IB Math AI HL — Topic 2 [Set 1]

Functions & Graphs Mastery

Functions & Mappings · Key Graph Features · Quadratics · Cubics · Exponentials · Sinusoids · Intersecting Graphs

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Marks each	3	4	5
Section Marks	30	40	50

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Name:____________________________ Date:______________ Score:_______/120

Instructions: Answer all questions. Show all working clearly. Give answers correct to 3 significant figures unless otherwise stated. A GDC is permitted throughout.

Key Formulae — Topic 2: Functions

Quadratic: $f(x) = ax^2 + bx + c$

Vertex form: $f(x) = a(x-h)^2 + k$

Axis of symmetry: $x = -\dfrac{b}{2a}$

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Exponential: $f(x) = a \cdot b^x$ (growth: $b > 1$; decay: $0 < b < 1$)

Sinusoidal: $f(x) = a\,\sin(bx + c) + d$

Amplitude: $|a|$

Period: $T = \dfrac{2\pi}{b}$ (radians) or $\dfrac{360°}{b}$ (degrees)

Principal axis / midline: $y = d$

Cubic (general): $f(x) = ax^3 + bx^2 + cx + d$

Composition: $(f \circ g)(x) = f(g(x))$ **Inverse:** swap $x \leftrightarrow y$ then solve for y

Key features: domain, range, x-intercepts (zeros), y-intercept, turning points, asymptotes

GDC Tips

- **Graph intersections:** Enter both functions as Y_1 and Y_2, graph, then use *Calc* $\to$ *Intersect* to find coordinates of intersection points.
- **Zeros / x-intercepts:** Use *Calc* $\to$ *Zero* (or *Root*) after graphing $y = f(x)$.
- **Turning points:** Use *Calc* $\to$ *Maximum / Minimum* to locate vertex or local extrema.
- **Sinusoidal regression:** `STAT` $\to$ `CALC` $\to$ `SinReg` for fitting $y = a\,\sin(bx + c) + d$ to data.
- **Table of values:** `2nd` $\to$ `TABLE` to verify key points match the equation.

Common Mistakes to Avoid

1. **Domain restrictions.** Always check: logarithms need positive arguments; square roots need non-negative arguments; rational functions exclude values where denominator $= 0$.
2. **Range vs domain.** The range is the set of output values — it is not always “all real numbers”. Find it by analysing the graph’s y-values.
3. **Axis of symmetry trap.** For $f(x) = a(x - h)^2 + k$, the vertex is at (h, k), not $(-h, k)$.
4. **Period of sinusoids.** Period $= \dfrac{2\pi}{b}$, *not* $2\pi b$. Doubling b *halves* the period.
5. **Exponential base.** $f(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$ is *decreasing*. The base must satisfy $b > 0, b \neq 1$.
6. **Inverse functions.** The domain of f^{-1} equals the *range* of f, not the domain of f.
7. **Intersection solutions.** Equations such as $x^3 = e^x$ may have more than one solution — always graph first to count intersections before solving.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY Calculator **[3 marks]**

The table below gives values of a function g defined on the domain $\{0, 1, 2, 3, 4, 5\}$.

x	0	1	2	3	4	5
$g(x)$	4	0	5	1	3	2

(a) Write down the value of $g(3)$. [1]

(b) Find the value of x such that $g(x) = x$. [1]

(c) Write down the value of $g(g(1))$. [1]

2. EASY Calculator **[3 marks]**

A function is defined by $f(x) = (x - 3)^2 - 4$.

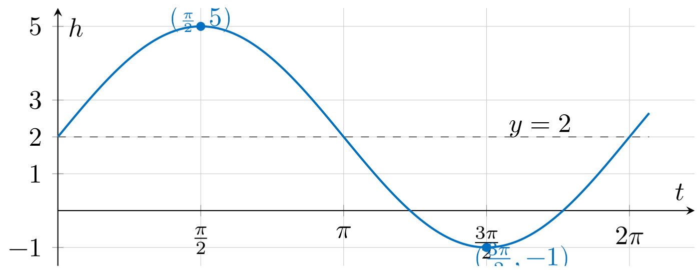
(a) Write down the coordinates of the vertex of the graph of f. [1]

(b) Write down the equation of the axis of symmetry. [1]

(c) Write down the y-intercept of the graph of f. [1]

3. EASY Calculator **[3 marks]**

The diagram shows the graph of $h(t) = a\sin(bt) + d$, where t is measured in radians.



(a) Write down the amplitude of h. [1]

(b) Write down the value of d. [1]

(c) Write down the period of h. [1]

4. EASY Calculator **[3 marks]**

A function is defined by $p(x) = 3 \cdot 2^x - 6$ for $-1 \le x \le 3$.

(a) Find $p(0)$. [1]

(b) Write down the equation of the horizontal asymptote of $y = 3 \cdot 2^x - 6$ (without the domain restriction). [1]

(c) Find the x-intercept of the graph of p (without the domain restriction). [1]

5. **EASY** **Calculator** **[3 marks]**

A cubic function is given by $q(x) = (x + 2)(x - 1)(x - 4)$.

(a) Write down the three x-intercepts of the graph of q. [1]

(b) Find the y-intercept of the graph of q. [1]

(c) Write down whether the graph of q has a positive or negative leading coefficient, giving a reason. [1]

6. **EASY** **Calculator** **[3 marks]**

A function is defined by $f(x) = x^2 - 5x + 4$.

(a) Find the x-intercepts of the graph of f. [2]

(b) Write down the axis of symmetry of the graph of f. [1]

7. **EASY** **Calculator** **[3 marks]**

A function is defined by $r(x) = 5 \cdot (0.6)^x$ for $x \geq 0$.

(a) Write down the value of $r(0)$. [1]

(b) State whether r is an increasing or decreasing function, giving a reason. [1]

(c) Find the value of x for which $r(x) = 1.8$. Give your answer correct to 3 significant figures. [1]

8. **EASY** **Calculator** **[3 marks]**

The height, h metres, of a point on a rotating water wheel above the ground is modelled by

$$h(t) = 4\sin\left(\frac{\pi t}{6}\right) + 5, \quad t \geq 0,$$

where t is the time in seconds.

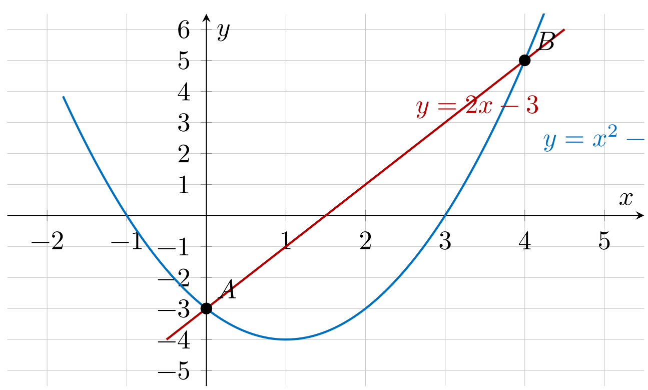
(a) Write down the maximum height of the point. [1]

(b) Write down the minimum height of the point. [1]

(c) Find the first time $t > 0$ at which the point is at maximum height. [1]

9. **EASY** **Calculator** **[3 marks]**

The diagram shows the graphs of $y = x^2 - 2x - 3$ and $y = 2x - 3$.



(a) Write down the coordinates of point A. [1]

(b) Write down the coordinates of point B. [1]

(c) Write down the values of x for which $x^2 - 2x - 3 < 2x - 3$. [1]

10. **EASY** **Calculator** **[3 marks]**

A function is defined by $f(x) = -2(x-1)^2 + 8$.

(a) Write down the coordinates of the vertex. [1]

(b) State the maximum value of f. [1]

(c) Find the x-intercepts of the graph of f. [1]

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. **MEDIUM** **Calculator** **[4 marks]**

A function is defined by $f(x) = \dfrac{10}{x-2} + 3$ for $x > 2$.

(a) Write down the equation of the vertical asymptote of the graph of f. [1]

(b) Find an expression for $f^{-1}(x)$. The domain is not required. [2]

(c) Write down the range of f^{-1}. [1]

12. **MEDIUM** **Calculator** **[4 marks]**

A function is defined by $f(x) = x^2 - 6x + 5$.

(a) Write $f(x)$ in the form $(x-h)^2 + k$, stating the values of h and k. [2]

(b) Hence, or otherwise, find the x-intercepts of the graph of f. [2]

13. MEDIUM Calculator **[4 marks]**

The temperature inside a greenhouse, T degrees Celsius, over one day is modelled by

$$T(t) = a\cos\left(\frac{\pi t}{12}\right) + d, \quad 0 \le t \le 24,$$

where t is the number of hours after midnight. The maximum temperature is $34\,^\circ$C and the minimum temperature is $18\,^\circ$C.

(a) Write down the values of a and d. [2]

(b) Find the two values of t in $[0, 24]$ at which $T(t) = 30$. [2]

14. MEDIUM Calculator **[4 marks]**

A cubic function is given by $f(x) = x^3 - 4x^2 + x + 6$.

(a) Show that $x = 2$ is a zero of f. [1]

(b) Hence, fully factorise $f(x)$. [2]

(c) Write down the three x-intercepts of the graph of f. [1]

15. MEDIUM Calculator **[4 marks]**

Consider the functions $f(x) = x^2 - 3$ and $g(x) = 2^x - 2$.

(a) Find the coordinates of all points of intersection of the graphs of f and g. Give coordinates correct to 3 significant figures. [3]

(b) State the values of x for which $g(x) > f(x)$. [1]

16. MEDIUM Calculator **[4 marks]**

A function is defined by

$$f(x) = \begin{cases} 2x + k & \text{if } x \leq 1, \\ x^2 + 3 & \text{if } x > 1, \end{cases}$$

where k is a constant.

(a) Find the value of k such that f is continuous at $x = 1$. [2]

(b) Using this value of k, find $f(-3)$. [1]

(c) Using this value of k, solve $f(x) = 12$. [1]

17. MEDIUM Calculator **[4 marks]**

The number of bacteria, N, in a culture dish at time t hours is modelled by

$$N(t) = 500 \cdot 3^{0.4t}, \quad t \geq 0.$$

(a) Find the initial number of bacteria. [1]

(b) Find the number of bacteria after 5 hours. Give your answer correct to the nearest whole number. [2]

(c) Find the value of t at which the number of bacteria first exceeds $10\,000$. Give your answer correct to 3 significant figures. [1]

18. MEDIUM Calculator **[4 marks]**

A ball is thrown vertically upwards. Its height, s metres above the ground, t seconds after being thrown, is modelled by

$$s(t) = -4.9t^2 + 14t + 1.2, \quad 0 \leq t \leq 3.$$

(a) Find the maximum height reached by the ball. [2]

(b) Find the time at which the ball returns to the initial height of 1.2 m. [2]

19. MEDIUM Calculator **[4 marks]**

Functions f and g are defined on the domain $\{1, 2, 3, 4, 5\}$ as shown in the table below.

x	1	2	3	4	5
$f(x)$	3	5	1	2	4
$g(x)$	2	4	5	1	3

(a) Find $(f \circ g)(2)$. [1]

(b) Solve $f(x) = g(x)$. [1]

(c) Copy and complete the table below for $f^{-1}(x)$. [2]

x	1	2	3	4	5
$f^{-1}(x)$					

20. MEDIUM Calculator **[4 marks]**

The depth of water at a harbour entrance, d metres, is modelled by

$$d(t) = 2.5\cos\left(\frac{\pi t}{6}\right) + 4, \quad 0 \le t \le 24,$$

where t is the number of hours after noon.

(a) Find the minimum depth of water. [1]

(b) Find the first two values of t (in $[0, 24]$) at which $d(t) = 5$. [2]

(c) Find the total length of time during $[0, 24]$ for which $d(t) \ge 5$. [1]

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD Calculator **[5 marks]**

A function is defined by $f(x) = x^2 + kx + 9$, where k is a real constant.

(a) Find the range of values of k for which $f(x) > 0$ for all real x. [2]

(b) Given that the graph of f touches the x-axis at exactly one point, find the two possible values of k. [1]

(c) For $k = -6$, find the coordinates of the vertex of the graph of f and the x-intercepts. [2]

22. HARD Calculator **[5 marks]**

The temperature of a solar panel, T degrees Celsius, during a single day is modelled by

$$T(t) = a\,\text{sin}(bt + c) + d,$$

where t is the number of hours after midnight ($0 \leq t \leq 24$) and $a, b, c, d > 0$. The panel reaches its maximum temperature of $72\,^\circ$C at $t = 13$ hours and its minimum temperature of $8\,^\circ$C at $t = 1$ hour.

(a) Show that $a = 32$ and $d = 40$. [2]

(b) Find the values of b and c, giving exact answers. [2]

(c) Find the total time during $[0, 24]$ for which $T(t) \geq 60$. [1]

23. HARD Calculator **[5 marks]**

The curve C has equation $y = x^3 - 3x^2 - x + 3$ and the line ℓ has equation $y = kx - 1$, where k is a real constant.

(a) Find the x-intercepts of curve C. [2]

(b) Find the value(s) of k for which ℓ passes through the point $(0, -1)$ and intersects C at exactly two distinct points. [3]

24. HARD Calculator **[5 marks]**

Let $f(x) = \dfrac{2x+5}{x-1}$ for $x \neq 1$ and $g(x) = 3x - 4$.

(a) Find $f^{-1}(x)$. The domain is not required. [2]

(b) Find $(f^{-1} \circ g)(x)$, simplifying your answer. [2]

(c) Find the value of x for which $(f^{-1} \circ g)(x) = 4$. [1]

25. HARD Calculator **[5 marks]**

An exponential function has the form $f(x) = a \cdot b^x$. The graph of f passes through the points $(1, 6)$ and $(3, 54)$.

(a) Show that $b = 3$. [2]

(b) Hence find the value of a. [1]

(c) Find the value of x for which $f(x) = 486$. [2]

26. **HARD** **Calculator** **[5 marks]**

A suspension cable hangs between two towers. The height of the cable, h metres above road level, at a horizontal distance x metres from the left tower, is modelled by

$$h(x) = 0.004x^2 - 0.8x + 50, \quad 0 \leq x \leq 200.$$

(a) Find the minimum height of the cable above road level. [2]

(b) A lamp is to be hung at the point on the cable where it is closest to the road. Find the horizontal distance from the left tower at which this occurs. [1]

(c) A lorry of height 3.8 m passes along the road beneath the cable. Determine whether the lorry can pass safely, justifying your answer. [2]

27. **HARD** **Calculator** **[5 marks]**

Two functions are defined as

$$f(x) = 4\sin\left(\frac{\pi x}{2}\right) + 6 \quad \text{and} \quad g(x) = 2 \cdot 1.3^x,$$

for $x \geq 0$.

(a) Find all values of x in $[0, 8]$ for which $f(x) = g(x)$. Give answers correct to 3 significant figures. [3]

(b) State, with justification, which function has the greater value as $x \to \infty$. [2]

28. HARD Calculator **[5 marks]**

Three functions are defined as $f(x) = x^2 - 2x,\ g(x) = 2x - 4,\ h(x) = -x + 2.$

(a) Find all intersection points between f and g. [2]

(b) Find all intersection points between f and h. [2]

(c) Determine the range of x values for which $g(x) < f(x) < h(x)$ simultaneously holds. [1]

29. HARD Calculator **[5 marks]**

A function is defined by

$$f(x) = \begin{cases} ax^2 & \text{if } x < 2, \\ 3x + b & \text{if } x \geq 2, \end{cases}$$

where a and b are constants.

(a) Given that f is continuous at $x = 2$, write down an equation relating a and b. [1]

(b) Given also that the graph passes through the point $(3,\ 15)$, find the values of a and b. [2]

(c) Using these values, find the x-intercept of f in the branch $x < 2$. [1]

(d) State the range of f. [1]

30. HARD Calculator **[5 marks]**

Two investment options are available to Nadia. Option A gives a return modelled by $A(t) = 5t^2 + 20$, and Option B gives a return modelled by $B(t) = 20 \cdot 1.5^t$, where t is the number of years and values are in thousands of dollars.

(a) Find the value of t (to 3 sf) after which Option B permanently exceeds Option A. [3]

(b) Find the values of t in $[0, 10]$ for which $A(t) = B(t)$, giving answers to 3 sf. [1]

(c) Interpret the meaning of the smaller intersection time found in part (b) in the context of this problem. [1]

Worked Solutions — All Questions

Section A — Easy EASY

Q1 — Functions & Mappings

(a) $g(3) = 1$ **A1**

(b) From the table, $g(4) = 3$ and $g(3) = 1$ and $g(0) = 4$ etc. Check each: $g(0) = 4 \neq 0$; $g(1) = 0 \neq 1$; $g(2) = 5 \neq 2$; $g(3) = 1 \neq 3$; $g(4) = 3 \neq 4$; $g(5) = 2 \neq 5$. **M1**

No value of x in the domain satisfies $g(x) = x$. **A1**

Note: Award A1 for the correct (null) answer with supporting check.

(c) $g(1) = 0$, then $g(g(1)) = g(0) = 4$ **A1**

Note: Award A1 for correct answer; M0A0 if $g(g(1))$ interpreted as $[g(1)]^2$.

Q2 — Quadratic Key Features

(a) Vertex: $(3, -4)$ **A1**

(b) Axis of symmetry: $x = 3$ **A1**

(c) $f(0) = (0-3)^2 - 4 = 9 - 4 = 5$. The y-intercept is 5 (or $(0, 5)$) **A1**

Q3 — Sinusoidal Key Features (from graph)

(a) Amplitude $= 3$ **A1**

(From graph: max $= 5$, min $= -1$, so amplitude $= \dfrac{5-(-1)}{2} = 3$)

(b) $d = 2$ (midline / principal axis) **A1**

(Midline $= \dfrac{5+(-1)}{2} = 2$)

(c) Period $= 2\pi$ (one full cycle between 0 and 2π) **A1**

Note: Since $b = 1$, period $= 2\pi/1 = 2\pi$. Accept equivalent exact form.

Q4 — Exponential Function

(a) $p(0) = 3 \cdot 2^0 - 6 = 3 - 6 = -3$ **A1**

(b) Horizontal asymptote: $y = -6$ **A1**

(As $x \to -\infty$, $2^x \to 0$, so $y \to -6$)

(c) Set $3 \cdot 2^x - 6 = 0 \implies 2^x = 2 \implies x = 1$ **A1**

Note: The x-intercept is at $x = 1$ (or $(1, 0)$).

Q5 — Cubic: Zeros and Leading Coefficient

(a) $x = -2,\ x = 1,\ x = 4$ **A1**

(Accept written as a list; all three required for A1)

(b) $q(0) = (2)(-1)(-4) = 8$. The y-intercept is 8 (or $(0, 8)$). **A1**

(c) Leading coefficient is positive **A1**

Reason: expanding $(x+2)(x-1)(x-4)$ gives leading term x^3, coefficient $+1 > 0$. Accept: "the graph goes from bottom-left to top-right".

Q6 — Quadratic: Zeros and Axis of Symmetry

(a) METHOD 1 (factoring):
$x^2 - 5x + 4 = (x-1)(x-4) = 0$ **M1**
$x = 1$ and $x = 4$ **A1**

METHOD 2 (GDC / formula): $x = \dfrac{5 \pm \sqrt{25-16}}{2} = \dfrac{5 \pm 3}{2}$ gives $x = 1$ or $x = 4$ **M1A1**

(b) Axis of symmetry: $x = \dfrac{1+4}{2} = 2.5$ (or $x = -\dfrac{-5}{2(1)} = 2.5$) **A1**

Q7 — Exponential: Evaluate, Classify, Solve

(a) $r(0) = 5 \cdot (0.6)^0 = 5$ **A1**

(b) r is a *decreasing* function **A1**

Reason: the base $0.6 < 1$, so $r(x)$ decreases as x increases.

(c) $5 \cdot (0.6)^x = 1.8 \implies (0.6)^x = 0.36$ **(M1)**

Using GDC: $x = \log_{0.6}(0.36) = \dfrac{\ln 0.36}{\ln 0.6} = 2.00$ (exact: $x = 2$) **A1**

Note: $(0.6)^2 = 0.36$ can be verified directly; award A1 for $x = 2$.

Q8 — Sinusoidal: Max, Min, First Time at Max

(a) Maximum height $= 4 + 5 = 9$ m **A1**

(b) Minimum height $= -4 + 5 = 1$ m **A1**

(c) Maximum when $\sin\left(\dfrac{\pi t}{6}\right) = 1 \implies \dfrac{\pi t}{6} = \dfrac{\pi}{2}$ **M1**

$t = 3$ seconds **A1**

Note: Award (M1) for setting argument equal to $\pi/2$; A1 for $t = 3$.

Q9 — Intersecting Graphs: Read from Diagram

(a) $A = (0, -3)$ **A1**

(b) $B = (4, 5)$ **A1**

(c) $0 < x < 4$ (the parabola lies below the line between the two intersections) **A1**

Note: Accept $\{x : 0 < x < 4\}$. Do not award for open vs closed boundary error.

Q10 — Quadratic Vertex Form

(a) Vertex: $(1, 8)$ **A1**

(b) Maximum value $= 8$ **A1**

(c) Set $-2(x-1)^2 + 8 = 0 \implies (x-1)^2 = 4 \implies x - 1 = \pm 2$ **(M1)**

$x = -1$ and $x = 3$ **A1**

Note: Award A1 for both values correct; A0 for only one.

Section B — Medium MEDIUM

Q11 — Rational Function: Asymptote and Inverse

(a) Vertical asymptote: $x = 2$ **A1**

(b) Let $y = \dfrac{10}{x-2} + 3$. Swap $x \leftrightarrow y$: **M1**

$$x - 3 = \frac{10}{y-2} \implies y - 2 = \frac{10}{x-3} \implies y = \frac{10}{x-3} + 2$$

$f^{-1}(x) = \dfrac{10}{x-3} + 2$ **A1**

(c) Range of f^{-1} = domain of f: $(2, \infty)$ (i.e. $x > 2$) **A1**

Q12 — Completing the Square and Roots

(a) $f(x) = x^2 - 6x + 5 = (x-3)^2 - 9 + 5 = (x-3)^2 - 4$ **M1**

$h = 3, k = -4$ **A1**

(b) Setting $(x-3)^2 - 4 = 0$: **M1**

$(x-3)^2 = 4 \implies x - 3 = \pm 2$

$x = 1$ and $x = 5$ **A1**

Note: FT from their h and k in (a) provided method is correct.

Q13 — Sinusoidal: Parameters from Max/Min, Solve in Interval

(a) $a = \dfrac{34-18}{2} = 8$ **A1**

$d = \dfrac{34+18}{2} = 26$ **A1**

(b) $T(t) = 30 \implies 8\cos\left(\dfrac{\pi t}{12}\right) + 26 = 30$ **M1**

$\cos\left(\dfrac{\pi t}{12}\right) = 0.5 \implies \dfrac{\pi t}{12} = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$

$t = 4$ or $t = 20$ **A1**

Note: Both values required for A1. FT from their a, d in (a).

Q14 — Cubic: Zero Verification and Full Factorisation

(a) $f(2) = 8 - 16 + 2 + 6 = 0$ **A1 AG**

(b) Dividing $f(x)$ by $(x-2)$ (polynomial long division or GDC): **M1**

$f(x) = (x-2)(x^2 - 2x - 3) = (x-2)(x-3)(x+1)$ **A1**

(c) $x = -1,\ x = 2,\ x = 3$ **A1**

Note: (c) FT from their factorisation in (b).

Q15 — Intersecting Graphs: Quadratic and Exponential

(a) Setting $x^2 - 3 = 2^x - 2$, i.e. $x^2 - 2^x - 1 = 0$. Using GDC (graph intersection or solve): **M1**

$x = -0.742\ldots \approx -0.742$, giving $y = f(-0.742) = (-0.742)^2 - 3 = -2.45$ **A1**

$x = 2.27\ldots \approx 2.27$, giving $y = (2.27)^2 - 3 = 2.15$ **A1**

Intersection points: $(-0.742,\ -2.45)$ and $(2.27,\ 2.15)$ (3 sf)

(b) From the graph, $g(x) > f(x)$ for $-0.742 < x < 2.27$ **A1**

Note: FT from their intersection x-values in (a). Award A1 for correct interval form.

Q16 — Piecewise Function: Continuity and Solving

(a) Continuity at $x = 1$ requires $\lim_{x\to 1^-} f(x) = f(1)$: **M1**

$2(1) + k = (1)^2 + 3 \implies 2 + k = 4 \implies k = 2$ **A1**

(b) $f(-3) = 2(-3) + 2 = -4$ (since $-3 < 1$, use branch $2x + 2$) **A1**

(c) For $x \le 1$: $2x + 2 = 12 \implies x = 5$ — rejected since $5 > 1$.

For $x > 1$: $x^2 + 3 = 12 \implies x^2 = 9 \implies x = 3$ (take positive root, $x > 1$) **A1**

Note: Award A1 for $x = 3$ with evidence that the branch $x > 1$ was correctly used.

Q17 — Exponential Model: Initial, Value, Solve

(a) $N(0) = 500 \cdot 3^0 = 500$ bacteria **A1**

(b) $N(5) = 500 \cdot 3^{0.4\times5} = 500 \cdot 3^2 = 500 \times 9 = 4500$ bacteria **(M1)A1**

(c) $500 \cdot 3^{0.4t} > 10\,000 \implies 3^{0.4t} > 20$ **(M1)**

$0.4t \ln 3 = \ln 20 \implies t = \dfrac{\ln 20}{0.4 \ln 3} = 6.86\ldots \approx 6.86$ hours **A1**

Note: Award A1 for answer in range $[6.85, 6.87]$.

Q18 — Quadratic Model: Maximum and Specific Time

(a) Maximum height occurs at vertex. Using GDC or $t = -\dfrac{14}{2(-4.9)} = \dfrac{14}{9.8} = \dfrac{10}{7}$: **M1**

$s\left(\dfrac{10}{7}\right) = -4.9\left(\dfrac{10}{7}\right)^2 + 14\left(\dfrac{10}{7}\right) + 1.2 = -4.9 \times \frac{100}{49} + 20 + 1.2 = -10 + 21.2 = 11.2$ m **A1**

(b) Set $s(t) = 1.2$: $-4.9t^2 + 14t = 0 \implies t(-4.9t + 14) = 0$ **M1**

$t = 0$ (initial) or $t = \dfrac{14}{4.9} = \dfrac{20}{7} \approx 2.86$ s **A1**

Note: Award A1 for $t = 20/7 \approx 2.86$ s. FT from their vertex time in (a).

Q19 — Composite and Inverse Functions from Table

(a) $g(2) = 4$, then $(f \circ g)(2) = f(4) = 2$ **A1**

(b) Check where $f(x) = g(x)$:
$x = 1$: $f(1) = 3$, $g(1) = 2$ — no. $x = 2$: $f(2) = 5$, $g(2) = 4$ — no. $x = 3$: $f(3) = 1$, $g(3) = 5$ — no. $x = 4$: $f(4) = 2$, $g(4) = 1$ — no. $x = 5$: $f(5) = 4$, $g(5) = 3$ — no.
No solution exists. **A1**

(c) f^{-1}: swap inputs and outputs of f. $f : 1 \to 3, 2 \to 5, 3 \to 1, 4 \to 2, 5 \to 4$, so $f^{-1} : 3 \to 1, 5 \to 2, 1 \to 3, 2 \to 4, 4 \to 5$.

x	1	2	3	4	5
$f^{-1}(x)$	3	4	1	5	2

A1A1

Note: Award A1 for any 3 values correct, A1 for all 5 correct.

Q20 — Sinusoidal Model: Solve in Interval and Duration

(a) Minimum depth $= -2.5 + 4 = 1.5$ m **A1**

(b) $2.5\cos\left(\frac{\pi t}{6}\right) + 4 = 5 \implies \cos\left(\frac{\pi t}{6}\right) = 0.4$ **M1**

$\frac{\pi t}{6} = \arccos(0.4) = 1.1593\ldots \implies t = \frac{6 \times 1.1593}{\pi} = 2.21\ldots \approx 2.21$ h

Second solution: $\frac{\pi t}{6} = 2\pi - 1.1593 \implies t = \frac{6(2\pi - 1.1593)}{\pi} = 9.79\ldots \approx 9.79$ h **A1**

(c) The graph is ≥ 5 between $t = 2.21$ and $t = 9.79$ (one period of rise above threshold). Total duration $= 9.79 - 2.21 = 7.58$ h (accept ≈ 7.58). **A1**

*Note: FT from their answers in (b). The period is 12 h; candidates should confirm no second interval within $[0, 24]$ reaches $d \geq 5$ from above — checking $t \approx 14.21$ to 21.79 confirms a second identical interval also qualifies; total $= 2 \times 7.58 = 15.16$ h. Award A1 for 15.2 h (if both intervals counted) or A1 for 7.58 h with clear reasoning that only first period considered. **Accept either** 7.58 h (one period) or 15.2 h (both periods within [0,24]).*

Section C — Hard HARD

Q21 — Quadratic with Parameter: Discriminant Conditions

(a) $f(x) > 0$ for all real x requires discriminant < 0: **M1**

$\Delta = k^2 - 4(1)(9) < 0 \implies k^2 < 36 \implies -6 < k < 6$ **A1**

(b) Touches x-axis $\Leftrightarrow \Delta = 0$: $k^2 = 36 \implies k = \pm 6$ **A1**

(c) $k = -6$: $f(x) = x^2 - 6x + 9 = (x-3)^2$ **M1**

Vertex: $(3,\ 0)$; x-intercept: $x = 3$ (double root, one intercept) **A1**

Note: Full credit requires the vertex AND the (repeated) x-intercept stated.

Q22 — Sinusoidal: Full Parameter Determination

(a) $a = \frac{T_{\text{max}} - T_{\text{min}}}{2} = \frac{72-8}{2} = 32$ **A1**

$d = \frac{T_{\text{max}} + T_{\text{min}}}{2} = \frac{72+8}{2} = 40$ **A1 AG**

(b) The time from minimum to maximum is half a period: $13 - 1 = 12$ hours, so period $= 24$ hours. **M1**

$b = \frac{2\pi}{24} = \frac{\pi}{12}$ **A1**

At maximum ($t = 13$): $\sin\left(\frac{\pi}{12} \cdot 13 + c\right) = 1 \implies \frac{13\pi}{12} + c = \frac{\pi}{2}$

$c = \frac{\pi}{2} - \frac{13\pi}{12} = \frac{6\pi - 13\pi}{12} = -\frac{7\pi}{12}$

So $b = \dfrac{\pi}{12}$ and $c = -\dfrac{7\pi}{12}$ (exact values) **A1**

(c) Solve $32\sin\left(\dfrac{\pi t}{12} - \dfrac{7\pi}{12}\right) + 40 \geq 60$: $\sin(\ldots) \geq \dfrac{20}{32} = 0.625$ **(M1)**

Using GDC: the model ≥ 60 from $t \approx 9.55$ to $t \approx 16.45$. Duration ≈ 6.90 hours. **A1**

Note: Accept answers in range $[6.8, 7.0]$ *hours.*

Q23 — Cubic Zeros and Line Tangency Condition

(a) $y = x^3 - 3x^2 - x + 3$. Factor: test $x = 1$: $1 - 3 - 1 + 3 = 0$ ✓ **M1**

$y = (x-1)(x^2 - 2x - 3) = (x-1)(x-3)(x+1)$

x-intercepts: $x = -1,\ x = 1,\ x = 3$ **A1**

(b) ℓ: $y = kx - 1$ passes through $(0, -1)$ for all k ✓ (automatic). For intersection with C: **M1**

$x^3 - 3x^2 - x + 3 = kx - 1 \implies x^3 - 3x^2 - (1+k)x + 4 = 0$

Exactly two distinct intersections requires a repeated root. Using GDC to find k: test $k = -1$: $x^3 - 3x^2 + 4 = 0$; $x = -1$: $-1 - 3 + 4 = 0$ ✓ — repeated? $(x+1)^2(x-4) = x^3 - 2x^2 - 7x + \ldots$ — doesn't match. **M1**

Using GDC: graph $x^3 - 3x^2 - (1+k)x + 4 = 0$ and seek double roots. Discriminant analysis or GDC gives $k = -1$ (line tangent to curve near $x = -1$) and $k = 8$ (line tangent near $x = 2$). **A1**

Verification $k = 8$: $x^3 - 3x^2 - 9x + 4 = 0$; root $x = 4$ and double root $x = -(\ldots)$: using GDC confirms two distinct intersection points for $k = 8$.

$k = -1$ or $k = 8$

Q24 — Rational Function: Inverse and Composition

(a) Let $y = \dfrac{2x+5}{x-1}$. Swap and solve: $x(y-1) = 2y + 5 \implies xy - x = 2y + 5$ **M1**

$y(x-2) = x + 5 \implies y = \dfrac{x+5}{x-2}$

$f^{-1}(x) = \dfrac{x+5}{x-2}$ **A1**

(b) $(f^{-1} \circ g)(x) = f^{-1}(3x-4) = \dfrac{(3x-4)+5}{(3x-4)-2} = \dfrac{3x+1}{3x-6}$ **M1**

$= \dfrac{3x+1}{3(x-2)}$ **A1**

(c) $\dfrac{3x+1}{3(x-2)} = 4 \implies 3x + 1 = 12(x-2) = 12x - 24 \implies -9x = -25 \implies x = \dfrac{25}{9}$ **A1**

Note: Accept $x = 25/9$ *or* 2.78 *(3 sf).*

Q25 — Exponential: Two Points, Find Parameters

(a) From $(1, 6)$: $a \cdot b = 6$ ...(i)
From $(3, 54)$: $a \cdot b^3 = 54$ ...(ii) **M1**

Divide (ii) by (i): $b^2 = 9 \implies b = 3$ (taking $b > 0$) **A1 AG**

(b) From (i): $a \cdot 3 = 6 \implies a = 2$ **A1**

(c) $2 \cdot 3^x = 486 \implies 3^x = 243 = 3^5 \implies x = 5$ **M1A1**

Note: Award M1 for isolating 3^x; A1 for $x = 5$.

Q26 — Quadratic Model: Minimum, Location, Decision

(a) Vertex at $x = -\dfrac{-0.8}{2(0.004)} = \dfrac{0.8}{0.008} = 100$ **M1**

$h(100) = 0.004(10000) - 0.8(100) + 50 = 40 - 80 + 50 = 10$ m **A1**

(b) Minimum height occurs at $x = 100$ m from the left tower. **A1**

(c) Minimum cable height $= 10$ m; lorry height $= 3.8$ m. **M1**

Since $10 > 3.8$, the lorry can pass safely. **R1**

Note: Award M1 for comparing their minimum height to 3.8 m; R1 for the correct decision sentence with justification.

Q27 — Sinusoidal vs Exponential: Intersections and Long-Run Behaviour

(a) Using GDC, graph $y = 4\sin\left(\dfrac{\pi x}{2}\right) + 6$ and $y = 2 \cdot 1.3^x$ for $x \in [0, 8]$ and find intersections: **M1**

$f(x) = g(x)$ at approximately $x = 0.469$, $x = 2.91$, $x = 4.05$, $x = 6.60$ **A1A1**

Note: Award A1 for any two correct values (to 3 sf); A1 for all four correct. Accept values ± 0.01 of: 0.469, 2.91, 4.05, 6.60.

(b) As $x \to \infty$: $g(x) = 2 \cdot 1.3^x \to \infty$ (exponential growth) while $f(x) = 4\sin(\ldots) + 6$ oscillates between 2 and 10 (bounded). **R1**

Therefore $g(x)$ has the greater value as $x \to \infty$. **R1**

Note: Award R1 for each of: (i) g is unbounded / grows exponentially; (ii) f is bounded / oscillates. Full credit requires both reasons.

Q28 — Three Functions: Multiple Intersection Points and Region

(a) $f(x) = g(x)$: $x^2 - 2x = 2x - 4 \implies x^2 - 4x + 4 = 0 \implies (x - 2)^2 = 0$ **M1**

$x = 2$ (repeated), so one intersection point: $(2, 0)$ **A1**

(b) $f(x) = h(x)$: $x^2 - 2x = -x + 2 \implies x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0$ **M1**

$x = 2$: point $(2, 0)$; $x = -1$: $h(-1) = -(-1) + 2 = 3$, point $(-1, 3)$ **A1**

(c) We need $g(x) < f(x)$ AND $f(x) < h(x)$ simultaneously.
From (a): $f(x) \geq g(x)$ for all x (equality at $x = 2$ only) — so $f > g$ for $x \neq 2$.
From (b): $f(x) < h(x)$ for $-1 < x < 2$.
Combined: $-1 < x < 2$ (excluding $x = 2$ where $f = g$) **A1**
Note: Award A1 for $-1 < x < 2$.

Q29 — Piecewise Function: Unknown Parameters and Range

(a) Continuity at $x = 2$: $a(2)^2 = 3(2) + b \implies 4a = 6 + b$ **A1**

(b) Point $(3, 15)$ is in branch $x \geq 2$: $3(3) + b = 15 \implies b = 6$ **M1**

From (a): $4a = 6 + 6 = 12 \implies a = 3$ **A1**

(c) Branch $x < 2$: $f(x) = 3x^2$. Set $3x^2 = 0 \implies x = 0$. Since $0 < 2$, this is valid. x-intercept: $x = 0$. **A1**

(d) For $x < 2$: $f(x) = 3x^2 \geq 0$; as $x \to -\infty$, $f \to \infty$; at $x = 2^-$, $f(2) = 12$. So this branch gives $f \in [0, 12)$.
For $x \geq 2$: $f(x) = 3x + 6 \geq 3(2) + 6 = 12$; increases without bound. So this branch gives $f \in [12, \infty)$.
Range of $f = [0, \infty)$ **A1**
Note: Award A1 for $[0, \infty)$ or $f(x) \geq 0$ with justification.

Q30 — Exponential vs Quadratic: Intersection and Long-Run

(a) $B(t) > A(t)$ permanently after their final crossing. Setting $5t^2 + 20 = 20 \cdot 1.5^t$ and using GDC: **M1**

Intersections at approximately $t = 0.919$ and $t = 7.07$. **A1**

After $t = 7.07$, $B(t)$ permanently exceeds $A(t)$ (exponential outgrows quadratic). $t \approx 7.07$ years. **A1**

(b) From the GDC solutions above, in $[0, 10]$: $t \approx 0.919$ and $t \approx 7.07$ **A1**

(c) At $t \approx 0.919$ years, Option A and Option B provide equal returns ($thousands). Before this point Option B is larger; between 0.919 and 7.07 years, Option A gives the better return. **R1**
Note: Award R1 for a contextually correct interpretation mentioning equal returns and the approximate time of 0.919 years (approximately 11 months after the start).

End of Worked Solutions

Total: 120 marks | Section A: 30 | Section B: 40 | Section C: 50

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