 3 significant figures. [2]

(c) Justify that this value gives a maximum, not a minimum. [1]

22. **HARD** **Calculator** **[5 marks]**

A curve has equation $y = x^3 + ax^2 + bx + c$. The curve passes through the point $M(1,\ 4)$, has a stationary point at $x = 1$, and a second stationary point at $x = -3$.

(a) Find the values of a and b. [3]

(b) Hence find the value of c. [1]

(c) Determine the nature of each stationary point. [1]

23. HARD Calculator **[5 marks]**

The tangent to the curve $y = kx^2 - 3x + 1$ at the point where $x = 2$ passes through the origin.

(a) Show that $4k - 3 = \dfrac{4k-5}{2}$ and hence find the value of k. [3]

(b) Write down the equation of the tangent. [1]

(c) Find the x-coordinates of any stationary points of the curve with this value of k. [1]

24. HARD Calculator **[5 marks]**

The function $f(x) = x^3 + px^2 + qx - 4$ has a local maximum at $x = -2$ and a local minimum at $x = 2$.

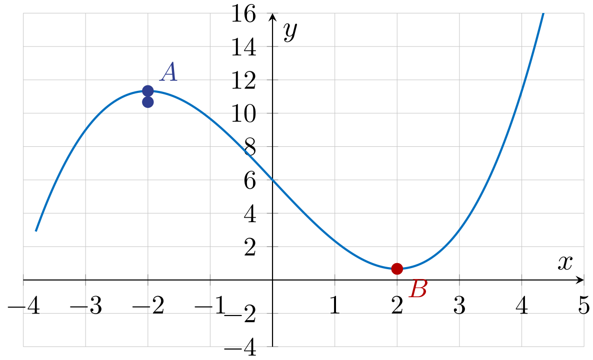
(a) Write down two equations in p and q by using $f'(-2) = 0$ and $f'(2) = 0$. [2]

(b) Find p and q. [1]

(c) Find $f(-2)$ and $f(2)$. [2]

25. HARD Calculator **[5 marks]**

The diagram below shows part of the curve $y = \dfrac{1}{3}x^3 - 4x + 6$.



The curve has stationary points A and B.

(a) Find the x-coordinates of A and B. [2]

(b) Find the coordinates of A and B. [1]

(c) The normal to the curve at B meets the line $y = 1$ at point C. Find the x-coordinate of C. [2]

26. **HARD** **Calculator** **[5 marks]**

A particle moves along a straight line. Its displacement s m from a fixed point O at time t seconds is given by

$$s(t) = t^3 - 6t^2 + 9t + 2, \quad t \geq 0.$$

(a) Find the velocity $v(t) = s'(t)$ of the particle. [1]

(b) Find all values of t for which the particle is at rest. [2]

(c) Find the acceleration when $t = 4$. [1]

(d) Determine whether the particle is speeding up or slowing down at $t = 4$. Justify your answer. [1]

27. **HARD** **Calculator** **[5 marks]**

Let $f(x) = x^3 - 6x^2 + 9x + k$, where k is a constant.
The local minimum of f occurs at the point $(3, m)$.

(a) Find $f'(x)$ and hence show that the stationary points are at $x = 1$ and $x = 3$. [2]

(b) Determine the nature of each stationary point, justifying your answer. [1]

(c) Given that the local minimum value of f is $m = -5$, find k. [1]

(d) Find the range of values of x for which $f(x) > f(1)$. [1]

28. **HARD** **No Calculator** **[5 marks]**

The curve $y = ax^3 + bx^2 + 3x - 2$ passes through the point $(1,\ 5)$ and has a stationary point at $x = -\dfrac{1}{3}$.

(a) Form two equations in a and b using the given information. [2]

(b) Solve the system to find a and b. [2]

(c) Determine whether the stationary point at $x = -\dfrac{1}{3}$ is a maximum or minimum. [1]

29. **HARD** **Calculator** **[5 marks]**

The tangent to the curve $y = x^2 - 4x + 7$ at the point $P(3, 4)$ and the normal to the curve at the point $Q(1, 4)$ intersect at a point T.

(a) Find the equations of the tangent at P and the normal at Q. [3]

(b) Find the coordinates of T. [2]

30. **HARD** **Calculator** **[5 marks]**

A company's daily revenue R (in hundreds of dollars) from selling n items is modelled by

$$R(n) = -2n^3 + 21n^2 - 36n + 5, \quad 0 \leq n \leq 8.$$

(a) Find the values of n at which the revenue is stationary. Give exact values. [2]

(b) Find the maximum revenue on $0 \leq n \leq 8$ and the number of items sold to achieve it. [2]

(c) Give one reason why this model may not be realistic for all values of n in $[0, 8]$. [1]

Common Mistakes to Avoid

1. **Power rule errors.** When differentiating x^n, multiply by n and reduce the power by 1. Don't forget: $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(c) = 0$ for any constant c.
2. **Tangent vs normal.** The tangent gradient is $f'(a)$. The normal gradient is $-\frac{1}{f'(a)}$. These are negative reciprocals. If $f'(a) = 0$, the normal is vertical (undefined slope).
3. **Stationary point $\neq$ maximum/minimum.** Setting $f'(x) = 0$ finds stationary points but you must confirm the nature with the second derivative test or a sign change of f'.
4. **Second derivative test direction.** $f''(a) > 0$ means the curve is concave up $\Rightarrow$ local *minimum*; $f''(a) < 0$ means concave down $\Rightarrow$ local *maximum*. Students commonly confuse these.
5. **Increasing/decreasing intervals.** State intervals as open or closed depending on context. The function is increasing where $f'(x) > 0$ (not ≥ 0), and decreasing where $f'(x) < 0$.
6. **Finding c or unknown constants.** After differentiating to find the gradient, remember to substitute back into the *original* (not differentiated) equation to find y-intercepts or unknown constants.
7. **Units.** In applied problems, always state units in your final answer: m, m/s, $, etc. Marks are allocated for correct units.

Worked Solutions

Official Mark-Scheme Style — M/A/R Notation

Section A — Easy

Q1 — Derivatives from a Quadratic Model — EASY

(a) $h'(t) = 18 - 10t$ **A1**

(b) $h'(2) = 18 - 10(2) = 18 - 20 = -2\,\text{m s}^{-1}$ **A1**

Note: Award A1 for -2 or $-2\,\text{m s}^{-1}$; units not required but condoned if wrong.

(c) $h(0) = 20\,\text{m}$ **A1**

Note: Award A1 for 20 or 20 m without working.

Q2 — Differentiating a Polynomial — EASY

(a) Attempting to differentiate term by term **M1**

$f'(x) = 12x^3 - 16x$ **A1**

(b) $f'(0) = 12(0)^3 - 16(0) = 0$ **A1**

Note: FT from their $f'(x)$ in (a).

Q3 — Rate of Change from a Cubic Model — EASY

(a) Attempting to differentiate **M1**

$N'(t) = 12t - 3t^2$ **A1**

(b) $N'(3) = 12(3) - 3(3)^2 = 36 - 27 = 9$ bacteria h^{-1} **A1**

Note: FT from their $N'(t)$.

Q4 — Gradient and Tangent Equation — EASY

(a) $\dfrac{dy}{dx} = 3x^2 - 6$ **A1**

(b) At $x = 2$: gradient $= 3(4) - 6 = 6$ **A1**

(c) Tangent: $y - (-2) = 6(x - 2) \implies y = 6x - 14$ **A1**

Note: Accept $y = 6x - 14$ or equivalent. FT from their gradient in (b).

Q5 — Increasing/Decreasing from a Profit Model — EASY

(a) Attempting to differentiate **M1**

$P'(x) = 6x^2 - 18x + 12$ **A1**

(b) $P'(1) = 6 - 18 + 12 = 0$
Since $P'(1) = 0$, the profit is neither increasing nor decreasing (stationary) at $x = 1$. **A1**

Note: Accept "stationary" or "at a turning point"; do not accept simply "zero".

Q6 — Increasing/Decreasing from a Graph — EASY

(a) Decreasing on $(-1, 3)$ (or $-1 < x < 3$) **A1**

(b) $g'(x) = 3x^2 - 6x - 9$ **A1**

(c) $g'(-1) = 3(1) - 6(-1) - 9 = 3 + 6 - 9 = 0$ **A1 AG**

Note: All three terms must be evaluated correctly for AG.

Q7 — Stationary Points of a Cubic — EASY

(a) $f'(x) = 6x^2 - 6x - 12$ **A1**

(b) Setting $f'(x) = 0$: $6x^2 - 6x - 12 = 0 \implies x^2 - x - 2 = 0$ **M1**

$(x - 2)(x + 1) = 0 \implies x = 2$ or $x = -1$ **A1**

Note: Award M1 for setting their $f'(x) = 0$ and attempting to solve; A1 for both correct x-values.

Q8 — Show Stationary Point and Classify — EASY

(a) $f'(x) = 3x^2 - 12x + 9$ **M1**

$f'(1) = 3(1) - 12(1) + 9 = 3 - 12 + 9 = 0$ **A1 AG**

(b) $f''(x) = 6x - 12$ **A1**

$f''(1) = 6 - 12 = -6 < 0 \implies$ local maximum at $x = 1$. **A1**

Note: Award final A1 for both $f''(1) = -6$ and the correct conclusion "local maximum".

Q9 — Basic Optimisation Setup — EASY

(a) $A = x(60 - 2x) = 60x - 2x^2$ **A1**

(b) $\dfrac{dA}{dx} = 60 - 4x$ **A1**

(c) Setting $\dfrac{dA}{dx} = 0$: $60 - 4x = 0 \implies x = 15$ m **A1**

Note: FT from their $\dfrac{dA}{dx}$ in (b).

Q10 — Normal to a Parabola — EASY

(a) $\dfrac{dy}{dx} = 2x + 3$; at $x = 1$: gradient of tangent $= 2(1) + 3 = 5$ **A1**

(b) Gradient of normal $= -\dfrac{1}{5}$ **A1**

(c) Normal at $Q(1, 3)$: $y - 3 = -\dfrac{1}{5}(x - 1) \implies y = -\dfrac{x}{5} + \dfrac{16}{5}$ **A1**

Note: Accept $5y + x = 16$ or $y = -0.2x + 3.2$ or equivalent. FT from (a) and (b).

Section B — Medium

Q11 — Maximum of a Quadratic Model — MEDIUM

(a) $d'(x) = -2x + 6$ **(M1)**

Setting $d'(x) = 0$: $-2x + 6 = 0 \implies x = 3$ **A1**

(b) $d(3) = -(9) + 18 - 5 = 4$ m **A1**

(c) $d'(4) = -2(4) + 6 = -2 < 0$, so d is **decreasing** at $x = 4$. **A1**

Note: R mark embedded in A1 — must state sign of $d'(4)$ and conclusion. FT from (a).

Q12 — Optimising a Power-Output Model — MEDIUM

(a) $P'(\theta) = 120 - 4\theta$ **A1**

(b) Setting $P'(\theta) = 0$: $120 - 4\theta = 0 \implies \theta = 30^\circ$ **M1 A1**

Check: within $(0, 60)$ ⍰; $P''(\theta) = -4 < 0$ confirms maximum.

(c) $P'(\theta)$ represents the rate of change of power output (in watts per degree) as the angle of inclination increases. **R1**

Note: Must reference context (angle) and rate; "how fast power changes" alone is insufficient.

Q13 — Tangent Equation (Show That) — MEDIUM

(a) $\dfrac{dy}{dx} = 4x - 5$ **A1**

(b) At $x = 2$: gradient $= 4(2) - 5 = 3$ **M1**

Tangent: $y - (-1) = 3(x - 2) \implies y = 3x - 6 - 1 = 3x - 7$ **A1 AG**

Note: Working must show gradient $= 3$ before forming the equation; substituting into $y = 3x + c$ with $c = -7$ shown is also acceptable.

(c) Setting $y = 0$: $3x - 7 = 0 \implies x = \dfrac{7}{3} \approx 2.33$ **A1**

Note: Accept exact $\frac{7}{3}$ or 2.33 (3 s.f.). FT from their tangent equation.

Q14 — Rate of Change and Minimum of a Cooling Model — MEDIUM

(a) $T'(t) = -12 + 2t$ **(M1)**

$T'(4) = -12 + 8 = -4\,°\text{C min}^{-1}$ **A1**

(b) The temperature decreases (i.e. $T'(t) < 0$) for $t < 6$, so it is decreasing most rapidly at $t = 0$ (start of the interval). **A1**

Note: Accept "at $t = 0$" with or without the reasoning. The question asks for the time, not the value.

(c) $T'(t) = 0 \implies t = 6$; $T(6) = 80 - 72 + 36 = 44\,°\text{C}$ **M1 A1**

Note: Award M1 for setting $T' = 0$ and finding $t = 6$; A1 for $T(6) = 44$.

Q15 — Stationary Points of a Quartic — MEDIUM

(a) $h'(x) = 4x^3 - 12x^2$ **A1**

$h''(x) = 12x^2 - 24x$ **A1**

(b) $h'(x) = 0$: $4x^2(x-3) = 0 \implies x = 0$ or $x = 3$

$h(0) = 3;\ \ h(3) = 81 - 108 + 3 = -24$ **A1**

Stationary points: $(0,\,3)$ and $(3,\,-24)$.

(c) $h''(0) = 0$ — second derivative test inconclusive at $x = 0$.

Sign of h': for $x < 0$, $h'(x) = 4x^2(x-3) < 0$; for $0 < x < 3$, $h' < 0$; so no sign change at $x = 0 \Rightarrow$ **inflection point** (not a local extremum). **A1**

$h''(3) = 12(9) - 24(3) = 108 - 72 = 36 > 0 \Rightarrow$ local **minimum** at $(3, -24)$. **A1**

Note: A1 for correct nature of each point; partial credit if one is correct.

Q16 — Increasing Intervals and Local Maximum — MEDIUM

(a) $f'(x) = 3x^2 - 3x - 18$ **A1**

(b) $f'(x) > 0$: $3x^2 - 3x - 18 > 0 \implies x^2 - x - 6 > 0 \implies (x-3)(x+2) > 0$ **M1**

f is increasing for $x < -2$ or $x > 3$. **A1**

(c) Local maximum at $x = -2$:
$f(-2) = (-8) - \frac{3}{2}(4) - 18(-2) + 5 = -8 - 6 + 36 + 5 = 27$ **A1**

Note: FT from (b). Award A1 for $f(-2) = 27$ with or without full working shown.

Q17 — Minimum Volume from a Quadratic Model — MEDIUM

(a) $V'(t) = -30 + \frac{3}{2}t$ **A1**

(b) $V'(t) < 0$ for $t < 20$ and $V'(t) > 0$ for $t > 20$, so volume decreases most rapidly at $t = 0$.
(The rate of melting $|V'(t)|$ is largest at $t = 0$ since $|-30| > |V'(t)|$ for $t > 0$.) **A1**

(c) $V'(t) = 0 \implies -30 + \frac{3}{2}t = 0 \implies t = 20\,\text{s}$ **M1**

$V(20) = 500 - 30(20) + \frac{3}{4}(400) = 500 - 600 + 300 = 200\,\text{cm}^3$ **A1**

Note: Check $t = 20$ is within $[0, 25]$ ✓. Evaluate endpoints too: $V(25) = 500 - 750 + 468.75 = 218.75 > 200$, so minimum is at $t = 20$.

Q18 — Stationary Points of a Cubic with Justification — MEDIUM

$\frac{dy}{dx} = 3x^2 - 6x$ **(M1)**

Setting $= 0$: $3x(x-2) = 0 \implies x = 0$ or $x = 2$ **A1**

$y(0) = 4,\ \ y(2) = 8 - 12 + 4 = 0$ **A1**

Stationary points: $(0,\ 4)$ and $(2,\ 0)$.

$\frac{d^2y}{dx^2} = 6x - 6$ **(M1)**

At $x = 0$: $y'' = -6 < 0 \Rightarrow$ local **maximum** at $(0, 4)$. **A1**

At $x = 2$: $y'' = 6 > 0 \Rightarrow$ local **minimum** at $(2, 0)$. **A1**

Note: Award A1 for both natures correctly stated with reference to sign of y''; partial A1 for one correct.

Q19 — Normal to a Cubic Meets the y-Axis — MEDIUM

(a) $\frac{dy}{dx} = 3x^2 - 4x$ **A1**

(b) At $x = 3$: tangent gradient $= 3(9) - 4(3) = 27 - 12 = 15$ **(M1)**

Normal gradient $= -\frac{1}{15}$ **A1**

(c) Normal at $R(3, 13)$: $y - 13 = -\frac{1}{15}(x - 3)$ **M1**

At $x = 0$: $y = 13 + \frac{3}{15} = 13 + \frac{1}{5} = \frac{66}{5} = 13.2$ **A1**

$S = \left(0,\ \frac{66}{5}\right) = (0,\ 13.2)$

Note: FT from their normal gradient in (b). Accept 13.2 or $\frac{66}{5}$.

Q20 — Reading a Derivative Table — MEDIUM

(a) $x = 0$ (since $f'(0) = 0$). **A1**

(b) $f'(x) > 0$ for $x = -2$ and $x = -1$ (from table), so f is increasing on $(-2, 0)$ (approximately). **A1**

Note: Accept "between $x = -2$ and $x = 0$" or "for $x < 0$" (consistent with table). Also increasing for $x > 1$ based on $f'(2) = 3 > 0$.

(c) Tangent at $x = -1$: point $(-1, 0)$, gradient $f'(-1) = 3$ **M1**

Tangent: $y - 0 = 3(x - (-1)) \implies y = 3x + 3$

Setting $y = 0$: $3x + 3 = 0 \implies x = -1$ **A1**

Note: The tangent passes through $(-1, 0)$ which is already on the x-axis (since $f(-1) = 0$), so the x-intercept of the tangent is $x = -1$.

Section C — Hard

Q21 — Box Optimisation — HARD

(a) After cutting, base dimensions are $(30 - 2x)$ by $(20 - 2x)$, height $= x$. **M1**

$V = x(30 - 2x)(20 - 2x)$

$= x(600 - 60x - 40x + 4x^2) = x(4x^2 - 100x + 600)$

$= 4x^3 - 100x^2 + 600x$ **A1 AG**

(b) $V'(x) = 12x^2 - 200x + 600$ **(M1)**

Setting $V'(x) = 0$: $12x^2 - 200x + 600 = 0 \implies 3x^2 - 50x + 150 = 0$

$$x = \frac{50 \pm \sqrt{2500 - 1800}}{6} = \frac{50 \pm \sqrt{700}}{6}$$

$x = \dfrac{50 - 26.457\ldots}{6} = 3.923\ldots \approx$ **3.92** cm (or $x \approx 12.7$, rejected since > 10) **A1**

(c) $V''(x) = 24x - 200$; at $x \approx 3.92$: $V'' = 24(3.92) - 200 = 94.08 - 200 = -105.9 < 0 \Rightarrow$ maximum. **R1**

Note: R1 for stating $V'' < 0$ and concluding maximum; also accept sign diagram of V' around $x \approx 3.92$.

Q22 — Determining Parameters from Stationary Conditions — HARD

(a) $f'(x) = 3x^2 + 2ax + b$ **(M1)**

Stationary at $x = 1$: $3 + 2a + b = 0$ **A1**

Stationary at $x = -3$: $27 - 6a + b = 0$ **A1**

Subtracting: $24 - 8a = 0 \implies a = 3$; then $b = -3 - 6 = -9$. **A1**

Note: Award M1 for forming $f'(x)$ and using $f'(1) = f'(-3) = 0$; A1 A1 for the two equations; A1 for both $a = 3$ and $b = -9$.

(b) Using $M(1, 4)$: $1 + a + b + c = 4 \implies 1 + 3 - 9 + c = 4 \implies c = 9$ **A1**

(c) $f''(x) = 6x + 2a = 6x + 6$

$f''(1) = 12 > 0 \Rightarrow$ local **minimum** at $x = 1$; $f''(-3) = -12 < 0 \Rightarrow$ local **maximum** at $x = -3$. **A1**

Q23 — Finding Parameter from Tangent Through Origin — HARD

(a) At $x = 2$: $y = k(4) - 3(2) + 1 = 4k - 5$ **M1**

$\dfrac{dy}{dx} = 2kx - 3$; at $x = 2$: gradient $= 4k - 3$ **M1**

Tangent through $(2,\ 4k-5)$ and $(0,0)$: gradient $= \dfrac{4k-5-0}{2-0} = \dfrac{4k-5}{2}$

Setting equal: $4k - 3 = \dfrac{4k-5}{2}$ **AG**

$2(4k-3) = 4k - 5 \implies 8k - 6 = 4k - 5 \implies 4k = 1 \implies k = \dfrac{1}{4}$ **A1**

(b) Gradient at $x = 2$: $4(\frac{1}{4}) - 3 = 1 - 3 = -2$; tangent through origin: $y = -2x$. **A1**

(c) $y' = 2(\frac{1}{4})x - 3 = \frac{x}{2} - 3 = 0 \implies x = 6$ **A1**

Note: FT from their k. The stationary point is at $x = 6$.

Q24 — Finding Parameters from Max/Min Conditions — HARD

(a) $f'(x) = 3x^2 + 2px + q$ **(M1)**

$f'(-2) = 0$: $12 - 4p + q = 0$ **A1**

$f'(2) = 0$: $12 + 4p + q = 0$ **A1**

(b) Adding the two equations: $24 + 2q = 0 \implies q = -12$

Subtracting: $-8p = 0 \implies p = 0$ **A1**

Note: $f(x) = x^3 - 12x - 4$.

(c) $f(-2) = -8 + 24 - 4 = 12$ **A1**

$f(2) = 8 - 24 - 4 = -20$ **A1**

Note: Local maximum value is 12; local minimum value is -20.

Q25 — Stationary Points and Normal to a Cubic — HARD

(a) $\dfrac{dy}{dx} = x^2 - 4$ **(M1)**

$x^2 - 4 = 0 \implies x = -2$ or $x = 2$ **A1 A1**

(b) $y(-2) = \frac{-8}{3} + 8 + 6 = \frac{-8}{3} + 14 = \frac{34}{3} \approx 11.3$; $\quad A = \left(-2,\ \frac{34}{3}\right)$ **A1**

$y(2) = \frac{8}{3} - 8 + 6 = \frac{8}{3} - 2 = \frac{2}{3} \approx 0.667$; $\quad B = \left(2,\ \frac{2}{3}\right)$

(c) Tangent at B: $\frac{dy}{dx}\Big|_{x=2} = 4 - 4 = 0$ (horizontal tangent), so normal at B is **vertical**: $x = 2$. **M1**

The vertical line $x = 2$ meets $y = 1$ at $C(2,\ 1)$. **A1**

Note: Award M1 for identifying the tangent gradient is 0 and concluding normal is vertical; A1 for $C(2, 1)$.

Q26 — Particle Motion: Velocity Rest and Acceleration HARD

(a) $v(t) = s'(t) = 3t^2 - 12t + 9$ **A1**

(b) $v(t) = 0$: $3t^2 - 12t + 9 = 0 \implies t^2 - 4t + 3 = 0 \implies (t-1)(t-3) = 0$ **M1**

$t = 1$ s and $t = 3$ s **A1**

(c) $a(t) = v'(t) = 6t - 12$; $a(4) = 24 - 12 = 12\,\text{m s}^{-2}$ **A1**

(d) $v(4) = 3(16) - 48 + 9 = 48 - 48 + 9 = 9 > 0$ (moving in positive direction).

$a(4) = 12 > 0$ (accelerating in positive direction).

Since v and a have the same sign, the particle is **speeding up**. **R1**

Note: R1 for both the sign check and the conclusion. Stating "speeding up" alone without justification scores R0.

Q27 — Cubic with Parameter k — HARD

(a) $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$ **M1**

$f'(x) = 0 \implies x = 1$ or $x = 3$ **A1 AG**

(b) $f''(x) = 6x - 12$

$f''(1) = 6 - 12 = -6 < 0 \Rightarrow$ local **maximum** at $x = 1$. **A1**

$f''(3) = 18 - 12 = 6 > 0 \Rightarrow$ local **minimum** at $x = 3$.

(c) Local minimum at $x = 3$: $f(3) = 27 - 54 + 27 + k = k$

$k = m = -5$ **A1**

(d) $f(1) = 1 - 6 + 9 + (-5) = -1$

$f(x) > -1$: since f decreases on $(1, 3)$ and increases for $x > 3$, the curve exceeds -1 for $x < 1$ and $x > 3$.

Range: $x < 1$ or $x > 3$ **A1**

Note: Accept interval notation $(-\infty, 1) \cup (3, \infty)$. *FT from (c).*

Q28 — Reconstructing a Cubic from Conditions — HARD

(a) Passes through $(1, 5)$: $a + b + 3 - 2 = 5 \implies a + b = 4$ **A1**

$\frac{dy}{dx} = 3ax^2 + 2bx + 3$; stationary at $x = -\frac{1}{3}$: $3a \cdot \frac{1}{9} + 2b \cdot (-\frac{1}{3}) + 3 = 0$ **M1**

$\dfrac{a}{3} - \dfrac{2b}{3} + 3 = 0 \implies a - 2b + 9 = 0 \implies a = 2b - 9$ **A1**

(b) Substituting into $a + b = 4$: $(2b - 9) + b = 4 \implies 3b = 13 \implies b = \dfrac{13}{3}$

$a = 2 \cdot \dfrac{13}{3} - 9 = \dfrac{26}{3} - \dfrac{27}{3} = -\dfrac{1}{3}$ **M1 A1**

(c) $\dfrac{d^2y}{dx^2} = 6ax + 2b = 6 \cdot (-\frac{1}{3})x + 2 \cdot \frac{13}{3} = -2x + \dfrac{26}{3}$

At $x = -\frac{1}{3}$: $f'' = \dfrac{2}{3} + \dfrac{26}{3} = \dfrac{28}{3} > 0 \Rightarrow$ local **minimum**. **A1**

Q29 — Tangent and Normal Intersection — HARD

$\dfrac{dy}{dx} = 2x - 4$ **(M1)**

(a) Tangent at $P(3,4)$**:** gradient $= 2(3) - 4 = 2$

$y - 4 = 2(x - 3) \implies y = 2x - 2$ **A1**

Normal at $Q(1,4)$**:** gradient of tangent $= 2(1) - 4 = -2$; gradient of normal $= \dfrac{1}{2}$

$y - 4 = \dfrac{1}{2}(x - 1) \implies y = \dfrac{x}{2} + \dfrac{7}{2}$ **A1 A1**

(A1 for gradient of normal $= \frac{1}{2}$*; A1 for correct normal equation)*

(b) Setting $2x - 2 = \dfrac{x}{2} + \dfrac{7}{2}$: $\dfrac{3x}{2} = \dfrac{11}{2} \implies x = \dfrac{11}{3}$ **M1**

$y = 2 \cdot \dfrac{11}{3} - 2 = \dfrac{22}{3} - \dfrac{6}{3} = \dfrac{16}{3}$

$T = \left(\dfrac{11}{3}, \dfrac{16}{3}\right) \approx (3.67,\ 5.33)$ **A1**

Q30 — Revenue Optimisation with Domain Analysis — HARD

(a) $R'(n) = -6n^2 + 42n - 36 = -6(n^2 - 7n + 6) = -6(n-1)(n-6)$ **M1**

Stationary at $n = 1$ and $n = 6$ **A1**

(b) Evaluate R at stationary points and endpoints:

$R(0) = 5$; $R(1) = -2 + 21 - 36 + 5 = -12$; $R(6) = -432 + 756 - 216 + 5 = 113$; $R(8) = -1024 + 1344 - 288 + 5 = 37$ **M1**

Maximum revenue is $\$113$ hundred (i.e. \$11 300) at $n = 6$ items. **A1**

(c) Any one valid reason, e.g.:

- n must be a positive integer, but the model is defined for continuous n. **R1**
- Revenue is negative for $n = 1$ (\$–1200), which is not realistic.

- A cubic model may give unrealistic (very large or very negative) values outside the stated domain.

Note: Award R1 for any one reasonable contextual comment.

End of Worked Solutions — IB Math AI HL Topic 5: Differentiation