IB Math AI HL — Topic 1: Number & Algebra [Set 1]

Exponentials & Logarithms

Laws of Indices · Introduction to Logarithms · Laws of Logarithms

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Marks each	3	4	5
Total Marks	30	40	50

Scan me for help

Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers to 3 sf unless stated. A GDC is permitted where indicated.

Key Formulae — Exponentials & Logarithms

Laws of Indices

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

$a^0 = 1 \quad (a \neq 0)$

$a^{-n} = \dfrac{1}{a^n}$

$a^{1/n} = \sqrt[n]{a}$

Logarithm Laws $(a, b > 0, a \neq 1)$

$\log_a(xy) = \log_a x + \log_a y$

$\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(x^n) = n \log_a x$

$\log_a a = 1; \quad \log_a 1 = 0$

$a^{\log_a x} = x; \quad \log_a(a^x) = x$

Change of base: $\log_a x = \dfrac{\ln x}{\ln a}$

Key link: $y = a^x \iff x = \log_a y$

GDC Tips

1. **Logarithms in any base:** Use `log(x)/log(a)` or the `logBASE` function if available.
2. **Solving exponential equations:** Graph $y =$ (left side) and $y =$ (right side); use the intersection feature.
3. **Verify solutions:** Always substitute back — GDCs can sometimes give extraneous roots near asymptotes.
4. **Natural log:** The `ln` key computes $\log_e$. Use it freely; log on your GDC is $\log_{10}$.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

Scan me for help

1. **EASY** **No Calculator** **[3 marks]**

Simplify each of the following expressions, writing your answer in the form a^n.

(a) $\dfrac{3^8 \times 3^{-2}}{3^3}$ [1]

(b) $(2^3)^4 \div 2^5$ [1]

(c) $\left(\dfrac{5^6}{5^2}\right)^{1/2}$ [1]

2. **EASY** **No Calculator** **[3 marks]**

Evaluate each expression without a calculator.

(a) $27^{2/3}$ [1]

(b) $16^{-3/4}$ [1]

(c) $8^0 + 4^{-1}$ [1]

3. EASY No Calculator **[3 marks]**

(a) Write $\log_4 64 = 3$ in exponential form. [1]

(b) Write $5^{-2} = \frac{1}{25}$ in logarithmic form. [1]

(c) Write down the value of $\log_3 1$. [1]

4. EASY No Calculator **[3 marks]**

Evaluate each of the following without using a calculator.

(a) $\log_2 32$ [1]

(b) $\log_5 \frac{1}{125}$ [1]

(c) $\ln e^4$ [1]

5. EASY No Calculator **[3 marks]**

Write each expression as a single logarithm.

(a) $\log 6 + \log 4$ [1]

(b) $\log_3 18 - \log_3 2$ [1]

(c) $3 \ln 5$ [1]

6. **EASY** **Calculator** **[3 marks]**

The temperature T (in °C) of a cup of tea t minutes after it is poured is modelled by

$$T(t) = 22 + 74 \times (0.96)^t, \quad t \geq 0.$$

(a) Write down the initial temperature of the tea. [1]

(b) Find the temperature after 10 minutes. Give your answer correct to 3 significant figures. [1]

(c) Write down the temperature that the tea approaches in the long term. [1]

7. **EASY** **No Calculator** **[3 marks]**

Solve each equation for x.

(a) $2^x = 64$ [1]

(b) $3^{x-1} = 81$ [1]

(c) $5^{2x} = 125$ [1]

8. **EASY** **No Calculator** **[3 marks]**

Simplify each expression, giving your answer with positive exponents only.

(a) $\dfrac{x^5 \cdot x^{-2}}{x^4}$ [1]

(b) $(4x^6)^{1/2}$ [1]

(c) $\dfrac{(2a^3)^2}{8a^4}$ [1]

9. EASY No Calculator **[3 marks]**

Expand and simplify each expression, expressing your answer in terms of $\log p$ and $\log q$.

(a) $\log\left(\frac{p^3}{q}\right)$ [1]

(b) $\log(\sqrt{p} \cdot q^2)$ [1]

(c) $\log(p^2 q^{-1})$ [1]

10. EASY Calculator **[3 marks]**

(a) Write $\log_7 50$ using natural logarithms. [1]

(b) Hence evaluate $\log_7 50$ correct to 3 significant figures. [1]

(c) Without a calculator, write down the value of $\log_7 7^3$. [1]

Section B — Medium **MEDIUM** (10 questions $\times$ 4 marks = 40 marks)

Scan me for help

11. **MEDIUM** **Calculator** **[4 marks]**

A radioactive isotope decays so that the mass M (in grams) remaining after t years is given by

$$M(t) = M_0\, e^{-kt},$$

where M_0 and k are positive constants. Initially there are 80 g of the isotope; after 12 years, 60 g remain.

(a) Find the value of k. Give your answer correct to 3 significant figures. [2]

(b) Find the half-life of the isotope, correct to 3 significant figures. [2]

12. **MEDIUM** **Calculator** **[4 marks]**

The pH of a solution is defined by pH $= -\log_{10}[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen-ion concentration in mol L^{-1}.

(a) A fruit juice has $[\text{H}^+] = 3.16 \times 10^{-4}$ mol L^{-1}. Find the pH of the fruit juice. [2]

(b) A cleaning solution has pH $= 11.3$. Find its hydrogen-ion concentration, giving your answer in the form $a \times 10^n$ where $1 \leq a < 10$. [2]

13. MEDIUM No Calculator **[4 marks]**

Solve the equation below, justifying that your solution is valid.

$$\log_2(x+3) + \log_2(x-1) = 5$$

14. MEDIUM Calculator **[4 marks]**

A scientist records the following data relating the intensity I (watts per square metre) of a signal to the distance d (metres) from the source. It is believed the data follow a model of the form $I = a \cdot d^b$.

d (m)	1.0	2.0	4.0	8.0
I ($\mathrm{W\,m^{-2}}$)	120	30.0	7.50	1.88

(a) Show that taking $\log_{10}$ of both sides gives a linear relationship between $\log_{10} I$ and $\log_{10} d$. [1]

(b) Using the data for $d = 1.0$ and $d = 4.0$, find the values of a and b. [3]

15. MEDIUM Calculator **[4 marks]**

The number of active users N (in thousands) of a new social media platform t months after launch is modelled by

$$N(t) = 12 \times 2^{0.35t}.$$

(a) Write down the number of active users at launch. [1]

(b) Find the number of active users after 6 months. Give your answer correct to 3 significant figures. [1]

(c) Find the number of months it takes for the platform to reach 500 thousand users. Give your answer correct to 3 significant figures. [2]

16. MEDIUM No Calculator **[4 marks]**

Solve the equation $5^{x+1} = 3^{2x-1}$, giving your answer in the form $\frac{\ln a}{\ln b}$ where $a, b \in \mathbb{Z}^+$. Simplify your answer fully.

17. MEDIUM No Calculator **[4 marks]**

(a) Show that $\log_a\left(\dfrac{x^3}{\sqrt{y}}\right) = 3\log_a x - \dfrac{1}{2}\log_a y$. [2]

(b) Given that $\log_a x = 4$ and $\log_a y = 6$, find the value of $\log_a\left(\dfrac{x^3}{\sqrt{y}}\right)$. [2]

18. MEDIUM Calculator **[4 marks]**

The loudness L (in decibels, dB) of a sound is given by $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$, where I is the sound intensity in W m^{-2} and $I_0 = 10^{-12}$ W m^{-2} is the reference intensity.

(a) A concert speaker produces a sound intensity of $I = 0.25$ W m^{-2} at a certain point. Find the loudness at that point. [2]

(b) A second speaker is added, doubling the total intensity. Find the increase in loudness in decibels, giving your answer correct to 3 significant figures. [2]

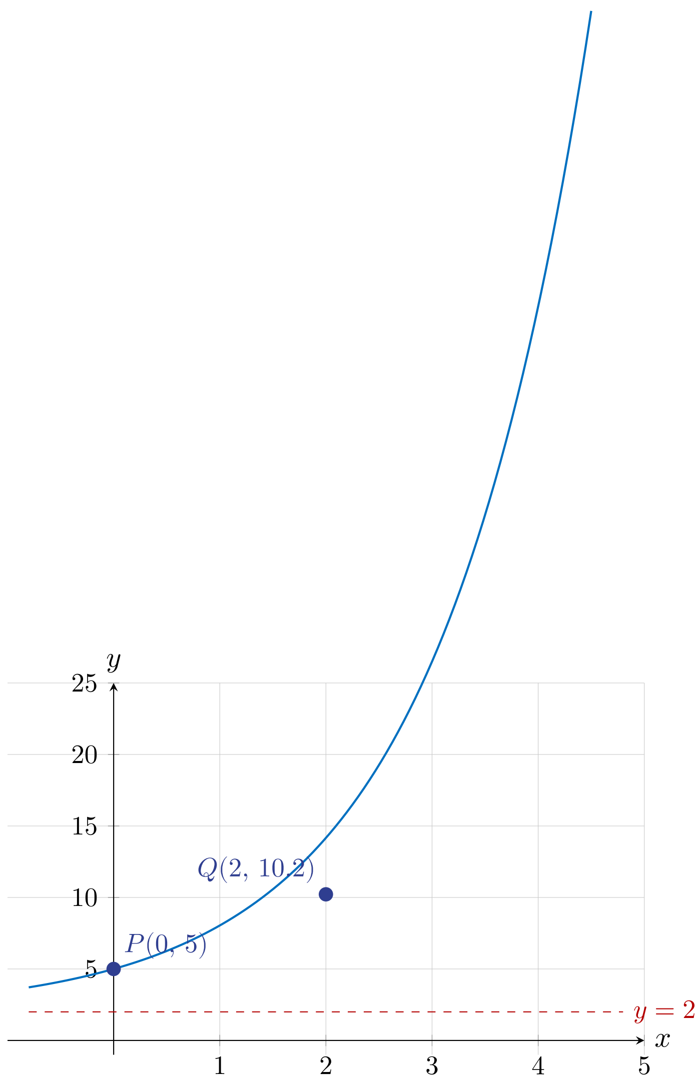
19. **MEDIUM** **No Calculator** **[4 marks]**

Solve the equation below, justifying that your solution is valid.

$$2\log_3 x - \log_3(x+6) = 1$$

20. **MEDIUM** **Calculator** **[4 marks]**

The graph below shows the function $f(x) = A \cdot e^{bx} + c$, where A, b, and c are constants.



The graph passes through the points $P(0,\ 5)$ and $Q(2,\ 10.2)$ and has a horizontal asymptote $y = 2$.

(a) Write down the value of c. [1]

(b) Using the point P, find the value of A. [1]

(c) Using the point Q, find the value of b. Give your answer correct to 3 significant figures. [2]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

Scan me for help

21. **HARD** **No Calculator** **[5 marks]**

Solve the following simultaneous equations.

$$\log_2 x + \log_2 y = 7 \qquad \text{and} \qquad \log_2 x - \log_2 y = 1$$

22. **HARD** **No Calculator** **[5 marks]**

A function is defined by $f(x) = k \cdot 3^x - 2$, where k is a positive constant.

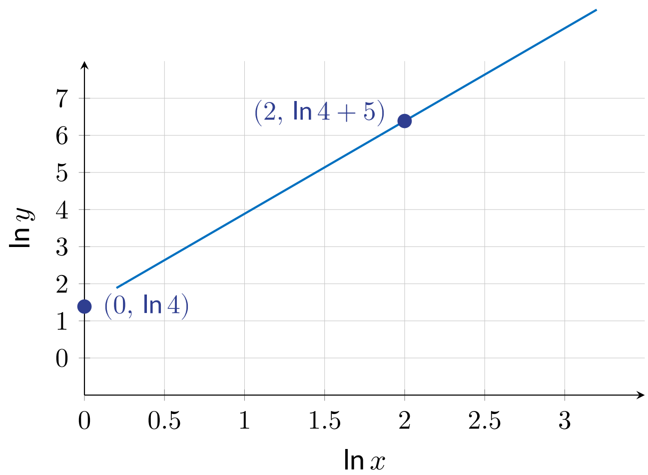
(a) The graph of f passes through the point $(1, 13)$. Find the value of k. [2]

(b) Find the x-intercept of the graph of f. Give your answer in exact form. [2]

(c) State the equation of the horizontal asymptote of f. [1]

23. HARD No Calculator **[5 marks]**

A researcher plots $\ln y$ against $\ln x$ and observes that all data points lie perfectly on the straight line shown below.



The straight line has gradient $\frac{5}{2}$ and passes through the point $(0,\ \ln 4)$.

(a) Show that $y = 4x^{5/2}$. [3]

(b) Find the exact value of y when $x = 9$. [2]

24. HARD Calculator **[5 marks]**

The population P of a colony of ants at time t weeks satisfies $P(t) = P_0\, e^{0.18t}$, where P_0 is the initial population. After 4 weeks the population is measured to be 7 400 ants.

(a) Find the value of P_0, correct to 3 significant figures. [2]

(b) Find the time at which the population first exceeds 50 000 ants. Give your answer correct to 3 significant figures. [2]

(c) State one assumption made in using this model. [1]

25. HARD No Calculator **[5 marks]**

(a) Show that $\log_a b = \dfrac{1}{\log_b a}$. [2]

(b) Hence solve $\log_x 81 + \log_{81} x = \dfrac{82}{9}$ for $x > 0, x \neq 1$. [3]

26. HARD No Calculator **[5 marks]**

(a) Show that $4^n \cdot 25^n = 100^n$. [1]

(b) Hence, or otherwise, show that $4^n \cdot 25^n$ always ends in the digit 0 for all $n \in \mathbb{Z}^+$. [1]

(c) Given that $4^n \cdot 25^n = 10^{18}$, find the value of n. [1]

(d) Find the number of digits in the integer $4^{13} \cdot 25^{13}$. [2]

27. HARD Calculator **[5 marks]**

Two competing populations of bacteria are grown in a laboratory. Population A starts with 200 bacteria and doubles every 3 hours. Population B starts with $8\,000$ bacteria and halves every 5 hours.

(a) Write expressions for $A(t)$ and $B(t)$, the sizes of populations A and B respectively at time t hours. [2]

(b) Find the time at which the two populations are equal. Give your answer correct to 3 significant figures. [3]

28. HARD No Calculator **[5 marks]**

Solve the equation below, justifying any rejected solutions.

$$(\log_3 x)^2 - 3\log_3 x - 10 = 0$$

29. HARD Calculator **[5 marks]**

The table below gives values of t (hours) and N (number of organisms), where it is suspected that $N = N_0 \cdot b^t$ for constants N_0 and b.

t (hours)	0	1	2	3	4
N	50	150	450	1350	4050

(a) Show that $\ln N = t \ln b + \ln N_0$. [1]

(b) Calculate $\ln N$ for each value of t and verify that the data lie on a straight line when $\ln N$ is plotted against t. Give the gradient and y-intercept of this line, each correct to 3 significant figures. [2]

(c) Hence find the values of N_0 and b. Give b as an exact integer. [2]

30. HARD No Calculator **[5 marks]**

The function g is defined by $g(x) = \log_3(2x - k)$, where k is a real constant.

(a) State the domain of g in terms of k. [1]

(b) The graph of g passes through the point $(5, 2)$. Find the value of k. [2]

(c) Find $g^{-1}(x)$, the inverse function of g, using the value of k found in part (b). [2]

Worked Solutions

Exponentials & Logarithms — All 30 Questions

Section A — Easy (Q1–Q10)

Q1 — Laws of Indices EASY

(a) $\dfrac{3^8 \times 3^{-2}}{3^3} = \dfrac{3^{8+(-2)}}{3^3} = \dfrac{3^6}{3^3} = 3^{6-3} = 3^3$ **A1**

(b) $(2^3)^4 \div 2^5 = 2^{12} \div 2^5 = 2^{12-5} = 2^7$ **A1**

(c) $\left(\dfrac{5^6}{5^2}\right)^{1/2} = (5^{6-2})^{1/2} = (5^4)^{1/2} = 5^{4\times\frac{1}{2}} = 5^2$ **A1**

Q2 — Fractional and Negative Indices EASY

(a) $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$ **A1**

(b) $16^{-3/4} = \dfrac{1}{(16^{1/4})^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$ **A1**

(c) $8^0 + 4^{-1} = 1 + \dfrac{1}{4} = \dfrac{5}{4}$ **A1**

Q3 — Introduction to Logarithms EASY

(a) $\log_4 64 = 3 \iff 4^3 = 64$ **A1**

(b) $5^{-2} = \dfrac{1}{25} \iff \log_5\left(\dfrac{1}{25}\right) = -2$ **A1**

(c) $\log_3 1 = 0$ **A1**

Note: In (c) no working required; award A1 for answer 0.

Q4 — Evaluating Logarithms EASY

(a) $\log_2 32 = \log_2 2^5 = 5$ **A1**

(b) $\log_5\left(\dfrac{1}{125}\right) = \log_5 5^{-3} = -3$ **A1**

(c) $\ln e^4 = 4$ **A1**

Q5 — Laws of Logarithms: Combining EASY

(a) $\log 6 + \log 4 = \log(6 \times 4) = \log 24$ **A1**

(b) $\log_3 18 - \log_3 2 = \log_3\left(\frac{18}{2}\right) = \log_3 9$ **A1**

(c) $3\ln 5 = \ln 5^3 = \ln 125$ **A1**

Q6 — Exponential Model EASY

(a) $T(0) = 22 + 74 \times (0.96)^0 = 22 + 74 = 96\,^\circ\text{C}$ **A1**

(b) $T(10) = 22 + 74 \times (0.96)^{10} = 22 + 74 \times 0.6648\ldots = 22 + 49.2\ldots = 71.2\,^\circ\text{C}$ **A1**

(c) As $t \to \infty$, $(0.96)^t \to 0$, so $T \to 22\,^\circ\text{C}$ **A1**

Note: (c) is the asymptote; accept "22 degrees" without justification.

Q7 — Solving Basic Exponential Equations EASY

(a) $2^x = 64 = 2^6 \;\Rightarrow\; x = 6$ **A1**

(b) $3^{x-1} = 81 = 3^4 \;\Rightarrow\; x - 1 = 4 \;\Rightarrow\; x = 5$ **A1**

(c) $5^{2x} = 125 = 5^3 \;\Rightarrow\; 2x = 3 \;\Rightarrow\; x = \frac{3}{2}$ **A1**

Q8 — Algebraic Index Simplification EASY

(a) $\frac{x^5 \cdot x^{-2}}{x^4} = \frac{x^{5-2}}{x^4} = \frac{x^3}{x^4} = x^{-1} = \frac{1}{x}$ **A1**

(b) $(4x^6)^{1/2} = 4^{1/2} \cdot x^{6\times\frac{1}{2}} = 2x^3$ **A1**

(c) $\frac{(2a^3)^2}{8a^4} = \frac{4a^6}{8a^4} = \frac{a^{6-4}}{2} = \frac{a^2}{2}$ **A1**

Q9 — Expanding Using Log Laws EASY

(a) $\log\left(\frac{p^3}{q}\right) = \log p^3 - \log q = 3\log p - \log q$ **A1**

(b) $\log(\sqrt{p} \cdot q^2) = \log p^{1/2} + \log q^2 = \frac{1}{2}\log p + 2\log q$ **A1**

(c) $\log(p^2 q^{-1}) = 2\log p - \log q$ **A1**

Q10 — Change of Base **EASY**

(a) $\log_7 50 = \dfrac{\ln 50}{\ln 7}$ **A1**

(b) $\log_7 50 = \dfrac{\ln 50}{\ln 7} = \dfrac{3.9120\ldots}{1.9459\ldots} = 2.0104\ldots \approx 2.01$ **A1**

(c) $\log_7 7^3 = 3$ **A1**

Note: (c) requires no working; A1 for answer 3.

Section B — Medium (Q11–Q20)

Q11 — Exponential Decay and Half-Life **MEDIUM**

(a) $M(0) = 80$, so $M_0 = 80$. **(A1)**

Substituting $M(12) = 60$:

$$60 = 80\, e^{-12k}$$

M1

$e^{-12k} = \dfrac{60}{80} = 0.75 \;\Rightarrow\; -12k = \ln(0.75) \;\Rightarrow\; k = \dfrac{-\ln 0.75}{12} = \dfrac{\ln(4/3)}{12}$

$k = 0.023978\ldots \approx 0.0240$ **A1**

(b) At half-life $t_{1/2}$: $M(t_{1/2}) = \dfrac{80}{2} = 40$:

$$40 = 80\, e^{-0.0240\, t_{1/2}}$$

M1

$e^{-0.0240\, t_{1/2}} = 0.5 \;\Rightarrow\; t_{1/2} = \dfrac{\ln 2}{0.023978\ldots} = 28.9\ldots \approx 28.9$ years **A1**

Note: FT from their k in (a). Accept $t_{1/2} = \dfrac{\ln 2}{k}$ evaluated with their k.

Q12 — pH Formula **MEDIUM**

(a) $\text{pH} = -\log_{10}(3.16 \times 10^{-4})$ **M1**

$= -(\log_{10} 3.16 + \log_{10} 10^{-4}) = -(0.4997\ldots + (-4)) = 3.500\ldots \approx 3.50$ **A1**

(b) $11.3 = -\log_{10}[\text{H}^+] \;\Rightarrow\; \log_{10}[\text{H}^+] = -11.3$ **M1**

$[\text{H}^+] = 10^{-11.3} = 5.012\ldots \times 10^{-12} \approx 5.01 \times 10^{-12}$ mol L^{-1} **A1**

Q13 — Log Equation: Sum of Logs **MEDIUM**

$\log_2(x+3) + \log_2(x-1) = 5$

$\log_2[(x+3)(x-1)] = 5$ **M1**

$(x+3)(x-1) = 2^5 = 32$ **M1**

$x^2 + 2x - 3 = 32 \;\Rightarrow\; x^2 + 2x - 35 = 0$

$(x+7)(x-5) = 0 \;\Rightarrow\; x = -7$ or $x = 5$ **A1**

Validity: $x = -7$ gives $\log_2(-4)$, which is undefined. Reject $x = -7$. **R1**

$x = 5$ **(A1 within R1)**

Note: Award M1 for combining logs; M1 for converting to exponential; A1 for both solutions from the quadratic; R1 for rejecting $x = -7$ with a valid reason.

Q14 — Linearisation: Power Model **MEDIUM**

(a) Taking $\log_{10}$ of $I = a \cdot d^b$: **M1**

$\log_{10} I = \log_{10} a + b \log_{10} d$

This is of the form $Y = mX + c$ with $Y = \log_{10} I$, $X = \log_{10} d$, slope $m = b$, intercept $c = \log_{10} a$. **AG**

(b) At $d = 1.0$: $\log_{10} I = \log_{10} 120 = 2.0792\ldots$ **(A1)**

At $d = 4.0$: $\log_{10} I = \log_{10} 7.50 = 0.8751\ldots$ **(A1)**

Slope (value of b): $b = \dfrac{0.8751 - 2.0792}{\log_{10} 4 - \log_{10} 1} = \dfrac{-1.204}{0.6021} = -1.999\ldots \approx -2.00$ **M1 A1**

$\log_{10} a = 2.0792 - (-2.00)(0) = 2.0792 \;\Rightarrow\; a = 10^{2.0792} = 120$

Model: $I = 120\, d^{-2}$ (equivalently $I = \dfrac{120}{d^2}$) **A1**

Note: Award final A1 for both $a = 120$ and $b = -2$ stated. Accept $b = -2$ exactly.

Q15 — Exponential Growth: Doubling (Reach a Threshold) **MEDIUM**

(a) $N(0) = 12 \times 2^0 = 12$ thousand users **A1**

(b) $N(6) = 12 \times 2^{0.35 \times 6} = 12 \times 2^{2.1} = 12 \times 4.2870\ldots = 51.4\ldots \approx 51.4$ thousand **A1**

(c) Solve $12 \times 2^{0.35t} = 500$: **M1**

$2^{0.35t} = \dfrac{500}{12} = 41.667\ldots$

$0.35t = \log_2 41.667\ldots = \dfrac{\ln 41.667\ldots}{\ln 2} = 5.3813\ldots$

$t = \dfrac{5.3813\ldots}{0.35} = 15.375\ldots \approx 15.4$ months **A1**

Q16 — Cross-base Exponential Equation MEDIUM

$5^{x+1} = 3^{2x-1}$

Take ln of both sides: **M1**

$(x+1)\ln 5 = (2x-1)\ln 3$ **M1**

$x\ln 5 + \ln 5 = 2x\ln 3 - \ln 3$

$\ln 5 + \ln 3 = 2x\ln 3 - x\ln 5 = x(2\ln 3 - \ln 5)$

$x = \dfrac{\ln 5 + \ln 3}{2\ln 3 - \ln 5} = \dfrac{\ln 15}{\ln 9 - \ln 5} = \dfrac{\ln 15}{\ln(9/5)}$ **A1 A1**

Note: Award A1 for $x = \dfrac{\ln 15}{2\ln 3 - \ln 5}$*; A1 for fully simplified exact form* $\dfrac{\ln 15}{\ln(9/5)}$*. Accept equivalent exact forms.*

Q17 — Log Identities: Show and Apply MEDIUM

(a) $\log_a\left(\dfrac{x^3}{\sqrt{y}}\right) = \log_a(x^3) - \log_a(y^{1/2})$ **M1**

$= 3\log_a x - \dfrac{1}{2}\log_a y$ **A1 AG**

(b) Substituting $\log_a x = 4$ and $\log_a y = 6$: **M1**

$\log_a\left(\dfrac{x^3}{\sqrt{y}}\right) = 3(4) - \dfrac{1}{2}(6) = 12 - 3 = 9$ **A1**

Note: FT from (a) using their expression. No marks in (b) if (a) was incorrect and not attempted.

Q18 — Decibel Formula MEDIUM

(a) $L = 10\log_{10}\left(\dfrac{0.25}{10^{-12}}\right) = 10\log_{10}(2.5 \times 10^{11})$ **M1**

$= 10(11 + \log_{10} 2.5) = 10(11 + 0.3979\ldots) = 10 \times 11.3979\ldots = 114$ dB **A1**

(b) New intensity $= 2 \times 0.25 = 0.50$ W m^{-2}

$L_{\text{new}} = 10\log_{10}\left(\dfrac{0.50}{10^{-12}}\right) = 10\log_{10}(5 \times 10^{11}) = 117.0\ldots$ dB **M1**

Increase $= 117.0 - 114.0 = 3.01$ dB **A1**

Note: The increase of $10\log_{10} 2 \approx 3.01$ *dB is exact; accept* 3.01 *or* $3.0103\ldots$

Q19 — Log Equation: Difference of Logs MEDIUM

$2\log_3 x - \log_3(x+6) = 1$

$\log_3 x^2 - \log_3(x+6) = 1$ **M1**

$\log_3\left(\dfrac{x^2}{x+6}\right) = 1 \;\Rightarrow\; \dfrac{x^2}{x+6} = 3$ **M1**

$x^2 = 3(x+6) = 3x + 18 \;\Rightarrow\; x^2 - 3x - 18 = 0$

$(x-6)(x+3) = 0 \;\Rightarrow\; x = 6$ or $x = -3$ **A1**

Validity: $\log_3 x$ requires $x > 0$. Reject $x = -3$. **R1**

$x = 6$ **(A1)**

Note: M1 for power law; M1 for converting to exponential; A1 for both values; R1 for correct rejection with domain reasoning.

Q20 — Exponential Graph: Finding Parameters MEDIUM

(a) The horizontal asymptote is $y = c$, so $c = 2$. **A1**

(b) Substituting $P(0,5)$ with $c = 2$: $5 = A\,e^{b(0)} + 2 = A + 2 \;\Rightarrow\; A = 3$ **A1**

(c) Substituting $Q(2, 10.2)$ with $A = 3$, $c = 2$: **M1**

$10.2 = 3\,e^{2b} + 2 \;\Rightarrow\; 3\,e^{2b} = 8.2 \;\Rightarrow\; e^{2b} = \dfrac{8.2}{3} = 2.7\overline{3}\ldots$

$2b = \ln\left(\dfrac{8.2}{3}\right) = 1.0051\ldots \;\Rightarrow\; b = 0.502\ldots \approx 0.502$ **A1**

Note: The graph shows $Q(2, 10.2)$ which is consistent with $b \approx 0.502$ (displayed as 10.2). FT from their A in (b).

Section C — Hard (Q21–Q30)

Q21 — Simultaneous Log Equations HARD

Let $u = \log_2 x$ and $v = \log_2 y$. The system becomes:

$$u + v = 7 \quad \text{and} \quad u - v = 1$$

M1

Adding: $2u = 8 \;\Rightarrow\; u = 4$; subtracting: $2v = 6 \;\Rightarrow\; v = 3$ **A1**

$\log_2 x = 4 \;\Rightarrow\; x = 2^4 = 16$ **A1**

$\log_2 y = 3 \;\Rightarrow\; y = 2^3 = 8$ **A1**

Check: $\log_2 16 + \log_2 8 = 4 + 3 = 7\,\checkmark$; $\log_2 16 - \log_2 8 = 1\,\checkmark$ **A1**

Note: Award M1 for adding/subtracting equations (or equivalent method); A1 for each of u, v; A1 each for $x = 16, y = 8$.

Q22 — Unknown Parameter in Exponential Function HARD

(a) $f(1) = k \cdot 3^1 - 2 = 13 \;\Rightarrow\; 3k = 15 \;\Rightarrow\; k = 5$ **M1 A1**

(b) x-intercept where $f(x) = 0$: **M1**

$5 \cdot 3^x = 2 \;\Rightarrow\; 3^x = \frac{2}{5} \;\Rightarrow\; x = \log_3\left(\frac{2}{5}\right) = \frac{\ln(2/5)}{\ln 3}$ **A1**

(c) Horizontal asymptote: $y = -2$ **A1**

Note: In (b) accept exact forms: $\log_3(\frac{2}{5})$, $\frac{\ln 2 - \ln 5}{\ln 3}$, *or* $-\log_3(\frac{5}{2})$.

Q23 — Log-Log Linearisation: Exact Power Model HARD

(a) The straight line has equation $\ln y = \frac{5}{2}\ln x + \ln 4$. **M1**

Apply the power law of logs: $\ln y = \ln x^{5/2} + \ln 4 = \ln(4x^{5/2})$ **M1**

Taking exponentials of both sides: $y = 4x^{5/2}$ **A1 AG**

(b) $y = 4 \times 9^{5/2} = 4 \times (9^{1/2})^5 = 4 \times 3^5 = 4 \times 243 = 972$ **M1 A1**

Note: In (b) M1 for substituting $x = 9$ *and attempting* $9^{5/2}$*; A1 for* $y = 972$.

Q24 — Exponential Growth: Unknown Initial Value HARD

(a) $P(4) = P_0\, e^{0.18\times 4} = P_0\, e^{0.72} = 7400$ **M1**

$P_0 = \frac{7400}{e^{0.72}} = \frac{7400}{2.0544\ldots} = 3602\ldots \approx 3600$ **A1**

(b) Solve $3602\, e^{0.18t} = 50\,000$: **M1**

$e^{0.18t} = \frac{50\,000}{3602\ldots} = 13.88\ldots$

$0.18t = \ln(13.88\ldots) = 2.630\ldots \;\Rightarrow\; t = 14.6\ldots \approx 14.6$ weeks **A1**

(c) Any one valid assumption, e.g.: "The growth rate remains constant"; or "There are unlimited resources / no limiting factors." **R1**

Note: FT in (b) from their P_0 *in (a). For (c) accept any contextually reasonable modelling assumption.*

Q25 — Log Identity Proof and Quadratic Solve HARD

(a) Let $y = \log_a b$, so $a^y = b$. Taking $\log_b$ of both sides: **M1**

$\log_b(a^y) = \log_b b \;\Rightarrow\; y\log_b a = 1 \;\Rightarrow\; \log_b a = \frac{1}{y} = \frac{1}{\log_a b}$ **A1 AG**

(b) Let $u = \log_x 81$. By part (a), $\log_{81} x = \frac{1}{\log_x 81} = \frac{1}{u}$. **M1**

Substituting: $u + \dfrac{1}{u} = \dfrac{82}{9}$ **M1**

$9u^2 - 82u + 9 = 0 \ \Rightarrow\ (9u-1)(u-9) = 0$

$u = \dfrac{1}{9}$ or $u = 9$ **A1**

$\log_x 81 = 9 \ \Rightarrow\ x^9 = 81 = 3^4 \ \Rightarrow\ x = 3^{4/9}$

$\log_x 81 = \dfrac{1}{9} \ \Rightarrow\ x^{1/9} = 81 \ \Rightarrow\ x = 81^9 = 3^{36}$

$x = 3^{4/9}$ or $x = 3^{36}$ **A1**

Note: M1 for using part (a); M1 for forming the quadratic; A1 for both values of u; A1 for both final x values.

Q26 — Indices Synthesis: Digit Count HARD

(a) $4^n \cdot 25^n = (4 \times 25)^n = 100^n$ **A1 AG**

(b) $100^n = (10^2)^n = 10^{2n}$, which is 1 followed by $2n$ zeros. The last digit is 0 for all $n \in \mathbb{Z}^+$. **R1**

(c) $100^n = 10^{18} = (10^2)^9 \ \Rightarrow\ n = 9$ **A1**

(d) $4^{13} \cdot 25^{13} = 100^{13} = 10^{26}$, a 1 followed by 26 zeros. **M1**

Number of digits $= 27$ **A1**

Note: In (d) M1 for deducing 10^{26}; A1 for 27 digits.

Q27 — Two Competing Exponential Populations HARD

(a) Population A doubles every 3 hours: $A(t) = 200 \times 2^{t/3}$ **A1**

Population B halves every 5 hours: $B(t) = 8000 \times 2^{-t/5} = 8000 \times \left(\frac{1}{2}\right)^{t/5}$ **A1**

(b) Set $A(t) = B(t)$: **M1**

$200 \times 2^{t/3} = 8000 \times 2^{-t/5}$

$2^{t/3+t/5} = 40 \ \Rightarrow\ 2^{8t/15} = 40$ **M1**

$\dfrac{8t}{15} = \log_2 40 = \dfrac{\ln 40}{\ln 2} = 5.3219\ldots$

$t = \dfrac{15 \times 5.3219\ldots}{8} = 9.978\ldots \approx 9.98$ hours **A1**

Note: M1 for setting equal; M1 for combining exponents or taking logs; A1 for $t = 9.98$.

Q28 — Quadratic in Log: Two Solutions HARD

Let $u = \log_3 x$. The equation becomes: **M1**

$u^2 - 3u - 10 = 0 \;\Rightarrow\; (u-5)(u+2) = 0$ **M1**

$u = 5$ or $u = -2$ **A1**

$\log_3 x = 5 \;\Rightarrow\; x = 3^5 = 243$ **A1**

$\log_3 x = -2 \;\Rightarrow\; x = 3^{-2} = \dfrac{1}{9}$

Both solutions are valid since $\log_3 x$ requires $x > 0$; $243 > 0$ and $\dfrac{1}{9} > 0$. **R1**

$x = 243$ or $x = \dfrac{1}{9}$

Note: Award M1 for substitution $u = \log_3 x$; M1 for factorising; A1 for $u = 5$ and $u = -2$; A1 for both x values; R1 for correctly accepting both with domain reasoning.

Q29 — Semi-Log Linearisation HARD

(a) $N = N_0 \cdot b^t \;\Rightarrow\; \ln N = \ln N_0 + \ln(b^t) = t\ln b + \ln N_0$ **A1 AG**

(b)

t	0	1	2	3	4
$\ln N$	3.912	5.011	6.109	7.208	8.307

A1

The differences in $\ln N$ are constant (≈ 1.099), confirming a straight line. **M1**

Gradient $= \ln b \approx 1.10$ (3 s.f.); y-intercept $= \ln N_0 \approx 3.91$ (3 s.f.) **A1**

(c) $N_0 = e^{3.912\ldots} = 50$ (exactly 50) **A1**

$b = e^{1.0986\ldots} = 3$ (exactly 3) **A1**

Note: Accept $b = 3$ without computation if candidate notes ratios $150/50 = 450/150 = 3$. Award second A1 in (b) for gradient and intercept stated; A1 for each of $N_0 = 50$ and $b = 3$.

Q30 — Domain, Parameter, and Inverse Function HARD

(a) $g(x) = \log_3(2x - k)$ requires $2x - k > 0 \;\Rightarrow\; x > \dfrac{k}{2}$ **A1**

(b) Substituting $(5, 2)$: $2 = \log_3(2(5) - k) = \log_3(10 - k)$ **M1**

$10 - k = 3^2 = 9 \;\Rightarrow\; k = 1$ **A1**

(c) With $k = 1$: $g(x) = \log_3(2x - 1)$. Let $y = \log_3(2x - 1)$: **M1**

$3^y = 2x - 1 \ \Rightarrow \ x = \dfrac{3^y + 1}{2}$

$g^{-1}(x) = \dfrac{3^x + 1}{2}$ **A1**

Note: In (a) accept "$x > k/2$" without specifying domain notation. In (c) M1 for swapping $x \leftrightarrow y$ and attempting to solve; A1 for correct final form.

Common Mistakes to Avoid

1. **Negative indices.** $a^{-n} = \dfrac{1}{a^n}$, not $-a^n$. Always rewrite before evaluating.
2. **Fractional indices.** $a^{m/n} = (\sqrt[n]{a})^m$. Take the root first, then the power — avoids working with large numbers.
3. **Log of a sum is not the sum of logs.** $\log(x+y) \neq \log x + \log y$. The product law applies to $\log(xy)$ only.
4. **Validity of log solutions.** Always check that each solution gives a positive argument inside every logarithm. Reject any value that makes the argument ≤ 0.
5. **Change of base.** $\log_a b = \dfrac{\ln b}{\ln a}$, not $\dfrac{\ln a}{\ln b}$. Remember: "what you want on top."
6. **Asymptote $\neq$ initial value.** In $f(t) = Ae^{bt} + c$, the constant c is the asymptote, and $A + c$ is the initial value. Do not confuse them.
7. **Log-log and semi-log.** When linearising $y = ax^b$, take log of *both sides* and compare with $Y = mX + c$: slope is b, not a.
8. **Quadratic substitution.** In equations like $(\log x)^2 - 3\log x - 10 = 0$, substitute $u = \log x$ and solve the quadratic first — do not try to solve directly.