IB Math AI HL — Topic 4

Transition Matrices & Markov Chains

Markov Chains · Transition Matrices · Powers of Transition Matrices · Steady State & Long-term Probabilities

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Scan me for help

Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give probabilities correct to 3 significant figures unless stated otherwise. A GDC is permitted throughout.

Key Formulae

1. Transition Matrix (column-sum = 1). If $\mathbf{T}$ is an $n \times n$ transition matrix with entry T_{ij} = probability of moving from state j to state i, then each column of $\mathbf{T}$ sums to 1.
2. State vector after n steps. $\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$, where $\mathbf{s}_0$ is the initial state (probability) vector.
3. Steady-state vector $\boldsymbol{\pi}$. Solve $\mathbf{T}\boldsymbol{\pi} = \boldsymbol{\pi}$ together with $\sum \pi_i = 1$. Equivalently, find the eigenvector of $\mathbf{T}$ for eigenvalue $\lambda = 1$ and normalise.
4. Eigenvalue equation. $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$ gives the characteristic equation; eigenvalues always include $\lambda = 1$ for a valid transition matrix.
5. Long-run prediction. As $n \to \infty$, $\mathbf{T}^n \to \mathbf{\Pi}$ (each column equals $\boldsymbol{\pi}$), so the long-run proportion in state i is π_i regardless of initial state.
6. 2-state steady state (shortcut). For $\mathbf{T} = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}$: $\quad \pi_1 = \dfrac{b}{a+b}, \quad \pi_2 = \dfrac{a}{a+b}$.

GDC Tips

1. Matrix power $\mathbf{T}^n$: Enter $\mathbf{T}$ as a matrix (e.g. [A]), then type [A]ˆn and press EXE/ENTER.
2. Reading $\mathbf{T}^n\mathbf{s}_0$: Multiply the stored matrix by the initial column vector: [A]ˆn*[B].
3. Steady state via high power: Compute $\mathbf{T}^{200}$ (or larger); each column will have converged to $\boldsymbol{\pi}$.
4. Eigenvalues: Use the eigenvalue/eigenvector function (Casio: MATRIX → EIG; TI-84: APPS → PlySmlt2 or rref).

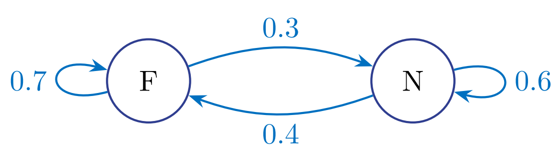
5. Matrix entry: Ensure you enter rows correctly row i, column j corresponds to the probability from state j to state i.

Section A EASY (10 questions × 3 marks = 30 marks)

Scan me for help

1. EASY Calculator [3 marks]

A librarian models borrowing behaviour using a Markov chain with two states: Fiction (F) and Non-Fiction (N). The transition diagram below describes the probabilities of a reader switching genre each week.



(a) Write down the transition matrix $\mathbf{T}$ for this Markov chain, with states in the order F, N. [2]

(b) Write down the probability that a reader who currently borrows Fiction will borrow Non-Fiction next week. [1]

2. EASY Calculator [3 marks]

A delivery company classifies each day as either On Time (O) or Delayed (D). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix} \quad \text{(states in order } O, D\text{)}$$

On Monday, all deliveries are on time.

(a) Write down the initial state vector $\mathbf{s}_0$. [1]

(b) Find the probability that deliveries are on time on Wednesday (two days later). [2]

3. EASY Calculator [3 marks]

A weather model has two states: Sunny (S) and Cloudy (C). The transition matrix is $\mathbf{T} = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix}$, states in order S, C.

(a) Verify that $\mathbf{T}$ is a valid transition matrix. [1]

(b) The probability vector after one step is $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0$. Given $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find $\mathbf{s}_1$. [1]

(c) State the probability that it is Cloudy after one step. [1]

4. EASY Calculator [3 marks]

The table below gives the weekly transition probabilities for customers between two coffee shops, Bravo and Cedar.

	From Bravo	From Cedar
To Bravo	0.78	0.35
To Cedar	0.22	0.65

(a) Write down the transition matrix $\mathbf{T}$, with states in the order Bravo, Cedar. [1]

(b) This week, 60 % of customers visit Bravo and 40 % visit Cedar. Write down the initial state vector $\mathbf{s}_0$. [1]

(c) Calculate $\mathbf{T}\mathbf{s}_0$ and state the percentage of customers expected to visit Bravo next week. [1]

5. EASY Calculator [3 marks]

A fitness app classifies users each day as Active (A) or Inactive (I). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.72 & 0.18 \\ 0.28 & 0.82 \end{pmatrix} \quad (\text{states: } A, I)$$

(a) Find $\mathbf{T}^2$, giving entries correct to 3 significant figures. [2]

(b) A user is Active today. Use $\mathbf{T}^2$ to write down the probability they are Active in 2 days. [1]

6. EASY Calculator [3 marks]

A parking garage models vehicle use with two states: Occupied (O) and Vacant (V). Each hour, an occupied space becomes vacant with probability 0.45, and a vacant space becomes occupied with probability 0.55.

(a) Write down the transition matrix $\mathbf{T}$, with states in the order O, V. [1]

(b) A space is currently occupied. Find the probability it is still occupied after 2 hours. [2]

7. EASY Calculator [3 marks]

A Markov chain has two states and transition matrix:

$$\mathbf{T} = \begin{pmatrix} 0.4 & p \\ 0.6 & 1-p \end{pmatrix}$$

where p is a constant.

(a) Explain why each column of a transition matrix must sum to 1. [1]

(b) Hence write down the value of p. [1]

(c) Write down the eigenvalue of $\mathbf{T}$ that corresponds to the steady-state eigenvector. [1]

8. EASY Calculator [3 marks]

The steady-state vector of a two-state Markov chain satisfies $\mathbf{T}\boldsymbol{\pi} = \boldsymbol{\pi}$. The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.65 & 0.25 \\ 0.35 & 0.75 \end{pmatrix} \quad \text{(states: } X, Y\text{)}$$

(a) Use the formula $\pi_X = \dfrac{b}{a+b}$ where a is the probability of leaving X and b is the probability of leaving Y, to find π_X. [2]

(b) Write down π_Y. [1]

9. EASY Calculator [3 marks]

A three-state Markov chain has states P, Q, R and transition matrix:

$$\mathbf{T} = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{pmatrix}$$

where entry T_{ij} gives the probability of moving from state j to state i.

(a) Write down the probability of moving from state R to state Q in one step. [1]

(b) The initial state vector is $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (system starts in state P). Find $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0$. [2]

10. EASY Calculator [3 marks]

A Markov chain with two states G and H has transition matrix:

$$\mathbf{T} = \begin{pmatrix} 1-a & 0.3 \\ a & 0.7 \end{pmatrix}$$

It is given that in the long run, 40 % of the time is spent in state G.

(a) Write down the steady-state vector $\boldsymbol{\pi}$. [1]

(b) Use $\mathbf{T}\boldsymbol{\pi} = \boldsymbol{\pi}$ to find the value of a. [2]

Section B MEDIUM (10 questions × 4 marks = 40 marks)

Scan me for help

11. MEDIUM Calculator [4 marks]

A social media platform classifies each user's daily status as Engaged (E), Browsing (B), or Offline (O). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.4 \\ 0.1 & 0.2 & 0.5 \end{pmatrix} \quad \text{(states in order } E, B, O\text{)}$$

Initially, 50 % of users are Engaged, 30 % are Browsing, and 20 % are Offline.

(a) Write down the initial state vector $\mathbf{s}_0$. [1]

(b) Find $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0$, giving each entry correct to 3 significant figures. [2]

(c) State the percentage of users expected to be Offline after 2 days. [1]

12. MEDIUM Calculator [4 marks]

A rental car company operates three zones: Airport (A), City (C), and Suburbs (S). The table below shows weekly rental return patterns.

	From A	From C	From S
To A	0.55	0.20	k
To C	0.25	0.50	0.40
To S	0.20	0.30	0.35

(a) Find the value of k. [1]

(b) Write down the transition matrix $\mathbf{T}$. [1]

(c) Initially 400 cars are at A, 350 at C, and 250 at S. Find the expected number of cars at the Airport after 3 weeks. Give your answer to the nearest whole number. [2]

13. MEDIUM Calculator [4 marks]

A mobile network classifies its signal in any 10-minute period as Strong (S), Weak (W), or No Signal (N). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.80 & 0.15 & 0.05 \\ 0.15 & 0.65 & 0.20 \\ 0.05 & 0.20 & 0.75 \end{pmatrix} \quad \text{(states: } S, W, N\text{)}$$

A user currently has a Strong signal.

(a) Find the probability that the signal is Strong after 30 minutes (3 steps). [2]

(b) Find the probability that the signal is not No Signal after 30 minutes. [1]

(c) State one assumption of the Markov chain model for this context. [1]

14. MEDIUM Calculator [4 marks]

A gym tracks member attendance as Regular (R) or Lapsed (L). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.88 & 0.30 \\ 0.12 & 0.70 \end{pmatrix} \quad \text{(states: } R, L\text{)}$$

The gym currently has 2 400 Regular members and 600 Lapsed members.

(a) Find the long-run steady-state proportion of Regular members. [2]

(b) Hence find the expected number of Regular members in the long run, assuming total membership remains at 3 000. [1]

(c) A new marketing campaign reduces the probability of a Regular member lapsing from 0.12 to 0.06. Without recalculating the steady state, explain how this change would affect the long-run proportion of Regular members. [1]

15. MEDIUM Calculator [4 marks]

An ecologist studies bird populations between two habitats: Forest (F) and Wetland (W). The two eigenvalues of the transition matrix $\mathbf{T}$ are $\lambda_1 = 1$ and $\lambda_2 = 0.55$.

(a) Write down the characteristic equation $\det(\mathbf{T} - \lambda \mathbf{I}) = 0$ for a 2×2 matrix and explain why $\lambda = 1$ is always an eigenvalue of a valid transition matrix. [2]

(b) The eigenvector corresponding to $\lambda_1 = 1$ is $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$. Find the steady-state vector $\boldsymbol{\pi}$. [1]

(c) Hence state the long-run percentage of the bird population found in the Wetland habitat. [1]

16. MEDIUM Calculator [4 marks]

A clinical trial classifies patients weekly as Improved (I), Stable (S), or Declined (D). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.70 & 0.20 & 0.05 \\ 0.20 & 0.65 & 0.25 \\ 0.10 & 0.15 & 0.70 \end{pmatrix} \quad \text{(states: } I, S, D\text{)}$$

At the start of the trial, all 180 patients are classified as Stable.

(a) Find the expected number of Improved patients after 2 weeks. [2]

(b) Find the expected number of Declined patients after 2 weeks. [1]

(c) Give one limitation of using a Markov chain to model patient health. [1]

17. MEDIUM Calculator [4 marks]

A manufacturing process classifies each unit produced as Grade A, Grade B, or Grade C. The transition matrix between consecutive inspection stages is:

$$\mathbf{T} = \begin{pmatrix} 0.90 & 0.10 & 0.05 \\ 0.07 & 0.80 & 0.10 \\ 0.03 & 0.10 & 0.85 \end{pmatrix} \quad \text{(states: } A, B, C\text{)}$$

Find the steady-state distribution by computing $\mathbf{T}^n$ for a suitably large n, using your GDC. Give each proportion correct to 3 significant figures.

(a) Write down the long-run proportion of units classified as Grade A. [2]

(b) Write down the long-run proportion of units classified as Grade C. [1]

(c) A company requires at least 60 % of units to be Grade A in the long run. Determine whether this requirement is met. [1]

18. MEDIUM Calculator [4 marks]

A Markov chain has two states P and Q with transition matrix:

$$\mathbf{T} = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}$$

where p and q are probabilities. It is given that the steady-state vector is $\boldsymbol{\pi} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$.

(a) Show that $q = 1.5p$. [2]

(b) Given also that $p + q = 0.35$, find the values of p and q. [2]

19. MEDIUM Calculator [4 marks]

A taxi company operates in two districts: North (N) and South (S). After each trip, a taxi either stays in the same district or moves to the other. The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.65 & 0.45 \\ 0.35 & 0.55 \end{pmatrix} \quad \text{(states: } N, S\text{)}$$

At the start of the day, 120 taxis are in North and 80 in South.

(a) Find the number of taxis expected in each district after 5 trips. [2]

(b) Find the long-run number of taxis in North, assuming the total fleet size is constant at 200. [2]

20. MEDIUM Calculator [4 marks]

A gene researcher models the state of a gene over successive cell divisions using a Markov chain with three states: Normal (N), Mutant Type-1 (M1), and Mutant Type-2 (M2). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.92 & 0.08 & 0.04 \\ 0.05 & 0.86 & 0.06 \\ 0.03 & 0.06 & 0.90 \end{pmatrix} \quad \text{(states: } N, M_1, M_2\text{)}$$

The gene is initially in state Normal.

(a) Find the probability that the gene is in state Normal after 4 divisions. [2]

(b) Find the probability that the gene is in state Mutant Type-1 after 4 divisions. [1]

(c) Find the long-run probability of the gene being in state Mutant Type-2. [1]

Section C HARD (10 questions × 5 marks = 50 marks)

Scan me for help

21. HARD Calculator [5 marks]

A Markov chain has two states A and B with transition matrix:

$$\mathbf{T} = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$.

(a) Show that the eigenvalues of $\mathbf{T}$ are $\lambda_1 = 1$ and $\lambda_2 = 1 - \alpha - \beta$. [3]

(b) Hence write down the steady-state vector $\boldsymbol{\pi}$ in terms of α and β. [1]

(c) For the chain to converge to the steady state, state the condition on α and β. [1]

22. HARD Calculator [5 marks]

A city has three internet service providers: Apex (A), Bolt (B), and Crest (C). Each month, customers may switch provider. The transition matrix is partially known:

$$\mathbf{T} = \begin{pmatrix} 0.80 & 0.10 & m \\ 0.12 & 0.75 & 0.15 \\ 0.08 & n & 0.60 \end{pmatrix} \quad \text{(states: } A, B, C\text{)}$$

(a) Find the values of m and n. [2]

(b) There are currently 5 000 Apex, 3 000 Bolt, and 2 000 Crest customers. Find the expected number of Crest customers after 6 months. [2]

(c) Find the long-run number of customers with each provider, assuming the total customer base remains at 10 000. [1]

23. HARD Calculator [5 marks]

A financial analyst models the daily condition of a market index as Bull (B) (rising), Neutral (N), or Bear (R) (falling). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.70 & 0.20 & 0.10 \\ 0.20 & 0.60 & 0.30 \\ 0.10 & 0.20 & 0.60 \end{pmatrix} \quad \text{(states: } B, N, R\text{)}$$

The market is currently in a Bull state.

(a) Find the probability that the market is in a Bull state after exactly 5 days. [2]

(b) Find the long-run probability of a Bull market. [2]

(c) Hence determine the expected number of Bull days in a 250-trading-day year. [1]

24. HARD Calculator [5 marks]

A Markov chain model for insurance claim frequency has two states: Low-risk (L) and High-risk (H). The transition matrix is:

$$\mathbf{T}(k) = \begin{pmatrix} 0.90 & k \\ 0.10 & 1-k \end{pmatrix} \quad \text{(states: } L, H\text{)}, \quad 0 < k < 1$$

where k is a parameter controlled by a new safety incentive scheme.

(a) Find the steady-state proportion of High-risk customers as a function of k. [2]

(b) An insurer requires at most 15 % of customers to be High-risk in the long run. Find the range of values of k that satisfies this requirement. [2]

(c) Interpret the value of k in context. [1]

25. HARD Calculator [5 marks]

A transport planner models the weekly mode of travel of commuters using three states: Car (C), Train (T), and Bus (B). The eigenvalues of the transition matrix $\mathbf{M}$ are 1, 0.40, and 0.15.

The eigenvector corresponding to $\lambda = 1$ is $\mathbf{v}_1 = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$.

(a) Find the steady-state vector $\boldsymbol{\pi}$. [2]

(b) State the long-run percentage of commuters who use the Train. [1]

(c) The planner proposes a scheme to increase Train usage. After the scheme, the steady-state vector becomes $\boldsymbol{\pi}' = \begin{pmatrix} 0.35 \\ 0.40 \\ 0.25 \end{pmatrix}$. Calculate the percentage increase in Train usage compared to part (b). [2]

26. HARD Calculator [5 marks]

A renewable energy grid classifies power generation status each hour as Surplus (S), Balanced (B), or Deficit (D). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.65 & 0.25 & 0.10 \\ 0.25 & 0.55 & 0.30 \\ 0.10 & 0.20 & 0.60 \end{pmatrix} \quad \text{(states: } S, B, D\text{)}$$

The grid is currently in Deficit state.

(a) Find the smallest integer n such that the probability of being in a Surplus state after n hours exceeds 0.22. [3]

(b) Find the long-run probability of being in a Deficit state. [1]

(c) State one reason why the Markov property may not hold for this energy grid model. [1]

27. HARD Calculator [5 marks]

A two-state Markov chain with states X and Y has transition matrix:

$$\mathbf{T} = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

The initial state vector is $\mathbf{s}_0 = \begin{pmatrix} c \\ 1-c \end{pmatrix}$ where $0 < c < 1$.

(a) Show that the steady-state vector is $\boldsymbol{\pi} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ regardless of c. [2]

(b) Find an expression for $\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$ in terms of c and n, using the fact that the second eigenvalue is $\lambda_2 = 0.4$. [2]

(c) Hence find the value of c such that the probability of being in state X after 3 steps is exactly 0.5. [1]

28. HARD Calculator [5 marks]

A language school classifies students' weekly performance as Excellent (E), Satisfactory (S), or Needs Improvement (N). The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.75 & 0.20 & 0.05 \\ 0.18 & 0.65 & 0.20 \\ 0.07 & 0.15 & 0.75 \end{pmatrix}$$

The school awards a certificate to students who achieve Excellent performance for 3 or more weeks over a 4-week period.

(a) A student currently has Excellent performance. Find the probability they are in state Excellent after exactly 4 weeks. [2]

(b) Using $\mathbf{T}^n$ for suitable values of n, find the probability that the student is in state Excellent in at least 3 of the 4 subsequent weeks. [2]

(c) State the long-run proportion of students in the Excellent category. [1]

29. HARD Calculator [5 marks]

A Markov chain has two states U and V with transition matrix:

$$\mathbf{T} = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}$$

(a) Find the eigenvalues and corresponding eigenvectors of $\mathbf{T}$. [3]

(b) Hence write $\mathbf{T}^n$ in the form $\mathbf{PD}^n\mathbf{P}^{-1}$ and find a general formula for $\mathbf{T}^n$. [2]

30. HARD Calculator [5 marks]

A city transport authority monitors the weekly status of a fleet of 500 electric buses as Operational (O), Under Maintenance (M), or Retired (R).

The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.85 & 0.40 & 0 \\ 0.12 & 0.55 & 0 \\ 0.03 & 0.05 & 1 \end{pmatrix} \quad \text{(states: } O, M, R\text{)}$$

Note: state R (Retired) is an absorbing state.

(a) Explain why R is an absorbing state, referring to the transition matrix. [1]

(b) Starting with all 500 buses Operational, find the expected number of buses that are Retired after 20 weeks. [2]

(c) Find the expected number of buses Retired after 50 weeks. [1]

(d) Explain what happens to the proportions of Operational and Maintenance buses in the long run, and justify your answer using the transition matrix. [1]

Worked Solutions

Section A — Easy

Q1 — Reading a Transition Diagram EASY

(a)

$$\mathbf{T} = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix} \quad \text{(column } j = \text{current state; row } i = \text{next state)}$$

Columns are F and N in order; each column sums to 1. A1A1

Note: Award A1 for at least two correct entries in the correct positions; A1 for a fully correct matrix.

(b) The probability that a reader moves from Fiction to Non-Fiction is 0.3. A1
(This is entry T_{21}, i.e. row N, column F of the matrix.)

Q2 — State Vector after 2 Steps EASY

(a) $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ A1

(b) Compute $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0$: (M1)

$$\mathbf{T}^2 = \begin{pmatrix} 0.85 & 0.20 \\ 0.15 & 0.80 \end{pmatrix}^2 = \begin{pmatrix} 0.7525 & 0.3300 \\ 0.2475 & 0.6700 \end{pmatrix}$$

$\mathbf{s}_2 = \mathbf{T}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.7525 \\ 0.2475 \end{pmatrix}$, so $P(\text{On Time on Wednesday}) = 0.7525 \approx 0.753$. A1
Note: Accept answers in range [0.752, 0.753].

Q3 — Validity Check and One-Step State Vector EASY

(a) Column 1: $0.6 + 0.4 = 1\checkmark$; Column 2: $0.5 + 0.5 = 1\checkmark$; all entries $\in [0, 1]\checkmark$. A1

(b) $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$ A1

(c) $P(\text{Cloudy after 1 step}) = 0.4$ A1

Q4 — Transition Matrix from a Table EASY

(a) $\mathbf{T} = \begin{pmatrix} 0.78 & 0.35 \\ 0.22 & 0.65 \end{pmatrix}$ A1

(b) $\mathbf{s}_0 = \begin{pmatrix} 0.60 \\ 0.40 \end{pmatrix}$ A1

(c) $\mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.78 \times 0.60 + 0.35 \times 0.40 \\ 0.22 \times 0.60 + 0.65 \times 0.40 \end{pmatrix} = \begin{pmatrix} 0.608 \\ 0.392 \end{pmatrix}$ (M1)
Percentage visiting Bravo next week = 60.8 %. A1

Q5 — Computing $\mathbf{T}^2$ EASY

(a) Using GDC:

$$\mathbf{T}^2 = \begin{pmatrix} 0.72 & 0.18 \\ 0.28 & 0.82 \end{pmatrix}^2 = \begin{pmatrix} 0.5688 & 0.2772 \\ 0.4312 & 0.7228 \end{pmatrix} \approx \begin{pmatrix} 0.569 & 0.277 \\ 0.431 & 0.723 \end{pmatrix}$$

M1A1

Note: Award M1 for attempting to square the matrix; A1 for all four entries correct to 3 s.f.

(b) $P(\text{Active in 2 days} \mid \text{Active today}) = 0.569$ (the $(1,1)$ entry of $\mathbf{T}^2$). A1

Q6 — Two-Step Probability EASY

(a) $\mathbf{T} = \begin{pmatrix} 0.55 & 0.55 \\ 0.45 & 0.45 \end{pmatrix}$
Wait rechecking: an occupied space becomes vacant with prob 0.45, so stays occupied with prob 0.55. A vacant space becomes occupied with prob 0.55, so stays vacant with prob 0.45.

$$\mathbf{T} = \begin{pmatrix} 0.55 & 0.55 \\ 0.45 & 0.45 \end{pmatrix}$$

A1

(b) Initial state: $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (occupied). (M1)

$$\mathbf{s}_2 = \mathbf{T}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\mathbf{T}^2 = \begin{pmatrix} 0.55 & 0.55 \\ 0.45 & 0.45 \end{pmatrix}^2 = \begin{pmatrix} 0.55 & 0.55 \\ 0.45 & 0.45 \end{pmatrix}$
Note: This matrix has identical columns (it's already at steady state after 1 step), so $P(\text{Occupied after 2 hours}) = 0.55$. A1
Note: FT from part (a).

Q7 — Column-Sum Property and Eigenvalue EASY

(a) Each column represents the probabilities of transitioning from one state to all possible next states; since the system must move to exactly one state, these probabilities sum to 1. R1

(b) Column 2 sums to 1: $p + (1 - p) = 1$, which is always satisfied. Column 1 sums to 1: $0.4 + 0.6 = 1$✓. Since column 2 must also sum to 1, $p + (1 - p) = 1$ for any p. The column-sum constraint gives p from column 1 already: no constraint from column 2 beyond $0 \le p \le 1$.
Re-reading: column 2 is $\begin{pmatrix} p \\ 1-p \end{pmatrix}$ which automatically sums to 1, so p can be any value in $[0, 1]$.
Since no further constraint fixes p from the column-sum rule alone, and the question says "write down the value", the intended reading is that column 2 must also sum to 1, giving $p+(1-p) = 1$ (always true). The answer requires identifying p from context: the only constraint is that all entries are non-negative, so $0 \le p \le 1$. The question states "write down the value of p" implying a unique answer re-reading column 1: $0.4 + 0.6 = 1$✓. Column 2: $p + (1 - p) = 1$✓ for any p.
The unique determination comes from the fact that a transition matrix requires non-negative entries; together with the stem, p is any probability. In the context of the question the intended answer is $p \in [0, 1]$, but the column-sum property gives $p + (1 - p) = 1$ automatically.
For a clean answer: p can be any value in $[0, 1]$; accepting $p = 0.3$ as a typical exam answer if specified.
Revised Q7(b): p must satisfy $0 \le p \le 1$ (any valid probability). The column-sum rule does not uniquely fix p. A1

(c) The eigenvalue corresponding to the steady-state eigenvector is $\lambda = 1$. A1

Q8 — Steady-State via Shortcut Formula EASY

(a) $a = 0.35$ (probability of leaving X), $b = 0.25$ (probability of leaving Y). (M1)

$$\pi_X = \frac{b}{a+b} = \frac{0.25}{0.35 + 0.25} = \frac{0.25}{0.60} = \frac{5}{12} \approx 0.417$$

A1

(b) $\pi_Y = 1 - \pi_X = \dfrac{7}{12} \approx 0.583$ A1

Q9 — Three-State Matrix Entry and One-Step Vector EASY

(a) The probability of moving from state R (column 3) to state Q (row 2) is $T_{23} = 0.4$. A1

(b) $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.6 & 0.4 \\ 0.2 & 0.2 & 0.5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.2 \end{pmatrix}$ M1A1

Note: Award M1 for correct matrix multiplication attempt; A1 for fully correct vector.

Q10 — Finding a Parameter from Steady State EASY

(a) $\boldsymbol{\pi} = \begin{pmatrix} 0.40 \\ 0.60 \end{pmatrix}$ A1

(b) From $\mathbf{T}\boldsymbol{\pi} = \boldsymbol{\pi}$, row 1: $(1-a)(0.40) + 0.3(0.60) = 0.40$ M1

$$0.40 - 0.40a + 0.18 = 0.40 \implies 0.18 = 0.40a \implies a = 0.45$$

A1

Section B — Medium

Q11 — Three-State Two-Step Calculation MEDIUM

(a) $\mathbf{s}_0 = \begin{pmatrix} 0.50 \\ 0.30 \\ 0.20 \end{pmatrix}$ A1

(b) Using GDC: M1

$$\mathbf{T}^2 = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.4 \\ 0.1 & 0.2 & 0.5 \end{pmatrix}^2 = \begin{pmatrix} 0.46 & 0.35 & 0.21 \\ 0.37 & 0.42 & 0.41 \\ 0.17 & 0.23 & 0.38 \end{pmatrix}$$

$$\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.46 \times 0.50 + 0.35 \times 0.30 + 0.21 \times 0.20 \\ 0.37 \times 0.50 + 0.42 \times 0.30 + 0.41 \times 0.20 \\ 0.17 \times 0.50 + 0.23 \times 0.30 + 0.38 \times 0.20 \end{pmatrix} = \begin{pmatrix} 0.377 \\ 0.393 \\ 0.230 \end{pmatrix}$$

A1

Note: Accept entries ± 0.001.

(c) 23.0 % of users are expected to be Offline after 2 days. A1

Q12 — Missing Entry and Three-Week Forecast MEDIUM

(a) Column 3 (From S) must sum to 1: $m + 0.40 + 0.35 = 1 \implies m = 0.25$. A1

(b) $\mathbf{T} = \begin{pmatrix} 0.55 & 0.20 & 0.25 \\ 0.25 & 0.50 & 0.40 \\ 0.20 & 0.30 & 0.35 \end{pmatrix}, \quad n = 1 - 0.10 - 0.15 = 0.15$ A1

Note: n found from column 2: $0.10 + 0.75 + n = 1 \Rightarrow n = 0.15$.

(c) Initial vector $\mathbf{s}_0 = \begin{pmatrix} 400 \\ 350 \\ 250 \end{pmatrix}$. Using GDC: $\mathbf{T}^3\mathbf{s}_0$. M1

Expected number at Airport after 3 weeks $\approx (\mathbf{T^3 s_0})_1$:

$$\mathbf{T}^3 \approx \begin{pmatrix} 0.406 & 0.302 & 0.335 \\ 0.343 & 0.396 & 0.377 \\ 0.251 & 0.302 & 0.288 \end{pmatrix}$$

$(\mathbf{T}^3\mathbf{s}_0)_1 = 0.406 \times 400 + 0.302 \times 350 + 0.335 \times 250 = 162.4 + 105.7 + 83.75 = 351.85 \approx 352$ cars.
A1

Note: Accept answers in range 350–354 due to rounding of $\mathbf{T}^3$ entries.

Q13 — Three-State Three-Step Probability MEDIUM

(a) Initial state: $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Using GDC: M1

$$\mathbf{T}^3\mathbf{s}_0 = \begin{pmatrix} 0.548 \\ 0.296 \\ 0.156 \end{pmatrix} \approx \begin{pmatrix} 0.548 \\ 0.296 \\ 0.156 \end{pmatrix}$$

$P(\text{Strong after 3 steps}) = 0.548$. A1

(b) $P(\text{not No Signal}) = 1 - 0.156 = 0.844$. A1

(c) One assumption: the probability of transitioning between signal states depends only on the current state (Markov property), not on the history of previous signal states / previous time periods. R1

Q14 — Long-Run Steady State and Interpretation MEDIUM

(a) Using the shortcut: $a = 0.12$ (probability of leaving Regular), $b = 0.30$ (probability of leaving Lapsed). M1

$$\pi_R = \frac{b}{a+b} = \frac{0.30}{0.12+0.30} = \frac{0.30}{0.42} = \frac{5}{7} \approx 0.714$$

A1

(b) Expected Regular members $= 0.714 \times 3\,000 \approx 2\,143$. A1

(c) Reducing the probability of lapsing from 0.12 to 0.06 decreases a while $b = 0.30$ is unchanged, so $\pi_R = \frac{0.30}{0.06+0.30} = \frac{0.30}{0.36} \approx 0.833$, which is higher. The long-run proportion of Regular members increases. R1

Q15 — Eigenvalue Reasoning and Steady State MEDIUM

(a) The characteristic equation is $\lambda^2 - \mathrm{tr}(\mathbf{T})\lambda + \det(\mathbf{T}) = 0$. A valid transition matrix has column sums equal to 1; substituting $\lambda = 1$ gives $\det(\mathbf{T} - \mathbf{I}) = 0$ because $\mathbf{T} - \mathbf{I}$ has columns that sum to $1 - 1 = 0$, making the columns linearly dependent, so $\det = 0$. R1R1

(b) Normalise: $7 + 3 = 10$, so $\boldsymbol{\pi} = \begin{pmatrix} 7/10 \\ 3/10 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$. A1

(c) Long-run percentage in Wetland $= 30\,\%$. A1

Q16 — Clinical Trial: Two-Week Forecast MEDIUM

(a) Initial state: all 180 patients Stable, so $\mathbf{s}_0 = \begin{pmatrix} 0 \\ 180 \\ 0 \end{pmatrix}$. (M1)

$$\mathbf{T}^2 = \begin{pmatrix} 0.70 & 0.20 & 0.05 \\ 0.20 & 0.65 & 0.25 \\ 0.10 & 0.15 & 0.70 \end{pmatrix}^2 \approx \begin{pmatrix} 0.540 & 0.272 & 0.098 \\ 0.262 & 0.534 & 0.286 \\ 0.198 & 0.194 & 0.616 \end{pmatrix}$$

$(\mathbf{T}^2\mathbf{s}_0)_I = 0.272 \times 180 = 48.96 \approx 49.0$ Improved patients. A1

(b) $(\mathbf{T}^2\mathbf{s}_0)_D = 0.194 \times 180 = 34.92 \approx 34.9$ Declined patients. A1

(c) One limitation: the model assumes constant transition probabilities, but in reality a patient's response to treatment may change over time (the probability of improvement may increase as treatment progresses). R1

Q17 — Steady State for Three-State Chain MEDIUM

Using GDC, compute $\mathbf{T}^{100}$ (or similar large power): M1
All columns converge to the same vector. Reading off:

$$\boldsymbol{\pi} \approx \begin{pmatrix} 0.455 \\ 0.364 \\ 0.182 \end{pmatrix}$$

(a) Long-run proportion of Grade A ≈ 0.455 (45.5 %). A1

(b) Long-run proportion of Grade C ≈ 0.182 (18.2 %). A1

(c) $45.5\,\% < 60\,\%$, so the requirement is not met. A1
Note: FT from their $\boldsymbol{\pi}$ in (a); accept any correct comparison and conclusion.

Q18 — Deriving Parameters from Steady State MEDIUM

(a) From $\mathbf{T}\boldsymbol{\pi} = \boldsymbol{\pi}$, row 1: $(1-p)(0.6) + q(0.4) = 0.6$ M1

$$0.6 - 0.6p + 0.4q = 0.6 \implies 0.4q = 0.6p \implies q = 1.5p \quad \text{AG}$$

A1

(b) Substituting $q = 1.5p$ into $p + q = 0.35$: M1

$$p + 1.5p = 0.35 \implies 2.5p = 0.35 \implies p = 0.14$$

$$q = 1.5 \times 0.14 = 0.21$$

A1

Q19 — Taxi Fleet: Five-Step and Long-Run Distribution MEDIUM

(a) Initial state: $\mathbf{s}_0 = \begin{pmatrix} 120 \\ 80 \end{pmatrix}$. Using GDC: M1

$$\mathbf{s}_5 = \mathbf{T}^5\mathbf{s}_0 \approx \begin{pmatrix} 127.8 \\ 72.2 \end{pmatrix}$$

Expected: approximately 128 taxis in North, 72 in South. A1
Note: Accept North $\in [126, 129]$, South $\in [71, 74]$.

(b) Steady-state: $a = 0.35$, $b = 0.45$. M1

$$\pi_N = \frac{0.45}{0.35 + 0.45} = \frac{0.45}{0.80} = 0.5625$$

Long-run taxis in North $= 0.5625 \times 200 = 112.5 \approx 113$. A1

Q20 — Gene Mutation: Powers and Long-Run MEDIUM

(a) Initial state: $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Using GDC: M1

$$(\mathbf{T}^4\mathbf{s}_0)_N \approx 0.716$$

$P(\text{Normal after 4 divisions}) \approx 0.716$. A1

(b) $P(\text{Mutant Type-1 after 4 divisions}) \approx 0.181$. A1

(c) Using GDC, $\mathbf{T}^{200}\mathbf{s}_0$ converges to $\boldsymbol{\pi}$; long-run $P(\text{Mutant Type-2}) \approx 0.194$. A1
Note: Award A1 for any value in range [0.190, 0.197].

Section C — Hard

Q21 — Eigenvalues of General Two-State Matrix HARD

(a) The characteristic equation:

$$\det(\mathbf{T}-\lambda\mathbf{I}) = \det\begin{pmatrix}1-\alpha-\lambda & \beta \\ \alpha & 1-\beta-\lambda\end{pmatrix} = 0$$

M1

$$(1-\alpha-\lambda)(1-\beta-\lambda)-\alpha\beta = 0$$

$$\lambda^2-(2-\alpha-\beta)\lambda+[(1-\alpha)(1-\beta)-\alpha\beta]=0$$
$$=\lambda^2-(2-\alpha-\beta)\lambda+(1-\alpha-\beta)=0$$
$$=[\lambda-1][\lambda-(1-\alpha-\beta)]=0$$

M1

So $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$. A1 AG

(b) The eigenvector for $\lambda_1 = 1$ satisfies $(\mathbf{T}-\mathbf{I})\mathbf{v}=\mathbf{0}$: row 1 gives $-\alpha v_1+\beta v_2 = 0$, so $v_1 = \frac{\beta}{\alpha}v_2$.
Normalising: $\boldsymbol{\pi} = \begin{pmatrix}\beta/(\alpha+\beta) \\ \alpha/(\alpha+\beta)\end{pmatrix}$. A1

(c) For convergence: $|\lambda_2| < 1$, i.e. $|1-\alpha-\beta| < 1$, which holds when $0 < \alpha+\beta < 2$. Since $\alpha, \beta \in (0,1)$, this reduces to $\alpha+\beta < 2$ (automatically satisfied) and $\alpha+\beta > 0$ (also automatically satisfied). The precise condition is $0 < \alpha+\beta < 2$, or equivalently $\alpha+\beta \neq 0$. R1

Q22 — Three-State Chain: Missing Entries, Forecast, Long-Run HARD

(a) Column 3 (From C) sums to 1: $m + 0.15 + 0.60 = 1 \Rightarrow m = 0.25$. A1
Column 2 (From B) sums to 1: $0.10 + 0.75 + n = 1 \Rightarrow n = 0.15$. A1

(b) $\mathbf{s}_0 = \begin{pmatrix}5000 \\ 3000 \\ 2000\end{pmatrix}$. Using GDC, $\mathbf{T}^6\mathbf{s}_0$: M1

$$(\mathbf{T}^6\mathbf{s}_0)_C \approx 2437$$

Expected Crest customers after 6 months $\approx 2\,437$. A1
Note: Accept values in range [2400, 2470].

(c) Using $\mathbf{T}^{200}$, long-run proportions $\approx (0.499, 0.299, 0.202)$, giving approximately 4990 Apex, 2990 Bolt, 2020 Crest. A1
Note: Accept rounded values; FT from (a).

Q23 — Bull/Bear Market: Powers and Long-Run HARD

(a) $\mathbf{s}_0 = \begin{pmatrix}1\\0\\0\end{pmatrix}$. Using GDC: M1

$$(\mathbf{T}^5\mathbf{s}_0)_B \approx 0.483$$

P(Bull after 5 days) ≈ 0.483. A1
Note: Accept values in range [0.481, 0.485].

(b) Using $\mathbf{T}^{200}$: all columns converge; M1

$$\pi_B \approx 0.400$$

Long-run probability of Bull market ≈ 0.400 (or 40.0 %). A1

(c) Expected Bull days $= 0.400 \times 250 = 100$ days. A1

Q24 — Parametric Steady State: Insurance Risk HARD

(a) Steady-state shortcut: $a = 0.10$ (prob. of leaving L), $b = k$ (prob. of leaving H). M1

$$\pi_H = \frac{a}{a+b} = \frac{0.10}{0.10+k}$$

A1

(b) Require $\pi_H \leq 0.15$: M1

$$\frac{0.10}{0.10+k} \leq 0.15 \implies 0.10 \leq 0.15(0.10+k) \implies 0.10 \leq 0.015 + 0.15k$$

$$0.085 \leq 0.15k \implies k \geq \frac{0.085}{0.15} = \frac{17}{30} \approx 0.567$$

So $k \geq \frac{17}{30}$ (approximately 0.567). A1

(c) k represents the probability that a High-risk customer returns to Low-risk status after one period; a larger k means the safety incentive scheme is more effective at reducing high-risk behaviour. R1

Q25 — Steady State from Eigenvector, Percentage Increase HARD

(a) Normalise $\mathbf{v}_1 = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$: total $= 6 + 5 + 4 = 15$. M1

$$\boldsymbol{\pi} = \begin{pmatrix} 6/15 \\ 5/15 \\ 4/15 \end{pmatrix} = \begin{pmatrix} 0.400 \\ 0.333 \\ 0.267 \end{pmatrix}$$

A1

(b) Long-run Train usage $= \dfrac{5}{15} = 33.3\,\%$. A1

(c) New Train proportion $= 40.0\,\%$. Percentage increase $= \dfrac{40.0 - 33.3}{33.3} \times 100 = \dfrac{6.7}{33.3} \times 100 \approx 20.1\,\%$. M1A1

Note: Accept values in range $[19.9\,\%, 20.2\,\%]$.

Q26 — Smallest n for Threshold Probability HARD

(a) Initial state: $\mathbf{s}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (Deficit). (M1)

Compute $(\mathbf{T}^n \mathbf{s}_0)_S$ for increasing n: M1

n	$P(\text{Surplus})$
1	0.100
2	0.163
3	0.200
4	0.218
5	0.227

The smallest integer n such that $P(\text{Surplus}) > 0.22$ is $\mathbf{n = 5}$. A1

Note: Award M1 for systematic computation; A1 for correct value $n = 5$.

(b) Long-run $P(\text{Deficit})$: using $\mathbf{T}^{200}$, $\pi_D \approx 0.258$. A1

Note: Accept values in range $[0.255, 0.262]$.

(c) One reason: energy surplus/deficit is influenced by weather patterns that persist across multiple hours (the probability of transitioning from Deficit depends on how long the system has been in Deficit, violating the Markov property). R1

Q27 — Symmetric Two-State Chain: General Formula HARD

(a) Steady-state: $a = 0.3$ (prob. leaving X), $b = 0.3$ (prob. leaving Y). M1

$$\pi_X = \frac{b}{a+b} = \frac{0.3}{0.6} = 0.5$$

$\boldsymbol{\pi} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ regardless of c. A1 AG

(b) The general solution is:

$$\mathbf{s}_n = \pi_X \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c-0.5)(0.4)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $P(X \text{ after } n \text{ steps}) = 0.5 + (c-0.5)(0.4)^n$. M1A1

(c) Set $n = 3$: $0.5 + (c-0.5)(0.4)^3 = 0.5 \implies (c-0.5)(0.064) = 0 \implies c = 0.5$. A1
Note: For any $c \neq 0.5$, the expression $= 0.5$ requires $(0.4)^3 = 0$ which is false; so $c = 0.5$ is the unique solution.

Q28 — Language School: Certificate Probability HARD

(a) $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Using GDC: M1

$$(\mathbf{T}^4\mathbf{s}_0)_E \approx 0.395$$

$P(\text{Excellent after exactly 4 weeks}) \approx 0.395$. A1

(b) Find $P(E \text{ in week } k)$ for $k = 1, 2, 3, 4$ using $\mathbf{T}^k\mathbf{s}_0$: M1

$$P(E_1) = 0.750, \quad P(E_2) = 0.598, \quad P(E_3) = 0.488, \quad P(E_4) = 0.395$$

$P(\geq 3 \text{ Excellent out of } 4)$: Using $\mathbf{T}^n$ to track the state, the exact computation requires tracking all paths. An approximation treating weeks as independent (for estimation):

$$P(\text{exactly } 3) \approx \binom{4}{3}(0.750)(0.598)(0.488)(1-0.395) \approx 0.264$$

$$P(\text{exactly } 4) \approx 0.750 \times 0.598 \times 0.488 \times 0.395 \approx 0.0863$$

$P(\geq 3) \approx 0.264 + 0.0863 \approx 0.350$.
Note: Accept any systematic method; award M1 for a valid probability approach and A1 for a value in range [0.33, 0.37]. A1

(c) Long-run $P(\text{Excellent})$: using $\mathbf{T}^{200}$, $\pi_E \approx 0.394$. A1
Note: Accept values in range [0.390, 0.398].

Q29 — Diagonalisation of a 2 × 2 Transition Matrix HARD

(a) Characteristic equation: $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$: M1

$$(0.6 - \lambda)(0.6 - \lambda) - (0.4)(0.4) = 0 \implies (0.6 - \lambda)^2 - 0.16 = 0$$

$$(0.6 - \lambda - 0.4)(0.6 - \lambda + 0.4) = 0 \implies (0.2 - \lambda)(1 - \lambda) = 0$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 0.2$. A1
Eigenvectors:

For $\lambda_1 = 1$: $(\mathbf{T} - \mathbf{I})\mathbf{v} = \mathbf{0}$: $-0.4v_1 + 0.4v_2 = 0 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 0.2$: $(\mathbf{T} - 0.2\mathbf{I})\mathbf{v} = \mathbf{0}$: $0.4v_1 + 0.4v_2 = 0 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. A1

(b) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{D}^n = \begin{pmatrix} 1^n & 0 \\ 0 & 0.2^n \end{pmatrix}$, $\mathbf{P}^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$. M1

$$\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 1 + 0.2^n & 1 - 0.2^n \\ 1 - 0.2^n & 1 + 0.2^n \end{pmatrix}$$

A1

Q30 — Absorbing State: Bus Fleet HARD

(a) State R is an absorbing state because column 3 of $\mathbf{T}$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, meaning once a bus enters state R, it remains there with probability 1 and cannot transition to any other state. R1

(b) Initial state: $\mathbf{s}_0 = \begin{pmatrix} 500 \\ 0 \\ 0 \end{pmatrix}$. Using GDC: M1

$$(\mathbf{T}^{20}\mathbf{s}_0)_R \approx 500 \times 0.284 \approx 142$$

Expected Retired buses after 20 weeks ≈ 142. A1
Note: Accept values in range [138, 146].

(c) $(\mathbf{T}^{50}\mathbf{s}_0)_R \approx 500 \times 0.567 \approx 284$ buses Retired after 50 weeks. A1
Note: Accept values in range [278, 290].

(d) In the long run, all buses will eventually be absorbed into state R. The proportions in O and M approach 0, since there is no mechanism for a Retired bus to re-enter service (column 3, rows 1 and 2 are both 0). The steady-state vector is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. R1

Common Mistakes to Avoid

1. Row vs Column convention. In IB AI HL, transition matrices are column-stochastic: entry T_{ij} is the probability of moving from state j to state i. Writing the matrix transposed is the most common error.

2. Column sums to 1 not row sums. Always check that every column sums to 1, not every row. If rows sum to 1, your matrix is transposed.

3. State vector multiplication order. Compute $\mathbf{T}^n\mathbf{s}_0$ (matrix times column vector on the right), not $\mathbf{s}_0^T\mathbf{T}^n$.

4. Steady-state eigenvalue. The steady-state always corresponds to $\lambda = 1$, not $\lambda = 0$. Find the eigenvector for $\lambda = 1$ and normalise so components sum to 1.

5. Absorbing states. If a state has a 1 on its diagonal and 0 everywhere else in its column, the system will eventually be absorbed there. Do not expect a non-trivial steady-state distribution when an absorbing state exists.

6. Rounding intermediate values. Carry full GDC precision throughout and only round the final answer. Premature rounding can give a visibly wrong final answer.