IB Math AI HL — Topic 5: Calculus

Integration

Trapezoidal Rule · Intro to Integration · Powers of x · Constant of Integration · Areas using GDC

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated. GDC permitted throughout.

Key Formulae

Trapezoidal Rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{h}{2}[y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n], \quad h = \frac{b-a}{n}$

Power Rule: $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$ **Constant Rule:** $\displaystyle\int k\,dx = kx + C$

Definite Integral (area): Area $= \displaystyle\int_a^b f(x)\,dx \quad (f(x) \geq 0)$ **Area between curves:** $\displaystyle\int_a^b [f(x) - g(x)]\,dx$

Finding C: substitute a known point (x_0, y_0) into the antiderivative to solve for C.

GDC Tips

1. Definite integral on TI-Nspire: `Menu` $\to$ `4:Calculus` $\to$ `3:Integral` then enter limits.
2. Casio fx-CG50: `MENU` $\to$ `Graph` mode, draw the curve, then `G-Solve` $\to$ $\int$`dx` and enter the limits.
3. For areas, always sketch the curve first to identify positive/negative regions.
4. For the trapezoidal rule, count the number of *strips* (n) and check $h = \frac{b-a}{n}$; the number of ordinates is $n+1$.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY GDC **[3 marks]**

The table gives the speed v (m s^{-1}) of a cyclist recorded at equal time intervals.

t (seconds)	0	3	6	9	12
v (m s^{-1})	4.0	5.6	6.8	7.2	6.0

Use the trapezoidal rule with four strips to estimate the distance travelled by the cyclist in the first 12 seconds.

2. EASY GDC **[3 marks]**

Find $\int (3x^2 - 4x + 5)\, dx$.

3. EASY GDC **[3 marks]**

(a) Find $\int_1^4 2\sqrt{x}\,dx$. [2]

(b) State the geometrical meaning of your answer. [1]

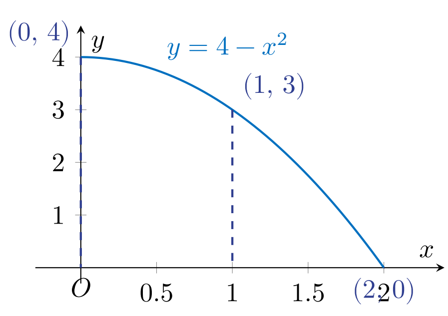
4. EASY GDC **[3 marks]**

A curve has gradient function $\frac{dy}{dx} = 6x^2 - 2$.

Given that the curve passes through the point $(1,\ 3)$, find the equation of the curve.

5. EASY GDC **[3 marks]**

The diagram shows part of the curve $y = 4 - x^2$ for $0 \le x \le 2$.



Use the trapezoidal rule with two strips of equal width to estimate $\int_0^2 (4 - x^2)\,dx$.

6. EASY GDC **[3 marks]**

Find $\int \left(5x^3 + \frac{2}{x^2} - 3\right) dx$.

7. EASY GDC **[3 marks]**

(a) Use your GDC to find $\int_0^3 (x^3 - 3x^2 + 2x + 1)\, dx$. [2]

(b) Determine whether this value equals the area of the region enclosed between the curve and the x-axis for $0 \le x \le 3$. Give a reason for your answer. [1]

8. EASY GDC **[3 marks]**

A curve has gradient function $\frac{dy}{dx} = 4x - 3$.

Given that the curve passes through the point $(0,\ 7)$, find the y-coordinate of the point on the curve where $x = 2$.

9. **EASY** **GDC** **[3 marks]**

The table gives values of $f(x)$.

x	1	3	5	7
$f(x)$	2.4	5.8	8.2	6.6

Use the trapezoidal rule with three strips of equal width to estimate $\int_1^7 f(x)\,dx$.

10. **EASY** **GDC** **[3 marks]**

(a) Find $\int (8x^3 - 6x)\,dx$. [2]

(b) Hence evaluate $\int_0^1 (8x^3 - 6x)\,dx$. [1]

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM GDC **[4 marks]**

The rate at which water flows into a tank, r (litres per minute), is recorded every two minutes.

t (min)	0	2	4	6	8	10
r (L min^{-1})	3.0	4.8	6.5	7.2	6.0	4.1

(a) Use the trapezoidal rule with five strips to estimate the total volume of water that flows into the tank in the first 10 minutes. [2]

(b) Use your GDC to evaluate $\int_0^{10} (3 + 0.9t - 0.07t^2)\, dt$ and hence find the percentage error of the trapezoidal estimate compared to this value. Give your answer correct to 1 decimal place. [2]

12. MEDIUM GDC **[4 marks]**

A curve $y = f(x)$ passes through the point $(2,\ 10)$. The gradient function of the curve is

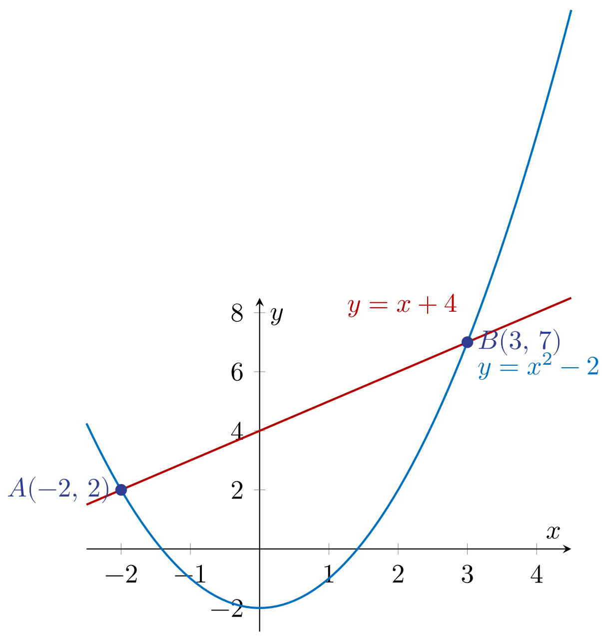
$$f'(x) = 3x^2 - 4x + 1.$$

(a) Find $f(x)$. [3]

(b) Find the x-coordinates of the stationary points of the curve. [1]

13. MEDIUM GDC **[4 marks]**

The diagram shows the curves $y = x^2 - 2$ and $y = x + 4$ for $-2 \leq x \leq 4$.



(a) Write down the x-coordinates of the points A and B where the curves intersect. [1]

(b) Find the area of the region enclosed between the two curves. [3]

14. MEDIUM GDC **[4 marks]**

(a) Find $\int \left(4x^3 - \frac{6}{x^3} + 2\right) dx.$ [3]

(b) Hence evaluate $\int_1^2 \left(4x^3 - \frac{6}{x^3} + 2\right) dx.$ [1]

15. MEDIUM GDC **[4 marks]**

A particle moves along a straight line. Its acceleration is given by $a(t) = 4t - 6$ m s^{-2}, where t is measured in seconds.

At $t = 0$, the particle has velocity 2 m s^{-1}.

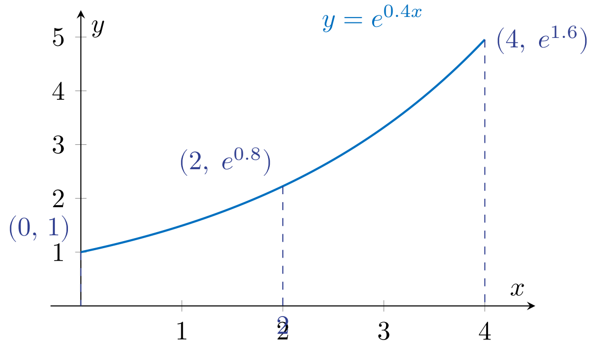
(a) Find an expression for the velocity of the particle at time t. [2]

(b) Find the velocity when $t = 4$ s. [1]

(c) Find the first time $t > 0$ at which the particle is momentarily at rest. [1]

16. MEDIUM GDC **[4 marks]**

The diagram shows part of the curve $y = e^{0.4x}$ for $0 \le x \le 4$.

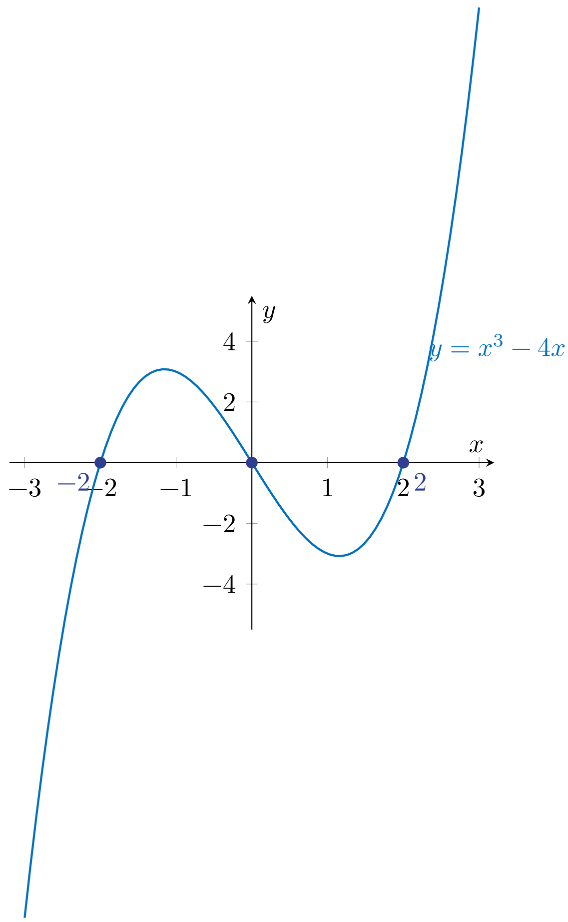


(a) Use the trapezoidal rule with two strips of equal width to estimate $\displaystyle\int_0^4 e^{0.4x}\,dx$. Give your answer correct to 3 significant figures. [2]

(b) Use your GDC to find the exact value of $\displaystyle\int_0^4 e^{0.4x}\,dx$ and hence determine whether the trapezoidal estimate is an over- or under-estimate. Justify your answer. [2]

17. MEDIUM GDC **[4 marks]**

The curve $y = x^3 - 4x$ is shown for $-3 \le x \le 3$.



(a) Write down the x-intercepts of the curve. [1]

(b) Find the total area of the two regions enclosed between the curve $y = x^3 - 4x$ and the x-axis for $-2 \le x \le 2$. [3]

18. MEDIUM GDC **[4 marks]**

A research team measures the cross-sectional area of a river. The width of the river is 12 m, and depths (in metres) are recorded at 3 m intervals from the left bank.

Distance from left bank (m)	0	3	6	9	12
Depth (m)	0	1.8	2.6	2.1	0

(a) Use the trapezoidal rule with four strips to estimate the cross-sectional area of the river. [2]

(b) A more accurate survey gives the cross-sectional area as 19.4 m^2. Calculate the percentage error of the trapezoidal estimate. [2]

19. MEDIUM GDC **[4 marks]**

A function $g(x)$ satisfies $g'(x) = 6x^2 - 10x + 4$ and $g(0) = -3$.

(a) Find $g(x)$. [2]

(b) Find the values of x for which $g(x) = 0$. Give your answers correct to 3 significant figures. [2]

20. MEDIUM GDC **[4 marks]**

The rate of heat loss from a building, H (watts), is modelled by

$$H(t) = 120 - 15\sin\left(\frac{\pi t}{12}\right), \quad 0 \leq t \leq 24,$$

where t is the time in hours after midnight.

(a) Use your GDC to find $\int_0^{24} H(t)\,dt$. [2]

(b) Interpret the meaning of your answer in context. [1]

(c) Find the average rate of heat loss over the 24-hour period. [1]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **GDC** **[5 marks]**

The curve C is defined by $y = x^2 - kx$, where $k > 0$. The line L has equation $y = x$.

(a) Show that C and L intersect when $x = 0$ and $x = k + 1$. [2]

(b) Find, in terms of k, the area of the region enclosed between C and L. [3]

22. HARD GDC **[5 marks]**

The table gives values of a function $f(x) = \dfrac{1}{1+x^2}$.

x	0	0.5	1.0	1.5	2.0
$f(x)$	1.000	0.800	0.500	0.308	0.200

(a) Use the trapezoidal rule with four strips to estimate $\displaystyle\int_0^2 f(x)\,dx$. [2]

(b) Use your GDC to find the exact value of $\displaystyle\int_0^2 \frac{1}{1+x^2}\,dx$. [1]

(c) Calculate the percentage error of the trapezoidal estimate compared to the GDC value. [1]

(d) State, with a reason, whether the trapezoidal estimate is an over- or under-estimate. [1]

23. HARD GDC **[5 marks]**

The curve $y = f(x)$ has gradient function $f'(x) = ax^2 + 4x$, where a is a constant.

The curve passes through $P(3,\,28)$ and $Q(1,\,q)$ where Q is the y-intercept of the tangent to the curve at P.

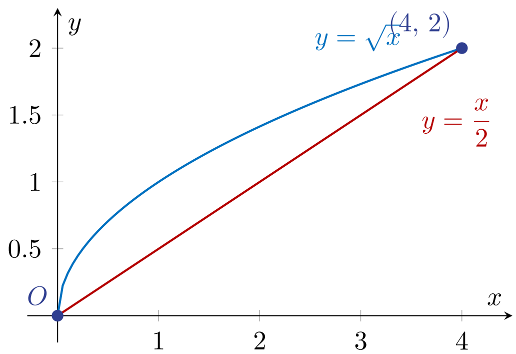
(a) Show that $f(x) = \dfrac{a}{3}x^3 + 2x^2 + C$. [1]

(b) The tangent to the curve at P has gradient 15. Find the value of a. [2]

(c) Hence find the equation of the curve. [2]

24. **HARD** **GDC** **[5 marks]**

The diagram shows the curve $y = \sqrt{x}$ and the line $y = \dfrac{x}{2}$ for $0 \leq x \leq 4$.



(a) Find the x-coordinates of the points of intersection of the two curves. [1]

(b) Find the area of the region enclosed between $y = \sqrt{x}$ and $y = \dfrac{x}{2}$. [2]

(c) The region is rotated $360°$ about the x-axis. Use your GDC to find the volume of the solid formed. Give your answer in terms of π. [2]

25. HARD GDC **[5 marks]**

A function is defined by $f(x) = x^3 - 6x^2 + 9x + c$, where c is a constant.

Given that $f(1) = 6$:

(a) Find the value of c. [1]

(b) Find the x-coordinates of the local maximum and local minimum of f. [2]

(c) Use your GDC to find the area of the region enclosed between the curve and the x-axis between the two stationary points. [2]

26. HARD GDC **[5 marks]**

A biologist models the rate of growth of a bacterial population as $r(t) = 2t^2 - 8t + 12$ thousand bacteria per hour, where t is measured in hours, $0 \leq t \leq 5$.

(a) Show that $r(t) > 0$ for all t in $[0, 5]$. [2]

(b) The biologist uses the trapezoidal rule with five strips of equal width to estimate the total increase in population over $[0, 5]$. Write down the value of h and calculate this estimate. [2]

(c) Use your GDC to evaluate $\int_0^5 r(t)\,dt$ exactly. State whether the trapezoidal estimate is an over- or under-estimate, justifying your answer using the concavity of $r(t)$. [1]

27. **HARD** **GDC** **[5 marks]**

The function $f(x) = 3x^2 - 2x$ is defined for $x \geq 0$.

(a) Find $\int_0^k f(x)\,dx$, giving your answer in terms of k. [2]

(b) Given that $\int_0^k f(x)\,dx = 10$, find the value of k. [3]

28. **HARD** **GDC** **[5 marks]**

A particle moves along a straight line. At time t seconds, its velocity is $v(t) = 6 - t^2$ m s^{-1}.

The particle starts at the origin when $t = 0$.

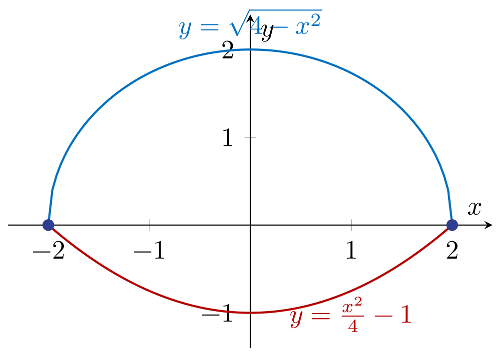
(a) Find an expression for the displacement $s(t)$ of the particle from the origin. [2]

(b) Find the time $t > 0$ at which the particle is momentarily at rest. [1]

(c) Find the total distance travelled by the particle in the first 3 seconds. [2]

29. HARD GDC **[5 marks]**

An engineer models the cross-sectional profile of an aerofoil. The upper surface is described by $y = \sqrt{4 - x^2}$ and the lower surface by $y = \dfrac{x^2}{4} - 1$, for $-2 \le x \le 2$.



(a) Show that the two curves meet at $x = -2$ and $x = 2$. [1]

(b) Use the trapezoidal rule with four strips of equal width to estimate the cross-sectional area of the aerofoil. [2]

(c) Use your GDC to find the exact cross-sectional area. [2]

30. HARD GDC **[5 marks]**

A curve C has equation $y = x^3 - 3x^2 - 9x + d$, where d is a constant.

Given that the curve passes through the point $(4, 10)$:

(a) Find the value of d. [1]

(b) Find the local maximum and local minimum values of y. [2]

(c) Use your GDC to find the area of the region enclosed between the curve and the line $y = 15$ for the interval where the curve lies below this line. [2]

Worked Solutions

Official Mark-Scheme Style — M/A/R notation

Section A — Easy

Q1 — Trapezoidal Rule – EASY [3 marks

] $h = 3$, four strips **(A1)**

Distance $\approx \frac{3}{2}[4.0 + 2(5.6 + 6.8 + 7.2) + 6.0]$ **M1**

$= \frac{3}{2}[4.0 + 39.2 + 6.0] = \frac{3}{2}(49.2) = 73.8$ m **A1**

Q2 — Power Rule – EASY [3 marks

] $\int (3x^2 - 4x + 5)\, dx$ **M1**

$= x^3 - 2x^2 + 5x + C$ **A1 A1**

Note: Award A1 for $x^3 - 2x^2 + 5x$ correct; A1 for $+C$. Penalise only once for missing C.

Q3 — Definite Integral (sqrt x) – EASY [3 marks

] **(a)** $\int_1^4 2x^{1/2}\, dx = \left[\frac{2x^{3/2}}{3/2}\right]_1^4 = \left[\frac{4x^{3/2}}{3}\right]_1^4$ **M1**

$= \frac{4(8)}{3} - \frac{4(1)}{3} = \frac{32}{3} - \frac{4}{3} = \frac{28}{3} \approx 9.33$ **A1**

(b) It is the area of the region bounded by $y = 2\sqrt{x}$, the x-axis, and the lines $x = 1$ and $x = 4$. **R1**

Q4 — Constant of Integration – EASY [3 marks

] Integrating: $y = 2x^3 - 2x + C$ **M1 A1**

Substituting $(1, 3)$: $3 = 2 - 2 + C \implies C = 3$ **M1**

$y = 2x^3 - 2x + 3$ **A1**

Note: Award M1 for attempting to integrate; A1 for correct $2x^3 - 2x$; final A1 for $C = 3$ correctly substituted.

Q5 — Trapezoidal Rule (2 strips) – EASY [3 marks

] $h = 1$, ordinates at $x = 0, 1, 2$: $y_0 = 4,\ y_1 = 3,\ y_2 = 0$ **(A1)**

$$\int_0^2 (4 - x^2)\,dx \approx \frac{1}{2}[4 + 2(3) + 0] = \frac{1}{2}(10) = 5$$ **M1 A1**

Q6 — Power Rule with Negative Exponent – EASY [3 marks

] $\int (5x^3 + 2x^{-2} - 3)\,dx$ **M1**

$= \dfrac{5x^4}{4} - \dfrac{2}{x} - 3x + C$ **A1 A1**

Note: Award A1 for $\frac{5x^4}{4} - \frac{2}{x} - 3x$; A1 for $+C$. Accept $-2x^{-1}$.

Q7 — GDC Definite Integral – EASY [3 marks

] **(a)** Using GDC: $\int_0^3 (x^3 - 3x^2 + 2x + 1)\,dx$ **(M1)**

$= 5.25$ **A1**

(b) Yes — the curve has no x-intercepts in $[0, 3]$ (the only real root is at $x \approx -0.32$, which lies outside this interval), so $f(x) > 0$ throughout and the integral equals the area. **R1**

Note: Award R1 for correct conclusion (yes) AND a valid reason (curve stays positive / no root in $[0, 3]$).

Q8 — Constant of Integration with Evaluation – EASY [3 marks

] Integrating: $y = 2x^2 - 3x + C$ **M1**

At $(0, 7)$: $7 = 0 - 0 + C \implies C = 7$ **A1**

At $x = 2$: $y = 2(4) - 3(2) + 7 = 8 - 6 + 7 = 9$ **A1**

Q9 — Trapezoidal Rule (3 strips) – EASY [3 marks

] $h = 2$, three strips **(A1)**

$$\int_1^7 f(x)\,dx \approx \frac{2}{2}[2.4 + 2(5.8 + 8.2) + 6.6]$$ **M1**

$= 1[2.4 + 28.0 + 6.6] = 37.0$ **A1**

Q10 — Indefinite then Definite Integral – EASY [3 marks

] **(a)** $\int (8x^3 - 6x)\,dx = 2x^4 - 3x^2 + C$ **M1 A1**

(b) $\int_0^1 (8x^3 - 6x)\,dx = \left[2x^4 - 3x^2\right]_0^1 = (2-3)-(0) = -1$ **A1**

Note: FT from their antiderivative in (a). No C required for definite integral.

Section B — Medium

Q11 — Trapezoidal Rule + Percentage Error – MEDIUM [4 marks

] **(a)** $h = 2$; five strips; ordinates: 3.0, 4.8, 6.5, 7.2, 6.0, 4.1 **(A1)**

$V \approx \frac{2}{2}\left[3.0 + 2(4.8+6.5+7.2+6.0)+4.1\right] = 1\left[3.0+48.0+4.1\right] = 55.1$ L **A1**

(b) GDC: $\int_0^{10} (3 + 0.9t - 0.07t^2)\,dt = \left[3t + 0.45t^2 - \frac{0.07}{3}t^3\right]_0^{10}$ **(M1)**

$= 30 + 45 - 23.3\overline{3} = 51.6\overline{6} \approx 51.7$ L

$\%$ error $= \dfrac{|55.1 - 51.67|}{51.67} \times 100 = 6.6\%$ **A1**

Note: Accept % error in range [6.5%, 6.7%] due to rounding. FT from (a).

Q12 — Constant of Integration + Stationary Points – MEDIUM [4 marks

] **(a)** $f(x) = \int (3x^2 - 4x + 1)\,dx = x^3 - 2x^2 + x + C$ **M1 A1**

At $(2, 10)$: $10 = 8 - 8 + 2 + C \implies C = 8$ **M1**

$f(x) = x^3 - 2x^2 + x + 8$ **A1**

(b) Set $f'(x) = 0$: $3x^2 - 4x + 1 = 0 \implies (3x-1)(x-1) = 0$

$x = \frac{1}{3}$ and $x = 1$ **A1**

Note: Award A1 for both values correct.

Q13 — Area Between Curves – MEDIUM [4 marks

] **(a)** $x = -2$ and $x = 3$ **A1**

(b) Area $= \int_{-2}^{3} \left[(x+4) - (x^2-2)\right] dx = \int_{-2}^{3} (6 + x - x^2)\,dx$ **M1**

$= \left[6x + \dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_{-2}^{3}$ **M1**

$= \left(18 + \frac{9}{2} - 9\right) - \left(-12 + 2 + \frac{8}{3}\right) = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{81}{6} + \frac{44}{6} = \dfrac{125}{6} \approx 20.8$ **A1**

Note: Award M1 for correct integrand (upper minus lower); M1 for correct antiderivative; A1 for $\frac{125}{6}$ *or* 20.8.

Q14 — Indefinite then Definite Integral (Negative Powers) – MEDIUM [4 marks

] **(a)** $\displaystyle\int (4x^3 - 6x^{-3} + 2)\,dx = x^4 + 3x^{-2} + 2x + C$ **M1 A1 A1**

Note: Award A1 for $x^4 + 2x$*; A1 for* $3x^{-2}$ *(or* $\frac{3}{x^2}$*). A1 for* $+C$.

(b) $\left[x^4 + \dfrac{3}{x^2} + 2x\right]_1^2 = \left(16 + \frac{3}{4} + 4\right) - (1 + 3 + 2) = 20.75 - 6 = 14.75$ **A1**

Note: FT from (a). Accept $\frac{59}{4}$.

Q15 — Constant of Integration in Kinematics – MEDIUM [4 marks

] **(a)** $v(t) = \displaystyle\int (4t - 6)\,dt = 2t^2 - 6t + C$ **M1**

At $t = 0, v = 2$: $2 = C$ **A1**

$v(t) = 2t^2 - 6t + 2$

(b) $v(4) = 2(16) - 24 + 2 = 32 - 24 + 2 = 10$ m s^{-1} **A1**

(c) $2t^2 - 6t + 2 = 0$ **(M1)**

$t = \dfrac{3 - \sqrt{5}}{2} \approx 0.382$ s **A1**

Note: Award A1 for $t \approx 0.382$ *s (accept* $\frac{3-\sqrt{5}}{2}$*). The second root* $t \approx 2.62$ *s is also a valid rest point.*

Q16 — Trapezoidal Rule + Over/Under-estimate – MEDIUM [4 marks

] **(a)** $h = 2$; ordinates: $y_0 = e^0 = 1$, $y_1 = e^{0.8} = 2.2255\ldots$, $y_2 = e^{1.6} = 4.9530\ldots$ **(A1)**

$\approx \dfrac{2}{2}[1 + 2(2.2255) + 4.9530] = 1(10.404) = 10.4$ **A1**

(b) GDC: $\displaystyle\int_0^4 e^{0.4x}\,dx = \left[\frac{e^{0.4x}}{0.4}\right]_0^4 = \frac{e^{1.6} - 1}{0.4} = \frac{4.9530 - 1}{0.4} = 9.883\ldots \approx 9.88$ **A1**

Since $10.4 > 9.88$, the trapezoidal rule gives an over-estimate.
$y = e^{0.4x}$ is concave up (convex), so the trapezoids lie above the curve. **R1**

Note: Award R1 for correct conclusion AND valid reason (concave upward / second derivative positive).

Q17 — GDC Area (Signed Regions) – MEDIUM [4 marks

] **(a)** $x = -2,\ 0,\ 2$ **A1**

(b) By symmetry, area of each region is equal.

$$\int_0^2 (x^3 - 4x)\,dx = \left[\frac{x^4}{4} - 2x^2\right]_0^2 = (4-8) - (0) = -4$$ **M1 (A1)**

Total area $= 2 \times |-4| = 8$ **A1**

Note: M1 for setting up integral; (A1) for antiderivative; A1 for total area = 8. FT if student computes both integrals separately and adds absolute values correctly.

Q18 — Trapezoidal Rule in Context – MEDIUM [4 marks

] **(a)** $h = 3$; four strips **(A1)**

$A \approx \frac{3}{2}[0 + 2(1.8 + 2.6 + 2.1) + 0] = \frac{3}{2}(13.0) = 19.5$ m^2 **A1**

(b) $\%$ error $= \frac{|19.5 - 19.4|}{19.4} \times 100 = \frac{0.1}{19.4} \times 100 = 0.515\ldots\% \approx 0.515\%$ **M1 A1**

Note: Award M1 for correct formula with 19.4 in denominator; A1 for answer in range [0.51%, 0.52%].

Q19 — Constant of Integration + GDC Zeros – MEDIUM [4 marks

] **(a)** $g(x) = \int (6x^2 - 10x + 4)\,dx = 2x^3 - 5x^2 + 4x + C$ **M1 A1**

$g(0) = -3 \implies C = -3$ **A1**

$g(x) = 2x^3 - 5x^2 + 4x - 3$

(b) Using GDC to solve $2x^3 - 5x^2 + 4x - 3 = 0$ **(M1)**

$x = 1.86$ (one real root; $x = 1.858\ldots$) **A1**

Note: Cubic has one real root at $x \approx 1.86$. Award A1 for $x = 1.86$ (accept 1.858–1.859).

Q20 — GDC Integral in Context – MEDIUM [4 marks

] **(a)** Using GDC: $\int_0^{24} (120 - 15\sin\frac{\pi t}{12})\,dt$ **(M1)**

$$= \left[120t + \frac{15 \cdot 12}{\pi}\cos\frac{\pi t}{12}\right]_0^{24}$$

$$= \left(2880 + \frac{180}{\pi}\right) - \left(0 + \frac{180}{\pi}\right) = 2880 \text{ W h}$$ **A1**

(b) The total energy lost from the building as heat over the 24-hour period is 2880 watt-hours. **R1**

(c) Average rate $= \dfrac{2880}{24} = 120$ W **A1**

Section C — Hard

Q21 — Area Between Curves with Parameter – HARD [5 marks

] **(a)** Setting $x^2 - kx = x$: $x^2 - (k+1)x = 0 \implies x[x-(k+1)] = 0$ **M1**

$\implies x = 0$ or $x = k+1$ **AG**

Note: Both roots must be clearly derived from factorisation for AG.

(b) $\displaystyle\int_0^{k+1} [x - (x^2 - kx)]\,dx = \int_0^{k+1} (x + kx - x^2)\,dx = \int_0^{k+1} [(k+1)x - x^2]\,dx$ **M1**

$= \left[\dfrac{(k+1)x^2}{2} - \dfrac{x^3}{3}\right]_0^{k+1}$ **A1**

$= \dfrac{(k+1)^3}{2} - \dfrac{(k+1)^3}{3} = (k+1)^3\left(\dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{(k+1)^3}{6}$ **M1 A1**

Q22 — Trapezoidal Rule + GDC Comparison – HARD [5 marks

] **(a)** $h = 0.5$; four strips; ordinates: 1.000, 0.800, 0.500, 0.308, 0.200 **(A1)**

$\approx \dfrac{0.5}{2}[1.000+2(0.800+0.500+0.308)+0.200] = 0.25[1.000+3.216+0.200] = 0.25(4.416) = 1.104$
A1

(b) GDC: $\displaystyle\int_0^2 \frac{1}{1+x^2}\,dx = \arctan(2) - \arctan(0) = \arctan(2) = 1.1071\ldots \approx 1.11$ **A1**

(c) % error $= \dfrac{|1.104 - 1.1071|}{1.1071} \times 100 = 0.278\ldots\% \approx 0.278\%$ **A1**

(d) Over-estimate. The function $f(x) = \dfrac{1}{1+x^2}$ is concave down (convex from above) on $[0, 2]$, so the trapezoids lie above the curve. **R1**

Note: Award R1 for correct conclusion AND valid concavity justification.

Q23 — Constant of Integration with Unknown Parameter – HARD [5 marks

] **(a)** $f(x) = \displaystyle\int (ax^2 + 4x)\,dx = \frac{a}{3}x^3 + 2x^2 + C$ **AG**

Note: Integration shown step-by-step; A0 if merely stated.

(b) $f'(3) = a(9) + 4(3) = 9a + 12 = 15 \implies 9a = 3 \implies a = \frac{1}{3}$ **M1 A1**

(c) With $a = \frac{1}{3}$: $f(x) = \dfrac{x^3}{9} + 2x^2 + C$

At $P(3, 28)$: $28 = \dfrac{27}{9} + 18 + C = 3 + 18 + C \implies C = 7$ **M1**

$f(x) = \dfrac{x^3}{9} + 2x^2 + 7$ **A1**

Note: Award M1 for substituting (3,28) with their a; A1 for correct equation.

Q24 — GDC Area + Volume of Revolution – HARD [5 marks

] **(a)** $\sqrt{x} = \dfrac{x}{2} \implies x = \dfrac{x^2}{4} \implies 4x = x^2 \implies x(x-4) = 0$

$x = 0$ and $x = 4$ **A1**

(b) Area $= \displaystyle\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \left[\frac{2x^{3/2}}{3} - \frac{x^2}{4}\right]_0^4$ **M1**

$= \left(\dfrac{16}{3} - 4\right) - 0 = \dfrac{4}{3} \approx 1.33$ **A1**

(c) $V = \pi \displaystyle\int_0^4 \left[(\sqrt{x})^2 - \left(\frac{x}{2}\right)^2\right] dx = \pi \int_0^4 \left(x - \frac{x^2}{4}\right) dx$ **M1**

$= \pi \left[\dfrac{x^2}{2} - \dfrac{x^3}{12}\right]_0^4 = \pi\left(8 - \dfrac{64}{12}\right) = \pi\left(8 - \dfrac{16}{3}\right) = \dfrac{8\pi}{3}$ **A1**

Note: Award M1 for setting up $\pi \int [(f)^2 - (g)^2]$ with correct functions; A1 for $\frac{8\pi}{3}$.

Q25 — Parameter Curve + Area – HARD [5 marks

] **(a)** $f(1) = 1 - 6 + 9 + c = 4 + c = 6 \implies c = 2$ **A1**

So $f(x) = x^3 - 6x^2 + 9x + 2$

(b) $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$ **M1**

Local maximum at $x = 1$; local minimum at $x = 3$ **A1**

(c) $f(1) = 6$, $f(3) = 27 - 54 + 27 + 2 = 2$; curve between $x = 1$ and $x = 3$ lies between these values.

Using GDC to evaluate area enclosed between curve and x-axis in $[1, 3]$: **M1**

$$\int_1^3 (x^3 - 6x^2 + 9x + 2)\, dx = \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} + 2x\right]_1^3$$

$= \left(\dfrac{81}{4} - 54 + \dfrac{81}{2} + 6\right) - \left(\dfrac{1}{4} - 2 + \dfrac{9}{2} + 2\right) = (20.25 - 54 + 40.5 + 6) - (0.25 - 2 + 4.5 + 2)$

$= 12.75 - 4.75 = 8.00$ **A1**

Note: Since $f(x) > 0$ on $[1, 3]$, the definite integral equals the area. Award M1 for GDC/analytical setup;

A1 for 8 (exact).

Q26 — Trapezoidal Rule + Concavity Justification – HARD [5 marks

] **(a)** Minimum of $r(t) = 2t^2 - 8t + 12$: vertex at $t = 2$, $r(2) = 8 - 16 + 12 = 4 > 0$ **M1**

Since the minimum value is $4 > 0$, $r(t) > 0$ for all $t \in [0, 5]$. **A1**

(b) $h = 1$ (five strips over $[0, 5]$) **A1**

Ordinates at $t = 0, 1, 2, 3, 4, 5$: $r = 12, 6, 4, 6, 12, 22$

Estimate $\approx \frac{1}{2}[12 + 2(6 + 4 + 6 + 12) + 22] = \frac{1}{2}[12 + 56 + 22] = \frac{1}{2}(90) = 45$ thousand bacteria **A1**

(c) GDC: $\int_0^5 (2t^2 - 8t + 12)\,dt = \left[\frac{2t^3}{3} - 4t^2 + 12t\right]_0^5 = \frac{250}{3} - 100 + 60 = \frac{250}{3} - 40 = \frac{130}{3} \approx 43.3$

$45 > 43.3$, so the estimate is an over-estimate.
$r''(t) = 4 > 0$: the function is concave up, so trapezoids lie above the curve. **R1**

Q27 — Integral with Unknown Upper Limit – HARD [5 marks

] **(a)** $\int_0^k (3x^2 - 2x)\,dx = \left[x^3 - x^2\right]_0^k = k^3 - k^2$ **M1 A1**

(b) Setting $k^3 - k^2 = 10$: $k^3 - k^2 - 10 = 0$ **M1**

Using GDC to solve: **(M1)**

$k = 2.54\ldots \approx 2.54$ **A1**

Verification: $2.5445^3 - 2.5445^2 \approx 16.48 - 6.47 = 10.01$✓

Note: Award M1 for correct equation $k^3 - k^2 - 10 = 0$; (M1) for GDC method; A1 for $k = 2.54$ (accept 2.544 to 2.545).

Q28 — Kinematics – Displacement and Total Distance – HARD [5 marks

] **(a)** $s(t) = \int (6 - t^2)\,dt = 6t - \frac{t^3}{3} + C$ **M1**

At $t = 0$, $s = 0$: $C = 0$; so $s(t) = 6t - \frac{t^3}{3}$ **A1**

(b) $v(t) = 0$: $6 - t^2 = 0 \implies t = \sqrt{6} \approx 2.45$ s **A1**

(c) For $0 \le t \le \sqrt{6}$: $v > 0$; for $\sqrt{6} < t \le 3$: $v < 0$ **(M1)**

$$d = \int_0^{\sqrt{6}} (6 - t^2)\,dt + \left|\int_{\sqrt{6}}^3 (6 - t^2)\,dt\right|$$

$= \left[6t - \frac{t^3}{3}\right]_0^{\sqrt{6}} + \left|\left[6t - \frac{t^3}{3}\right]_{\sqrt{6}}^3\right|$

$= \left(6\sqrt{6} - \frac{6\sqrt{6}}{3}\right) + |(18-9) - (6\sqrt{6} - 2\sqrt{6})|$

$= 4\sqrt{6} + |9 - 4\sqrt{6}| = 4\sqrt{6} + |9 - 9.798\ldots| = 4\sqrt{6} + 0.798\ldots$

$= 9.798\ldots + 0.798\ldots = 10.6$ m **A1**

Note: Award M1 for recognising the need to split at $t = \sqrt{6}$*; A1 for* 10.6 *m (accept* $10.5 - 10.6$*).*

Q29 — Aerofoil Area – Trapezoidal + GDC – HARD [5 marks

] **(a)** At $x = \pm 2$: upper $y = \sqrt{4-4} = 0$; lower $y = \frac{4}{4} - 1 = 0$. Both curves meet at $(\pm 2, 0)$. **A1**

Note: AG — both intersection values clearly shown.

(b) $h = 1$; four strips over $[-2, 2]$; $g(x) = \sqrt{4-x^2} - (\frac{x^2}{4} - 1) = \sqrt{4-x^2} - \frac{x^2}{4} + 1$

Ordinates of g: $g(-2) = 0,\ g(-1) = \sqrt{3} - \frac{1}{4} + 1 = 2.482,\ g(0) = 2 + 1 = 3,\ g(1) = 2.482,\ g(2) = 0$
(A1)

Area $\approx \frac{1}{2}[0 + 2(2.482 + 3 + 2.482) + 0] = \frac{1}{2}(2 \times 7.964) = 7.964 \approx 7.96$ **A1**

(c) Exact area $= \int_{-2}^{2} \left(\sqrt{4-x^2} - \frac{x^2}{4} + 1\right) dx$

$=$ [(area of semicircle radius 2)] $- \left[\frac{x^3}{12} - x\right]_{-2}^{2}$

$= 2\pi - \left[\left(\frac{8}{12} - 2\right) - \left(-\frac{8}{12} + 2\right)\right] = 2\pi - [(-\frac{4}{3}) - (\frac{4}{3})] = 2\pi + \frac{8}{3}$

$= 2\pi + \frac{8}{3} \approx 6.283 + 2.667 = 8.950 \approx 8.95$ **M1 A1**

Note: Award M1 for GDC evaluation or recognising semicircle; A1 for $2\pi + \frac{8}{3}$ *or 8.95.*

Q30 — Parameter Cubic – d-value / Extrema / Area – HARD [5 marks]

(a) $f(4) = 64 - 48 - 36 + d = d - 20 = 10 \implies d = 30$ **A1**

So $y = x^3 - 3x^2 - 9x + 30$; line $y = d - 15 = 15$

(b) $y' = 3x^2 - 6x - 9 = 3(x-3)(x+1) = 0$ **M1**

Local max at $x = -1$: $y = (-1) - 3 - (-9)(-..)$, $y = -1 - 3 + 9 + 30 = 35$

At $x = -1$: $y = (-1)^3 - 3(-1)^2 - 9(-1) + 30 = -1 - 3 + 9 + 30 = 35$

Local min at $x = 3$: $y = 27 - 27 - 27 + 30 = 3$ **A1**

(c) Set $y = x^3 - 3x^2 - 9x + 30 = 15$: $x^3 - 3x^2 - 9x + 15 = 0$ **M1**

GDC roots: $x \approx -2.62,\ 1.34,\ 4.28$

Local min value $y = 3 < 15$, confirming curve lies below $y = 15$ in interval $[1.34,\ 4.28]$

Area $= \displaystyle\int_{1.337}^{4.284} [15 - (x^3 - 3x^2 - 9x + 30)]\,dx$ **(M1)**

$= \displaystyle\int_{1.337}^{4.284} (-x^3 + 3x^2 + 9x - 15)\,dx \approx 23.2$ **A1**

Note: Award M1 for setting up $\int[15 - f(x)]\,dx$ over the correct interval; A1 for 23.2 (accept 23.1–23.2).

Common Mistakes to Avoid

1. **Trapezoidal rule — counting ordinates vs strips.** With n strips there are $n+1$ ordinates $(y_0, y_1, \dots, y_n)$. Check: inner ordinates are doubled; first and last are not.
2. **Missing constant of integration.** Any indefinite integral must include $+C$. Always use a given point to find C before evaluating the function elsewhere.
3. **Negative areas.** $\displaystyle\int_a^b f(x)\,dx$ gives a *signed* value. For total area, split at x-intercepts and add absolute values of each piece.
4. **Over/under-estimate.** A concave up (convex) curve $\Rightarrow$ trapezoidal rule over-estimates. A concave down curve $\Rightarrow$ under-estimates. Use $f''(x) > 0$ or < 0 as the test.
5. **Power rule for $n = -1$.** The rule $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ requires $n \neq -1$. Attempting to use it for $\frac{1}{x}$ is a common error.
6. **Definite integral of area between curves.** Always subtract *lower* function from *upper* function. Reversing the order gives a negative result.