IB Math AI HL — Topic 5 [Set 1]

Further Differentiation

Chain Rule · Product Rule · Quotient Rule · Related Rates · Second Derivatives · Stationary Points · Concavity & Inflection

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Marks each	3	4	5
Total Marks	30	40	50

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Name:____________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working.
Give answers correct to 3 significant figures unless an exact form is requested. Calculator permitted throughout.

Key Formulae — Further Differentiation

Special Derivatives

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Product Rule

$$\frac{d}{dx}[u\,v] = u\,v' + v\,u'$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\,u' - u\,v'}{v^2}$$

Related Rates

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

Second Derivative & Concavity

$f''(x) > 0 \Rightarrow$ concave up (min)

$f''(x) < 0 \Rightarrow$ concave down (max)

Point of inflection: $f''(x) = 0$ & sign change

Stationary Points: $\dfrac{dy}{dx} = 0$

GDC Tips

- **Numerical derivative:** Use `nDeriv(f(x),x,a)` to verify $f'(a)$; graph $y = f'(x)$ to locate stationary points.
- **Stationary points:** Graph $f'(x)$ and find zeros; or graph $f(x)$ and use *minimum/maximum* tools.
- **Second derivative:** Compute `nDeriv(nDeriv(f(x),x,x),x,a)` for $f''(a)$.
- **Related rates:** Set up the connecting formula analytically; use GDC only to evaluate the final expression at the given instant.

- **Chain/product/quotient:** Always differentiate by hand and use GDC to verify the derivative at a specific value.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. EASY Hand **[3 marks]**

Let $f(x) = 5e^{3x} - 4\ln x$.

(a) Find $f'(x)$. **[2]**

(b) Hence find the exact value of $f'(1)$. **[1]**

2. EASY Hand **[3 marks]**

Let $g(x) = 3\sin x - 2\cos x + x^2$.

(a) Write down $g'(x)$. **[2]**

(b) Find the exact value of $g'(\pi)$. **[1]**

3. EASY Hand **[3 marks]**

Let $h(x) = (4x - 1)^5$.

(a) Find $h'(x)$ using the chain rule. [2]

(b) Find $h'(0)$. [1]

4. EASY Hand **[3 marks]**

Let $p(x) = e^{2x^2-1}$.

(a) Find $p'(x)$. [2]

(b) Find the exact value of $p'(1)$. [1]

5. EASY Hand **[3 marks]**

Let $f(x) = x^3e^x$.

(a) Find $f'(x)$ using the product rule. [2]

(b) Find $f'(0)$. [1]

6. EASY Hand **[3 marks]**

Let $q(x) = \dfrac{x^2 + 3}{2x - 1}$.

(a) Find $q'(x)$ using the quotient rule. [2]

(b) Find $q'(2)$. [1]

7. EASY Hand **[3 marks]**

Let $y = \ln(3x^2 + 2)$.

(a) Find $\dfrac{dy}{dx}$. [2]

(b) Find the value of $\dfrac{dy}{dx}$ when $x = 1$. Give your answer as an exact fraction. [1]

8. EASY Hand **[3 marks]**

The function $f(x) = 2x^3 - 9x^2 + 12x - 4$ has stationary points.

(a) Find $f'(x)$. [1]

(b) Find the x-coordinates of both stationary points. [2]

9. EASY Hand **[3 marks]**

Let $f(x) = x^4 - 8x^2 + 3$.

(a) Find $f''(x)$. [2]

(b) Determine whether the point $(0, 3)$ is a local maximum, local minimum, or point of inflection. Justify your answer. [1]

10. EASY Hand **[3 marks]**

A spherical balloon is being inflated. At the instant when its radius is $r = 6$ cm, the radius is increasing at a rate of 0.4 cm s^{-1}.

The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

(a) Write down $\frac{dV}{dr}$. [1]

(b) Find $\frac{dV}{dt}$ at the instant when $r = 6$ cm. Give your answer correct to 3 significant figures. [2]

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. MEDIUM Hand **[4 marks]**

Let $f(x) = x^2 \ln(2x)$, for $x > 0$.

(a) Find $f'(x)$. [2]

(b) Find the equation of the tangent to the curve at the point where $x = 1$. [2]

12. MEDIUM Hand **[4 marks]**

Let $g(x) = \dfrac{\sin x}{e^x}$.

(a) Find $g'(x)$. [2]

(b) Show that $g'(x) = 0$ when $\tan x = 1$. [1]

(c) Hence write down one value of x in $[0, 2\pi]$ for which the curve has a stationary point. [1]

13. MEDIUM Hand **[4 marks]**

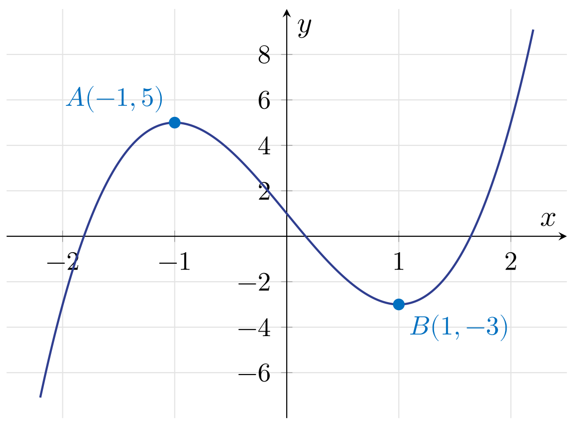
Let $y = \dfrac{3x+2}{\sqrt{x^2+1}}$, for $x \in \mathbb{R}$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3-2x}{(x^2+1)^{3/2}}$. **[3]**

(b) Hence write down the x-coordinate of the stationary point of the curve. **[1]**

14. MEDIUM Hand **[4 marks]**

The diagram shows the curve $y = f(x)$ where $f(x) = 2x^3 - 6x + 1$.



Points $A(-1, 5)$ and $B(1, -3)$ are the stationary points of the curve.

(a) Find $f'(x)$ and verify that the x-coordinates of A and B are correct. **[2]**

(b) Use the second derivative to determine the nature of each stationary point. **[2]**

15. MEDIUM Hand **[4 marks]**

A curve has equation $y = x^2 e^{-2x}$.

(a) Find $\frac{dy}{dx}$. **[2]**

(b) Find the coordinates of the stationary points of the curve. **[2]**

16. MEDIUM GDC **[4 marks]**

A water tank is in the shape of a cone (vertex downward) with a half-angle of $30°$. When the depth of water is h metres, the volume of water is $V = \dfrac{\pi h^3}{9}$ m^3.

Water pours in at a constant rate of 0.05 m^{3}min^{-1}.

(a) Find $\dfrac{dV}{dh}$. [1]

(b) Find the rate at which the depth of water is increasing when $h = 1.2$ m. Give your answer correct to 3 significant figures. [3]

17. MEDIUM Hand **[4 marks]**

Let $f(x) = 3x^4 - 8x^3 + 6x^2$.

(a) Find $f''(x)$. [2]

(b) Find the x-coordinates of the points of inflection of the curve. [2]

18. MEDIUM Hand **[4 marks]**

The position of a particle moving along a straight line at time t seconds (where $t \geq 0$) is given by

$$s(t) = t^3 e^{-t}, \quad t \geq 0.$$

(a) Find the velocity $v(t) = s'(t)$. [2]

(b) Find the times at which the particle is instantaneously at rest. [2]

19. MEDIUM Hand **[4 marks]**

Let $f(x) = \sin(3x)\cos x$ for $0 \leq x \leq \pi$.

(a) Find $f'(x)$. [2]

(b) Find the x-coordinate of the first stationary point of f in $\left(0, \dfrac{\pi}{2}\right)$. Give your answer correct to 3 significant figures. [2]

20. MEDIUM Hand **[4 marks]**

The table below gives values of a differentiable function f and its derivative for several values of x.

x	-2	-1	0	1
$f(x)$	-5	0	3	4
$f'(x)$	6	4	0	-3

Let $h(x) = x^2 f(x)$.

(a) Show that $h'(x) = 2x\, f(x) + x^2 f'(x)$. **[1]**

(b) Find $h'(-1)$. **[1]**

(c) Find the equation of the tangent to $y = h(x)$ at $x = -1$. **[2]**

Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)

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21. **HARD** **Hand** **[5 marks]**

Let $f(x) = \dfrac{e^{2x}}{x^2 + k}$ where $k > 0$ is a constant.

(a) Find $f'(x)$, simplifying your answer. [2]

(b) The curve $y = f(x)$ has a stationary point at $x = \dfrac{1}{2}$. Find the value of k. [2]

(c) Determine whether this stationary point is a local maximum or local minimum. [1]

22. **HARD** **Hand** **[5 marks]**

A curve is defined by $y = x^2 \ln x - \dfrac{5}{2}x^2 + 3x$ for $x > 0$.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [3]

(b) Find the coordinates of the point of inflection of the curve. [2]

23. HARD Hand **[5 marks]**

Two variables x and y are connected by the relation $y = x\,e^{-x}$.

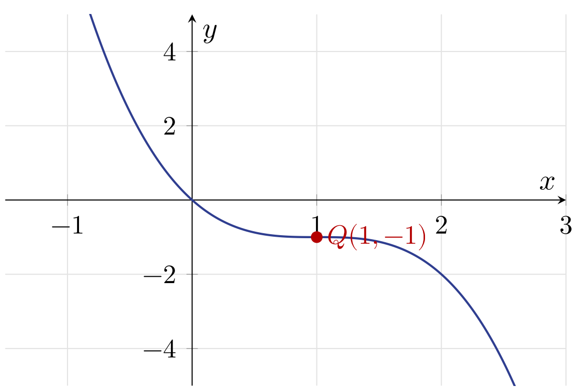
A point P moves along this curve such that $\dfrac{dx}{dt} = 2$ units s^{-1} at all times.

(a) Find $\dfrac{dy}{dt}$ in terms of x. [3]

(b) Find the value of x at which $\dfrac{dy}{dt} = 0$. Interpret this in context. [2]

24. HARD Hand **[5 marks]**

The diagram shows the curve $y = f(x)$ where $f(x) = ax^3 + bx^2 - 3x$ for constants a and b.



The curve has a stationary point at $Q(1, -1)$ and passes through the origin.

(a) Use the condition $f(0) = 0$ to show that there is no additional constant term, then show $f'(1) = 0$ and $f(1) = -1$ give the system: $3a + 2b = 3$ and $a + b = 2$. [2]

(b) Solve the system to find a and b. [2]

(c) Determine the nature of the stationary point at $x = 1$. [1]

25. HARD Hand **[5 marks]**

The function $f(x) = \dfrac{\ln x}{x^n}$, where n is a positive integer and $x > 0$, is to be analysed.

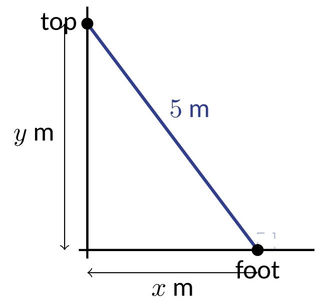
(a) Show that $f'(x) = \dfrac{1 - n\ln x}{x^{n+1}}$. [2]

(b) Hence show that the x-coordinate of the stationary point of f is $x = e^{1/n}$. [1]

(c) For the case $n = 2$, find the exact coordinates of the stationary point and determine whether it is a maximum or minimum. [2]

26. HARD GDC **[5 marks]**

A ladder of fixed length 5 m rests with its foot on horizontal ground and its top against a vertical wall. At a certain instant the foot of the ladder is 3 m from the wall and is sliding away from the wall at a rate of 0.6 m s^{-1}.



Let x be the horizontal distance of the foot from the wall and y the height of the top of the ladder.

(a) Write down the relationship between x, y, and the length of the ladder. **[1]**

(b) Show that $\dfrac{dy}{dt} = -\dfrac{x}{y} \cdot \dfrac{dx}{dt}$. **[2]**

(c) Find the rate at which the top of the ladder is sliding down the wall at this instant. Give your answer correct to 3 significant figures. **[2]**

27. HARD Hand **[5 marks]**

Let $f(x) = \sin^3 x$ for $0 \leq x \leq 2\pi$.

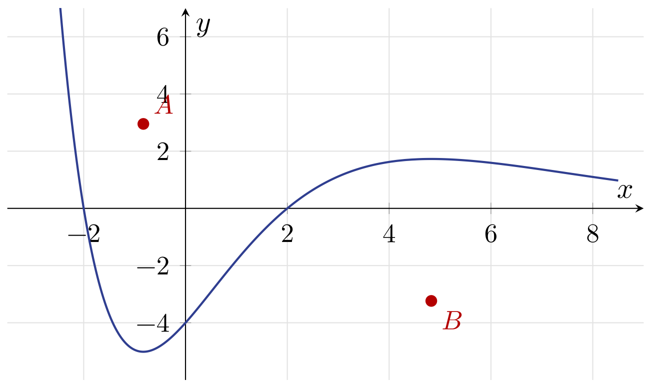
(a) Use the chain rule to show that $f'(x) = 3\sin^2 x \cos x$. **[1]**

(b) Find $f''(x)$. **[2]**

(c) Find all x-values in $[0, 2\pi]$ where the curve has a point of inflection. Justify that each is a point of inflection by showing the concavity changes sign. **[2]**

28. HARD Hand **[5 marks]**

A function is defined by $f(x) = (x^2 - 4)e^{-x/2}$ for $x \in \mathbb{R}$.
The diagram below shows a sketch of $y = f(x)$.



(a) Find $f'(x)$, giving your answer in the form $\frac{1}{2}e^{-x/2}(-x^2 + px + q)$ for integers p and q. **[2]**

(b) Hence find the exact x-coordinates of the stationary points A and B. **[2]**

(c) State, with justification, the nature of the stationary point B. **[1]**

29. HARD Hand **[5 marks]**

Let $f(x) = e^x - kx^2$ where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$. [1]

(b) The function f has a local minimum. Show that $f'(x) = 0$ can be written as $e^x = 2kx$. [1]

(c) When $k = 2$, use a GDC to find the x-coordinate of the local minimum, giving your answer correct to 3 significant figures. [1]

(d) For general $k > 0$, determine using f'' whether a stationary point satisfying $e^x = 2kx$ with $x > 0$ is always a minimum. [2]

30. HARD Hand **[5 marks]**

A manufacturer models the profit (in thousands of dollars) from selling x thousand units by

$$P(x) = \frac{12x}{x^2+4} - 0.1x, \quad x > 0.$$

(a) Show that $P'(x) = \dfrac{-12x^2+48}{(x^2+4)^2} - 0.1$. [2]

(b) Find the value of x that maximises profit, giving your answer correct to 3 significant figures. [2]

(c) Determine the maximum profit, giving your answer correct to 3 significant figures. [1]

Common Mistakes to Avoid

1. **Chain rule omission.** When differentiating $e^{g(x)}$ or $\ln(g(x))$, always multiply by $g'(x)$. A very common error is writing $\frac{d}{dx}(e^{3x}) = e^{3x}$ instead of $3e^{3x}$.
2. **Product rule sign errors.** The product rule is $uv' + vu'$, not $u'v'$. Write down u, u', v, v' separately before substituting.
3. **Quotient rule confusion.** The numerator is $vu' - uv'$ (not $uv' - vu'$). A mnemonic: "low d-high minus high d-low, over the square of what's below."
4. **Related rates: forgetting to use chain rule.** $\dfrac{dV}{dt} \neq \dfrac{dV}{dr}$; you must write $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$.
5. **Point of inflection without sign check.** Finding $f''(x) = 0$ alone is not sufficient; you must verify that f'' changes sign at that point.
6. **Stationary vs. inflection.** A point where $f'(x) = 0$ may still be a point of inflection (e.g. $y = x^3$ at $x = 0$). Always use f'' or sign analysis of f' to classify.

Worked Solutions

Mark scheme notation: **M1** = method mark, **A1** = accuracy/answer mark, **R1** = reasoning mark, **AG** = answer given (no mark)

Section A — Easy

Q1 — Special Derivatives EASY

(a) Differentiating term by term: **M1**

$f'(x) = 15e^{3x} - \dfrac{4}{x}$ **A1**

Note: M1 for applying $\frac{d}{dx}(e^{kx}) = ke^{kx}$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$. Award A1 for both terms correct.

(b) $f'(1) = 15e^3 - 4$ **A1**

Note: Accept exact form only.

Q2 — Special Derivatives EASY

(a) $g'(x) = 3\cos x + 2\sin x + 2x$ **A1A1**

Note: A1 for $3\cos x + 2\sin x$; A1 for $+2x$. Penalise wrong sign on $\sin x$ term.

(b) $g'(\pi) = 3\cos\pi + 2\sin\pi + 2\pi$

$= 3(-1) + 2(0) + 2\pi = -3 + 2\pi$ **A1**

Note: Accept $2\pi - 3$ or exact equivalent.

Q3 — Chain Rule EASY

(a) Let $u = 4x - 1$, so $h = u^5$.

$h'(x) = 5(4x-1)^4 \cdot 4 = 20(4x-1)^4$ **M1A1**

Note: M1 for attempting chain rule with correct outer derivative $5(\ldots)^4$; A1 for multiplying by 4.

(b) $h'(0) = 20(4(0)-1)^4 = 20(-1)^4 = 20$ **A1**

Q4 — Chain Rule EASY

(a) $p'(x) = e^{2x^2-1} \cdot 4x$ **M1A1**

Note: M1 for chain rule structure $e^{2x^2-1} \cdot (\ldots)$; A1 for $4x$ as the inner derivative.

(b) $p'(1) = 4(1) \cdot e^{2(1)^2-1} = 4e^1 = 4e$ **A1**

Note: Accept $4e$ only; do not accept decimal unless stated.

Q5 — Product Rule EASY

(a) Let $u = x^3$, $v = e^x$, so $u' = 3x^2$, $v' = e^x$.

$f'(x) = x^3e^x + 3x^2e^x = x^2e^x(x+3)$ **M1A1**

Note: M1 for product rule structure $uv' + vu'$ with both terms present; A1 for correct expression (factored or unfactored).

(b) $f'(0) = 0^2 \cdot e^0 \cdot (0+3) = 0$ **A1**

Q6 — Quotient Rule EASY

(a) $u = x^2 + 3$, $v = 2x - 1$, $u' = 2x$, $v' = 2$.

$$q'(x) = \frac{(2x-1)(2x) - (x^2+3)(2)}{(2x-1)^2}$$ **M1**

$$= \frac{4x^2 - 2x - 2x^2 - 6}{(2x-1)^2} = \frac{2x^2 - 2x - 6}{(2x-1)^2}$$ **A1**

(b) $$q'(2) = \frac{2(4) - 2(2) - 6}{(2(2)-1)^2} = \frac{8-4-6}{9} = \frac{-2}{9}$$ **A1**

Note: Accept $-\frac{2}{9}$ or -0.222.

Q7 — Chain Rule with Logarithm EASY

(a) $$\frac{dy}{dx} = \frac{1}{3x^2+2} \cdot 6x = \frac{6x}{3x^2+2}$$ **M1A1**

Note: M1 for $\frac{1}{3x^2+2} \cdot (\ldots)$; A1 for correct $6x$ in numerator.

(b) $$\left.\frac{dy}{dx}\right|_{x=1} = \frac{6(1)}{3(1)+2} = \frac{6}{5}$$ **A1**

Note: Must be exact fraction $\frac{6}{5}$; accept 1.2.

Q8 — Stationary Points EASY

(a) $f'(x) = 6x^2 - 18x + 12$ **A1**

(b) Setting $f'(x) = 0$: **M1**

$6x^2 - 18x + 12 = 0 \implies x^2 - 3x + 2 = 0 \implies (x-1)(x-2) = 0$

$x = 1$ or $x = 2$ **A1**

Note: M1 for setting $f'(x) = 0$ and attempting to solve; A1 for both correct values.

Q9 — Second Derivative & Classification EASY

(a) $f'(x) = 4x^3 - 16x$

$f''(x) = 12x^2 - 16$ **A1A1**

Note: A1 for $4x^3 - 16x$; A1 for $12x^2 - 16$.

(b) $f'(0) = 0$ ✓ $f''(0) = 12(0) - 16 = -16 < 0$ **R1**

Since $f''(0) < 0$, the point $(0, 3)$ is a **local maximum**.

Note: R1 requires both evaluating $f''(0)$ and a correct conclusion. FT from their f''.

Q10 — Related Rates EASY

(a) $\dfrac{dV}{dr} = 4\pi r^2$ **A1**

(b) By chain rule: $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$ **M1**

$= 4\pi(6)^2 \times 0.4 = 4\pi \times 36 \times 0.4 = 57.6\pi$

$= 180.956\ldots \approx 181$ cm^{3}s^{-1} **A1**

Note: M1 for chain rule structure; A1 for 181 (accept 57.6π). Unit cm^3 s^{-1} required for final A.

Section B — Medium

Q11 — Product Rule; Tangent MEDIUM

(a) Let $u = x^2$, $v = \ln(2x)$, $u' = 2x$, $v' = \dfrac{1}{x}$.

$f'(x) = x^2 \cdot \dfrac{1}{x} + \ln(2x) \cdot 2x = x + 2x\ln(2x)$ **M1A1**

Note: M1 for product rule with both terms; A1 for $x + 2x\ln(2x)$ (or equivalent such as $x(1 + 2\ln(2x))$).

(b) At $x = 1$: $f(1) = 1^2 \ln 2 = \ln 2$; $f'(1) = 1 + 2\ln 2$ **(M1)**

Tangent: $y - \ln 2 = (1 + 2\ln 2)(x - 1)$ **M1**

$y = (1 + 2\ln 2)x - 1 - 2\ln 2 + \ln 2 = (1 + 2\ln 2)x - 1 - \ln 2$ **A1**

Note: Accept any correct equivalent form. FT from their $f'(1)$ provided method is correct.

Q12 — Quotient Rule; Stationary Points MEDIUM

(a) $u = \sin x$, $v = e^x$, $u' = \cos x$, $v' = e^x$.

$g'(x) = \dfrac{e^x \cos x - e^x \sin x}{e^{2x}} = \dfrac{\cos x - \sin x}{e^x}$ **M1A1**

Note: M1 for quotient rule structure; A1 for simplification to $(\cos x - \sin x)/e^x$.

(b) Setting $g'(x) = 0$: since $e^x > 0$, we need $\cos x - \sin x = 0$ **M1**

$\cos x = \sin x \implies \tan x = 1$ **AG**

(c) $\tan x = 1$ for $x \in [0, 2\pi]$: $x = \dfrac{\pi}{4}$ (or $x = \dfrac{5\pi}{4}$) **A1**

Note: Accept either value.

Q13 — Quotient Rule; Show That MEDIUM

(a) $u = 3x + 2$, $v = (x^2 + 1)^{1/2}$, $u' = 3$, $v' = \dfrac{x}{(x^2+1)^{1/2}}$ **M1**

$$\frac{dy}{dx} = \frac{3(x^2+1)^{1/2} - (3x+2) \cdot \dfrac{x}{(x^2+1)^{1/2}}}{x^2+1}$$ **A1**

$$= \frac{3(x^2+1) - (3x+2)x}{(x^2+1)^{3/2}} = \frac{3x^2 + 3 - 3x^2 - 2x}{(x^2+1)^{3/2}} = \frac{3 - 2x}{(x^2+1)^{3/2}}$$ **A1AG**

Note: M1 for quotient rule; A1 for correct unsimplified numerator; final A1 for algebra leading to AG.

(b) $\dfrac{dy}{dx} = 0 \implies 3 - 2x = 0 \implies x = \dfrac{3}{2}$ **A1**

Q14 — Stationary Points; Second Derivative Test MEDIUM

(a) $f'(x) = 6x^2 - 6$ **A1**

$f'(-1) = 6(1) - 6 = 0$ ✓; $f'(1) = 6(1) - 6 = 0$ ✓ **A1**

Note: First A1 for correct $f'(x)$; second A1 for verifying both points (must show substitution).

(b) $f''(x) = 12x$

$f''(-1) = -12 < 0 \Rightarrow A(-1, 5)$ is a **local maximum** **A1**

$f''(1) = 12 > 0 \Rightarrow B(1, -3)$ is a **local minimum** **A1**

Note: A1 for each correct classification with justification by sign of f''.

Q15 — Product Rule; Stationary Points MEDIUM

(a) $u = x^2$, $v = e^{-2x}$, $u' = 2x$, $v' = -2e^{-2x}$.

$\dfrac{dy}{dx} = 2xe^{-2x} + x^2(-2e^{-2x}) = 2xe^{-2x}(1 - x)$ **M1A1**

Note: M1 for product rule; A1 for factored form $2xe^{-2x}(1 - x)$ or equivalent.

(b) Setting $\dfrac{dy}{dx} = 0$: since $e^{-2x} > 0$, we need $2x(1 - x) = 0$ **M1**

$x = 0$ or $x = 1$

At $x = 0$: $y = 0$; at $x = 1$: $y = e^{-2}$ **A1**

Stationary points: $(0, 0)$ and $(1,\ e^{-2})$.
Note: M1 for setting derivative to zero and factoring; A1 for both coordinate pairs.

Q16 — Related Rates; Conical Tank MEDIUM

(a) $\dfrac{dV}{dh} = \dfrac{3\pi h^2}{9} = \dfrac{\pi h^2}{3}$ **A1**

(b) Chain rule: $\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt}$ **M1**

$0.05 = \dfrac{\pi(1.2)^2}{3} \cdot \dfrac{dh}{dt}$ **A1**

$\dfrac{dh}{dt} = \dfrac{0.05 \times 3}{\pi(1.44)} = \dfrac{0.15}{1.44\pi}$

$= 0.033157\ldots \approx 0.0332$ m min^{-1} **A1**

Note: M1 for chain rule; first A1 for correct equation with $h = 1.2$; final A1 for 0.0332. FT if their $\frac{dV}{dh}$ is consistent.

Q17 — Second Derivative; Points of Inflection MEDIUM

(a) $f'(x) = 12x^3 - 24x^2 + 12x$

$f''(x) = 36x^2 - 48x + 12$ **A1A1**

Note: A1 for correct $f'(x)$; A1 for correct $f''(x)$.

(b) Set $f''(x) = 0$: $36x^2 - 48x + 12 = 0 \implies 3x^2 - 4x + 1 = 0$ **M1**

$(3x - 1)(x - 1) = 0 \implies x = \dfrac{1}{3}$ or $x = 1$ **A1**

Note: M1 for setting $f'' = 0$ and solving; A1 for both correct values. Candidates should note a sign check is required to confirm inflection; accept without it at this difficulty level.

Q18 — Product Rule; Velocity MEDIUM

(a) $u = t^3, v = e^{-t}, u' = 3t^2, v' = -e^{-t}$.

$v(t) = 3t^2e^{-t} - t^3e^{-t} = t^2e^{-t}(3 - t)$ **M1A1**

Note: M1 for product rule; A1 for factored form $t^2e^{-t}(3 - t)$.

(b) $v(t) = 0 \implies t^2e^{-t}(3 - t) = 0$; since $e^{-t} > 0$: **M1**

$t = 0$ or $t = 3$ **A1**

Note: M1 for factoring and reasoning $e^{-t} > 0$; A1 for both values (accept $t = 0$ as boundary if candidate notes $t \geq 0$).

Q19 — Product Rule; Trig Stationary Points MEDIUM

(a) $u = \sin(3x)$, $v = \cos x$, $u' = 3\cos(3x)$, $v' = -\sin x$.

$f'(x) = 3\cos(3x)\cos x - \sin(3x)\sin x$ **M1A1**

Note: M1 for product rule; A1 for both correct terms.

(b) Set $f'(x) = 0$ in $\left(0, \frac{\pi}{2}\right)$: using GDC **(M1)**

$x = 0.785\ldots \approx 0.785$ (i.e. $x = \pi/4$) **A1**

Note: (M1) for using GDC/graph of f'; A1 for $x \approx 0.785$.

Q20 — Product Rule with Table Data; Tangent MEDIUM

(a) $h(x) = x^2 f(x)$. By product rule: $h'(x) = 2x\,f(x) + x^2 f'(x)$ **A1AG**

(b) $h'(-1) = 2(-1)f(-1) + (-1)^2 f'(-1)$

$= 2(-1)(0) + (1)(4) = 0 + 4 = 4$ **A1**

(c) $h(-1) = (-1)^2 f(-1) = (1)(0) = 0$ **(M1)**

Tangent at $(-1, 0)$ with gradient 4: $y - 0 = 4(x - (-1))$

$y = 4x + 4$ **A1**

Note: (M1) for finding $h(-1) = 0$; A1 for correct tangent equation.

Section C — Hard

Q21 — Quotient Rule; Unknown Parameter HARD

(a) $u = e^{2x}$, $v = x^2 + k$, $u' = 2e^{2x}$, $v' = 2x$.

$f'(x) = \dfrac{2e^{2x}(x^2+k) - e^{2x}(2x)}{(x^2+k)^2} = \dfrac{2e^{2x}(x^2 - x + k)}{(x^2+k)^2}$ **M1A1**

Note: M1 for quotient rule; A1 for correct simplification.

(b) At $x = \frac{1}{2}$: $f'\left(\frac{1}{2}\right) = 0 \implies x^2 - x + k = 0$ at $x = \frac{1}{2}$ **M1**

$\left(\frac{1}{2}\right)^2 - \frac{1}{2} + k = 0 \implies \frac{1}{4} - \frac{1}{2} + k = 0 \implies k = \frac{1}{4}$ **A1**

(c) With $k = \frac{1}{4}$, check sign of f' either side of $x = \frac{1}{2}$, or use f'':

The numerator factor $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 \geq 0$, so $f'(x) \geq 0$ for all x (equals 0 only at $x = \frac{1}{2}$).

This means f' does not change sign: the stationary point is a **point of inflection**. **R1**

Note: R1 for correct conclusion with valid reasoning (sign analysis or f'' evaluation).

Q22 — Product Rule; Point of Inflection HARD

(a) $\dfrac{dy}{dx} = 2x \ln x + x^2 \cdot \dfrac{1}{x} - 5x + 3 = 2x \ln x + x - 5x + 3 = 2x \ln x - 4x + 3$ **M1A1**

$\dfrac{d^2y}{dx^2} = 2 \ln x + 2x \cdot \dfrac{1}{x} - 4 = 2 \ln x + 2 - 4 = 2 \ln x - 2$ **A1**

Note: M1 for product rule on $x^2 \ln x$; A1 for correct $\frac{dy}{dx}$; A1 for correct $\frac{d^2y}{dx^2}$.

(b) Set $\dfrac{d^2y}{dx^2} = 0$: $2 \ln x - 2 = 0 \implies \ln x = 1 \implies x = e$ **M1**

$y(e) = e^2 \ln e - \dfrac{5}{2}e^2 + 3e = e^2 - \dfrac{5}{2}e^2 + 3e = -\dfrac{3}{2}e^2 + 3e$ **A1**

Point of inflection: $\left(e,\ 3e - \dfrac{3}{2}e^2\right)$

Note: Confirm sign change: $x < e$, $2 \ln x - 2 < 0$; $x > e$, $2 \ln x - 2 > 0$

$\checkmark$

. Accept decimal $\approx (2.72, -4.26)$.

Q23 — Related Rates with Chain Rule HARD

(a) $y = xe^{-x}$, so $\dfrac{dy}{dx} = e^{-x} + x(-e^{-x}) = e^{-x}(1 - x)$ **M1A1**

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = e^{-x}(1 - x) \cdot 2 = 2e^{-x}(1 - x)$ **A1**

Note: M1 for product rule on xe^{-x}; A1 for $\frac{dy}{dx}$; A1 for $\frac{dy}{dt}$.

(b) $\dfrac{dy}{dt} = 0 \implies 2e^{-x}(1 - x) = 0$; since $e^{-x} > 0$, $x = 1$ **A1**

At $x = 1$ the y-coordinate of P is momentarily not changing; P is at the turning point of the curve $y = xe^{-x}$. **R1**

Note: R1 requires a contextual statement (not just "y is stationary").

Q24 — System from Stationary Point Conditions HARD

(a) Since $f(0) = 0$: $a(0) + b(0) - 3(0) = 0$

$\checkmark$

(satisfied for any a, b).

$f'(x) = 3ax^2 + 2bx - 3$.

$f'(1) = 0$: $3a + 2b - 3 = 0 \implies 3a + 2b = 3$ **A1**

$f(1) = -1$: $a + b - 3 = -1 \implies a + b = 2$ **A1AG**

(b) From $a + b = 2$: $a = 2 - b$. Substituting: **M1**

$3(2-b)+2b=3 \implies 6-3b+2b=3 \implies b=3; a=2-3=-1$ **A1**

(c) $f''(x)=6ax+2b=-6x+6; f''(1)=-6+6=0.$

Since $f''(1)=0$, check sign of f': $f'(x)=-3x^2+6x-3=-3(x-1)^2 \le 0$ for all x.

f' does not change sign at $x=1$, so $Q(1,-1)$ is a **point of inflection**. **R1**

Note: R1 for correct classification with sign-analysis justification.

Q25 — Quotient Rule; General Parameter HARD

(a) $u=\ln x, v=x^n, u'=\dfrac{1}{x}, v'=nx^{n-1}$.

$$f'(x)=\frac{\frac{1}{x}\cdot x^n-\ln x\cdot nx^{n-1}}{x^{2n}}=\frac{x^{n-1}-nx^{n-1}\ln x}{x^{2n}}=\frac{x^{n-1}(1-n\ln x)}{x^{2n}}=\frac{1-n\ln x}{x^{n+1}}$$ **M1A1AG**

Note: M1 for quotient rule; A1 for correct simplification leading to AG.

(b) $f'(x)=0 \implies 1-n\ln x=0 \implies \ln x=\dfrac{1}{n} \implies x=e^{1/n}$ **A1AG**

(c) For $n=2$: $x=e^{1/2}=\sqrt{e}$; $y=\dfrac{\ln\sqrt{e}}{e}=\dfrac{\frac{1}{2}}{e}=\dfrac{1}{2e}$ **A1**

$f''(x)=\dfrac{d}{dx}\left[\dfrac{1-2\ln x}{x^3}\right]$; at $x=\sqrt{e}$, using quotient rule or substituting into f'':

$$f''(\sqrt{e})=\frac{-\frac{1}{\sqrt{e}}\cdot(\sqrt{e})^3-(1-2\cdot\frac{1}{2})\cdot 3(\sqrt{e})^2}{(\sqrt{e})^6}=\frac{-e-0}{e^3}=\frac{-1}{e^2}<0$$ **A1**

Since $f''(\sqrt{e})<0$, the stationary point $\left(\sqrt{e},\ \dfrac{1}{2e}\right)$ is a **local maximum**.

Note: A1 for coordinates; A1 for $f''<0$ with correct conclusion.

Q26 — Related Rates; Ladder HARD

(a) $x^2+y^2=25$ (Pythagoras) **A1**

(b) Differentiating with respect to t: **M1**

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0 \implies y\frac{dy}{dt}=-x\frac{dx}{dt} \implies \frac{dy}{dt}=-\frac{x}{y}\cdot\frac{dx}{dt}$$ **A1AG**

Note: M1 for implicit differentiation; A1 for rearrangement to AG form.

(c) When $x=3$: $y=\sqrt{25-9}=4$ **(A1)**

$\dfrac{dy}{dt}=-\dfrac{3}{4}\times 0.6=-0.45$ m s^{-1} **A1**

The top of the ladder slides down at 0.450 m s^{-1}.

Note: (A1) implied for $y=4$; A1 for -0.45 (negative sign indicates downward; accept 0.45 if direction stated).

Q27 — Chain Rule; Inflection Points of $\sin^3 x$ HARD

(a) Let $u = \sin x$, so $f = u^3$. $\dfrac{df}{dx} = 3u^2 \cdot \cos x = 3\sin^2 x \cos x$ **A1AG**

(b) $f''(x) = \dfrac{d}{dx}[3\sin^2 x \cos x]$

Apply product rule: $u = 3\sin^2 x$, $v = \cos x$, $u' = 6\sin x\cos x$, $v' = -\sin x$ **M1**

$f''(x) = 6\sin x\cos^2 x - 3\sin^3 x = 3\sin x(2\cos^2 x - \sin^2 x)$ **A1**

Note: M1 for product rule attempt; A1 for $3\sin x(2\cos^2 x - \sin^2 x)$ or equivalent.

(c) $f''(x) = 0$ when $\sin x = 0$ or $2\cos^2 x = \sin^2 x$, i.e. $\tan^2 x = 2$, $\tan x = \pm\sqrt{2}$ **M1**

In $[0, 2\pi]$: $\sin x = 0$ gives $x = 0, \pi, 2\pi$; $\tan x = \sqrt{2}$ gives $x = \arctan\sqrt{2} \approx 0.955$ and $x \approx 0.955 + \pi \approx 4.10$; $\tan x = -\sqrt{2}$ gives $x \approx \pi - 0.955 \approx 2.19$ and $x \approx 2\pi - 0.955 \approx 5.33$.

Sign changes of f'' confirm inflections at $x \approx 0.955,\ 2.19,\ \pi,\ 4.10,\ 5.33$ (and boundaries $0, 2\pi$ excluded as endpoints). **A1**

Note: A1 for identifying the five interior inflection x-values with justification of sign change. Accept $\arctan\sqrt{2}$ as exact form.

Q28 — Product Rule; Stationary Points in Exact Form HARD

(a) $u = x^2 - 4$, $v = e^{-x/2}$, $u' = 2x$, $v' = -\frac{1}{2}e^{-x/2}$.

$f'(x) = 2x\,e^{-x/2} + (x^2 - 4)\left(-\frac{1}{2}\right)e^{-x/2}$ **M1**

$= e^{-x/2}\left(2x - \dfrac{x^2 - 4}{2}\right) = \dfrac{1}{2}e^{-x/2}(4x - x^2 + 4) = \dfrac{1}{2}e^{-x/2}(-x^2 + 4x + 4)$ **A1**

So $p = 4$, $q = 4$.

(b) $f'(x) = 0 \implies -x^2 + 4x + 4 = 0 \implies x^2 - 4x - 4 = 0$ **M1**

$x = \dfrac{4 \pm \sqrt{16 + 16}}{2} = \dfrac{4 \pm 4\sqrt{2}}{2} = 2 \pm 2\sqrt{2}$ **A1**

$x_A = 2 - 2\sqrt{2} \approx -0.828$; $\quad x_B = 2 + 2\sqrt{2} \approx 4.83$.

(c) For $x > x_B = 2 + 2\sqrt{2}$: $-x^2 + 4x + 4 < 0$, so $f'(x) < 0$ (decreasing). For $x_A < x < x_B$: $f' > 0$.

f' changes from positive to negative at x_B, so B is a **local maximum**. **R1**

Note: R1 for sign analysis of f' around x_B leading to correct conclusion.

Q29 — General Parameter; Minimum Classification HARD

(a) $f'(x) = e^x - 2kx$; $\quad f''(x) = e^x - 2k$ **A1**

Note: A1 for both derivatives correct.

(b) $f'(x) = 0 \implies e^x - 2kx = 0 \implies e^x = 2kx$ **A1AG**

(c) For $k = 2$: solve $e^x = 4x$ using GDC **(M1)**

$x \approx 0.357$ **A1**

Note: Full precision $x = 0.35740\ldots$. Accept 3 s.f. value.

(d) At a stationary point with $x > 0$: $e^x = 2kx$, so $f''(x) = e^x - 2k = 2kx - 2k = 2k(x-1)$. **M1**

If $x > 1$: $f''(x) = 2k(x-1) > 0 \Rightarrow$ minimum. If $0 < x < 1$: $f''(x) < 0 \Rightarrow$ maximum.

So a stationary point with $x > 0$ is a minimum if and only if $x > 1$; it is not always a minimum. **A1**

Note: M1 for substituting $e^x = 2kx$ into f''; A1 for correct conclusion with reasoning.

Q30 — Quotient Rule; Optimisation HARD

(a) $u = 12x$, $v = x^2 + 4$, $u' = 12$, $v' = 2x$.

$$\frac{d}{dx}\left(\frac{12x}{x^2+4}\right) = \frac{12(x^2+4) - 12x(2x)}{(x^2+4)^2} = \frac{12x^2 + 48 - 24x^2}{(x^2+4)^2} = \frac{-12x^2+48}{(x^2+4)^2}$$ **M1A1**

$$P'(x) = \frac{-12x^2+48}{(x^2+4)^2} - 0.1$$ **AG**

Note: M1 for quotient rule; A1 for correct numerator $-12x^2 + 48$.

(b) Setting $P'(x) = 0$ using GDC (graph P' or solve numerically) **M1**

$P'(x) = 0$ at $x = 1.68\ldots \approx 1.68$ (thousands of units) **A1**

Note: Full precision: solving gives $x \approx 1.6839$. Award A1 for 3 s.f.

(c) $$P(1.684) = \frac{12(1.684)}{(1.684)^2+4} - 0.1(1.684) \approx \frac{20.208}{6.836} - 0.1684 \approx 2.957 - 0.168 \approx 2.79$$ **A1**

Maximum profit $\approx \$2\,790$ (i.e. 2.79 thousand dollars).

Note: A1 for 2.79 (or equivalent). FT from their x in part (b).

Common Mistakes to Avoid

1. **Chain rule omission.** When differentiating $e^{g(x)}$ or $\ln(g(x))$, always multiply by $g'(x)$. A very common error is writing $\frac{d}{dx}(e^{3x}) = e^{3x}$ instead of $3e^{3x}$.
2. **Product rule sign errors.** The product rule is $uv' + vu'$, not $u'v'$. Write down u, u', v, v' separately before substituting.
3. **Quotient rule confusion.** The numerator is $vu' - uv'$ (not $uv' - vu'$). A mnemonic: "low d-high minus high d-low, over the square of what's below."
4. **Related rates: forgetting to use chain rule.** $\dfrac{dV}{dt} \neq \dfrac{dV}{dr}$; you must write $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$.
5. **Point of inflection without sign check.** Finding $f''(x) = 0$ alone is not sufficient; you must verify that f'' changes sign at that point.

6. **Stationary vs. inflection.** A point where $f'(x) = 0$ may still be a point of inflection (e.g. $y = x^3$ at $x = 0$, or Q24 in this worksheet). Always use f'' or sign analysis of f' to classify.