# IB Math AI HL — Topic 2

## Modelling with Functions

Linear · Quadratic · Cubic · Exponential · Direct & Inverse Variation · Sinusoidal · Strategy

| Difficulty | EASY | MEDIUM | HARD |
|---|---|---|---|
| **Questions** | 10 | 10 | 10 |
| **Marks each** | 3 | 4 | 5 |
| **Total Marks** | 30 | 40 | 50 |

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Name:____________________________ Date:______________

Score:________/120

**Instructions:** Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. A GDC is permitted throughout.

### Key Formulae — Modelling with Functions

**Linear:** $f(x) = mx + c$

**Quadratic:** $f(x) = ax^2 + bx + c$; vertex at $x = -\frac{b}{2a}$

**Cubic:** $f(x) = ax^3 + bx^2 + cx + d$

**Exponential growth:** $P(t) = P_0\, e^{rt}$ or $P_0\, b^t$

**Exponential decay:** $A(t) = A_0\left(\frac{1}{2}\right)^{t/T}$

**Direct variation:** $y = kx^n$

**Inverse variation:** $y = \frac{k}{x^n}$

**Sinusoidal:** $f(t) = a\sin(bt + c) + d$

**Sinusoidal parameters:** amplitude $= |a|$; period $= \frac{2\pi}{b}$; midline $= d$; phase shift $= -\frac{c}{b}$

### GDC Tips

- **Solve equations:** use `Solve()` or graph intersection (`G-Solv` → `ISCT`) to find where two models are equal.
- **Regression:** `STAT` → `CALC` → choose model type (`LinReg`, `QuadReg`, `ExpReg`, `SinReg`).
- **Maximum/minimum:** `G-Solv` → `MAX` or `MIN`, or use `fMax()`/`fMin()` with domain.
- **Solving sinusoidal inequalities:** graph both sides and read the $x$-interval where one curve lies above the other.
- **Make sure** your GDC is in the correct angle mode (radians for calculus/sinusoidal models, degrees for geometry).

## Common Mistakes to Avoid

- **Period vs frequency:** period $= 2\pi/b$ — do *not* confuse $b$ with the period itself.
- **Exponential base:** $P = P_0(1+r)^t$ uses growth *factor* $(1+r)$, not the rate $r$.
- **Vertex form:** axis of symmetry is $x = -b/(2a)$, NOT $x = b/(2a)$.
- **Inverse function:** swap $x$ and $y$ *before* rearranging — a common order error.
- **Direct vs inverse:** $y \propto x^n$ (direct) gives larger $y$ for larger $x$; $y \propto 1/x^n$ (inverse) gives smaller $y$ for larger $x$.
- **Model limitations:** always check whether the domain makes physical sense (e.g. exponential growth cannot continue indefinitely).

## Section A — Easy EASY (Questions 1–10, 3 marks each)

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**1.** EASY Calculator **[3 marks]**

A courier company charges a delivery fee $C$ (in dollars) that can be modelled by a linear function of the package weight $w$ (in kg). When $w = 1$, the fee is \$14; when $w = 4$, the fee is \$23.

(a) Find the equation of $C$ in terms of $w$. [2]

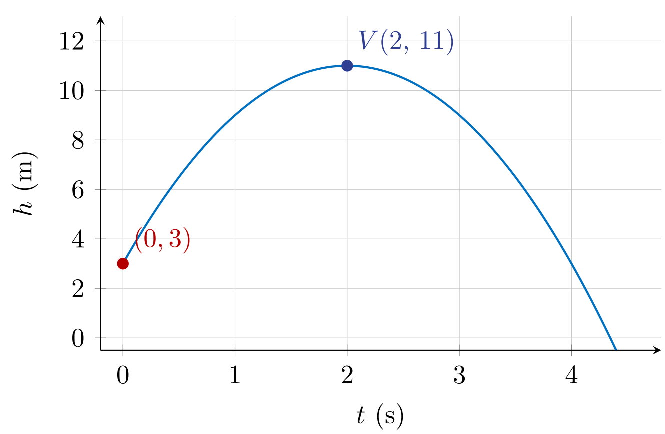
(b) Find the fee for a package weighing 7 kg. [1]

**2.** EASY Calculator **[3 marks]**

A ball is thrown upward. Its height $h$ metres above the ground at time $t$ seconds is modelled by

$$h(t) = -2t^2 + 8t + 3, \quad 0 \le t \le 4.5.$$

The graph of $h$ is shown below.




(a) Write down the coordinates of the vertex of the graph. [1]

(b) Write down the initial height of the ball (when $t = 0$). [1]

(c) Find the maximum height reached by the ball. [1]

**3.** **EASY** **Calculator** **[3 marks]**

The number of visitors $V$ to a nature reserve varies inversely with the average daily temperature $T\,°\text{C}$. When $T = 4$, there are $V = 150$ visitors.

(a) Find the constant of variation. [1]

(b) Write down the model for $V$ in terms of $T$. [1]

(c) Find the number of visitors when $T = 10$. [1]

**4.** **EASY** **Calculator** **[3 marks]**

The braking distance $d$ metres of a car varies directly with the square of its speed $v$ km h$^{-1}$. When $v = 30$, the braking distance is $d = 45$ m.

(a) Find the constant of proportionality $k$. [1]

(b) Find the braking distance when the car is travelling at 50 km h$^{-1}$. [1]

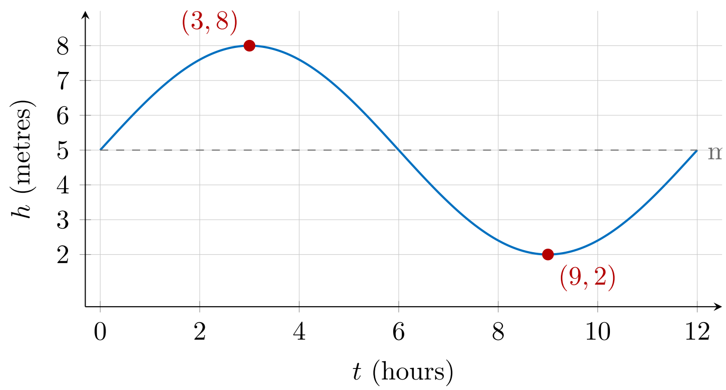
(c) Find the speed at which the braking distance is 80 m. [1]

**5.** **EASY** **Calculator** **[3 marks]**

The water depth $h$ metres in a harbour is modelled by

$$h(t) = 3\sin\left(\frac{\pi t}{6}\right) + 5, \quad t \geq 0,$$

where $t$ is the time in hours after midnight. The graph of $h$ for one complete cycle is shown below.




(a) Write down the amplitude of $h$. [1]

(b) Write down the period of $h$. [1]

(c) Write down the midline depth of the harbour. [1]

**6.** **EASY** **Calculator** **[3 marks]**

The number of bacteria $P$ in a culture is modelled by $P(t) = 200\,e^{0.05t}$, where $t$ is the time in hours.

(a) Find the number of bacteria after 10 hours. Give your answer to the nearest whole number. [1]

(b) Find the time at which there are 500 bacteria. Give your answer correct to 3 significant figures. [2]

**7.** **EASY** **Calculator** **[3 marks]**

A taxi company charges a fare $F$ (in AED) modelled by

$$F(n) = 120 + 7.5n,$$

where $n$ is the number of kilometres travelled, for $n \geq 0$.

(a) Write down the fixed charge (the fare when $n = 0$). [1]

(b) Interpret the meaning of the value 7.5 in the context of this model. [1]

(c) Find the number of kilometres travelled when the fare is AED 270. [1]

**8.** **EASY** **Calculator** **[3 marks]**

A rectangular garden has area $A$ m$^2$ modelled by $A(x) = -x^2 + 6x - 5$, where $x$ metres is one side length $(1 \leq x \leq 5)$.

(a) Write down the values of $x$ for which $A(x) = 0$. [1]

(b) Find the maximum area and the value of $x$ at which it occurs. [2]

**9.** **EASY** **Calculator** **[3 marks]**

A radioactive substance decays according to the model

$$A(t) = 800\left(\frac{1}{2}\right)^{t/30},$$

where $A$ is the mass in grams and $t$ is the time in years.

(a) Write down the initial mass of the substance. [1]

(b) Find the mass after 60 years. [1]

(c) Find the time at which the mass is 100 grams. [1]

**10.** **EASY** **Calculator** **[3 marks]**

The height $h$ metres of a seat on a Ferris wheel is modelled by

$$h(t) = 4\sin\left(\frac{\pi t}{4}\right) + 7,$$

where $t$ is the time in minutes.

(a) Write down the maximum height of the seat. [1]

(b) Write down the minimum height of the seat. [1]

(c) Find the height of the seat when $t = 2$ minutes. [1]

## Section B — Medium MEDIUM (Questions 11–20, 4 marks each)

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**11.** MEDIUM Calculator **[4 marks]**

A plumber charges a call-out fee plus a fixed hourly rate. Her total charge $C$ (in \$) is a linear function of the number of hours worked $h$. When $h = 2$, she charges \$190; when $h = 7$, she charges \$390.

(a) Find the values of the gradient and the $C$-intercept of the model. [2]

(b) Write down the equation for $C$. [1]

(c) Interpret the gradient in the context of this model. [1]

**12.** MEDIUM Calculator [4 marks]

A parabolic arch has height $h(x)$ metres above the ground at horizontal distance $x$ metres from one end. The arch is symmetric with vertex at $x = 3$ and $h(3) = 23$. The arch meets the ground at $x = 0$ where $h(0) = 5$.

(a) Show that $h(x) = -2x^2 + 12x + 5$. [2]

(b) Find the maximum height of the arch. [1]

(c) Find the value of $x$ at the other end of the arch where $h = 5$, giving your answer correct to 3 significant figures. [1]

**13.** MEDIUM Calculator [4 marks]

The volume $V$ cm$^3$ of a type of spherical fruit varies directly with the cube of its radius $r$ cm. A fruit with radius 3 cm has volume 270 cm$^3$.

(a) Find the constant of proportionality $k$. [1]

(b) Find the volume of a fruit with radius 5 cm. [1]

(c) A second fruit has radius 6 cm (twice the radius of the first). Without calculating, state the ratio of the volume of the second fruit to the volume of the first. Justify your answer. [2]

**14.** MEDIUM Calculator [4 marks]

The fuel efficiency $f$ (litres per 100 km) of a vehicle is modelled by

$$f(v) = \frac{60}{v+2}, \quad v \geq 0,$$

where $v$ is the speed in $\mathrm{km\,h^{-1}}$.

(a) Write down the horizontal asymptote of $f$. State what it represents. [1]

(b) Find $f(0)$ and interpret its value. [1]

(c) Find the inverse function $f^{-1}(v)$. [1]

(d) Use $f^{-1}$ to find the speed at which fuel efficiency is 12 litres per 100 km. [1]

**15.** MEDIUM Calculator [4 marks]

The daily temperature $T\,°\mathrm{C}$ in a coastal city is modelled by

$$T(t) = 8\sin\left(\frac{\pi t}{6}\right) + 20, \quad 0 \leq t \leq 12,$$

where $t$ is the time in months after January.

(a) Write down the maximum temperature predicted by the model. [1]

(b) Find the values of $t$ for which $T(t) = 24$. [2]

(c) Hence state the number of months during which $T > 24$. [1]

**16.** MEDIUM Calculator **[4 marks]**

The table below gives the depth $D$ cm of water in a tank at time $t$ minutes after filling began.

| $t$ **(min)** | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $D$ **(cm)** | 2 | 5 | 14 | 35 |

The depth is modelled by a cubic function $D(t) = t^3 + pt + q$.

(a) Use the table to find the values of $p$ and $q$. [2]

(b) Write down the model for $D(t)$. [1]

(c) Use your GDC to find the time at which $D = 80$ cm. Give your answer correct to 3 significant figures. [1]

**17.** MEDIUM Calculator **[4 marks]**

The maximum daily temperature in a desert town reaches 32 °C at the end of month 2 and falls to a minimum of 8 °C at the end of month 14. The temperature can be modelled by a sinusoidal function.

(a) Write down the amplitude and midline of the model. [2]

(b) Find the period of the model. [1]

(c) Write down the equation of the model in the form $T(t) = a\cos(b(t-c)) + d$, where $a, b, c, d$ are values you have found. [1]

**18.** MEDIUM Calculator **[4 marks]**

A city's population was 3200 at the start of the year $t = 0$ and has grown by 6% each year.

(a) Write down a model for the population $P(t)$ after $t$ years. [1]

(b) Find the population after 10 years. Give your answer to the nearest whole number. [1]

(c) Find the number of years it takes for the population to reach 8000. Give your answer correct to 3 significant figures. [2]

**19.** MEDIUM Calculator **[4 marks]**

A car park charges a parking fee $C$ (in \$) according to the following piecewise model:

$$C(t) = \begin{cases} 2t, & 0 \le t \le 3 \\ 6 + 1.5(t-3), & t > 3 \end{cases}$$

where $t$ is the number of hours parked.

(a) Show that $C$ is continuous at $t = 3$. [1]

(b) Find the fee for parking for 5 hours. [1]

(c) Find the number of hours parked when the fee is \$12. [2]

**20.** MEDIUM Calculator **[4 marks]**

The table below shows the number of downloads $N$ (in thousands) of a song at the end of each week $w$.

| **Week** $w$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **Downloads** $N$ | 2 | 8 | 18 | 32 | 50 |

Two models are proposed: a quadratic model $N = aw^2$ and an exponential model $N = c \cdot r^w$.

(a) Show that the quadratic model $N = 2w^2$ fits all five data points exactly. [1]

(b) Find the second differences of the $N$ values and state what they tell you about the model type. [2]

(c) Explain why the quadratic model is preferable to an exponential model for these data. [1]

## Section C — Hard HARD (Questions 21–30, 5 marks each)

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**21.** HARD Calculator **[5 marks]**

A market stall sells hand-crafted items. The revenue (in \$) when selling $x$ items is modelled by $R(x) = -x^2 + 32x$ and the cost of producing $x$ items is $C(x) = 50 + 12x$.

(a) Find the values of $x$ at which revenue equals cost (the break-even points). Give your answers in exact form. [2]

(b) Find the profit function $P(x) = R(x) - C(x)$. [1]

(c) Find the maximum profit and the number of items that should be sold to achieve it. [2]

**22.** HARD Calculator **[5 marks]**

A spring oscillates so that its displacement $h$ cm from rest is modelled by $h(t) = a\sin(bt) + 5$, where $a$ and $b$ are positive constants and $t$ is in seconds.

It is known that $h\left(\frac{\pi}{2}\right) = 9$ and $h(\pi) = 5$.

(a) Show that $b = 1$. [2]

(b) Hence find the value of $a$. [1]

(c) Write down the period of the oscillation. [1]

(d) State one limitation of this model. [1]

**23.** HARD Calculator **[5 marks]**

A colony of insects grows according to the model $P(t) = P_0 \cdot b^t$, where $P$ is the population and $t$ is the time in days. The colony has 36 insects on day 2 and 972 insects on day 5.

(a) Show that $b = 3$. [2]

(b) Find the value of $P_0$. [1]

(c) Find the day on which the colony first reaches 2916 insects. [1]

(d) State one assumption that must hold for this model to be valid. [1]

**24.** **HARD** **Calculator** **[5 marks]**

The temperature $D\,^\circ$C inside a chemical reactor is modelled by

$$D(t) = t^3 - 4t^2 + pt + q,$$

where $t$ is time in minutes and $p, q$ are constants. The table below gives observed values.

| $t$ **(min)** | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $D$ **(°C)** | 2 | 5 | 6 | 11 |

(a) Use the table to write down two equations in $p$ and $q$. [2]

(b) Solve the system to find $p$ and $q$. [2]

(c) Find the value of $D$ at $t = 4$. [1]

**25.** **HARD** **Calculator** **[5 marks]**

The depth $d$ metres of water in a bay is modelled by

$$d(t) = 5\sin\left(\frac{\pi t}{4}\right) + 8, \quad 0 \le t \le 16,$$

where $t$ is time in hours after noon.

(a) Find the maximum depth and the first time it occurs. [2]

(b) Solve the inequality $d(t) \ge 10.5$ over $0 \le t \le 16$, giving exact values. [2]

(c) Hence find the total number of hours during which $d(t) \ge 10.5$. [1]

**26.** HARD Calculator **[5 marks]**

The intensity $I$ (lux) of light from a source varies inversely with the square of the distance $r$ metres from the source. When $r = 2\,\text{m}$, $I = 25$ lux.

(a) Find the model for $I$ in terms of $r$. [2]

(b) Find the intensity at $r = 5\,\text{m}$. [1]

(c) A photographer needs $I \geq 1$ lux. Find the maximum distance from the source at which this is achieved. [1]

(d) Find the percentage change in intensity if $r$ increases by 50% from its initial value of 2 m. [1]

**27.** HARD Calculator **[5 marks]**

The table below shows the distance $s$ (metres) a remote-controlled vehicle travels in $t$ seconds.

| $t$ (s) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $s$ (m) | 0 | 2 | 8 | 18 | 32 | 50 |

(a) Calculate the first and second differences of $s$. [1]

(b) State, with a reason, the most appropriate type of model for these data. [1]

(c) Find the equation of the model. [1]

(d) Use the model to predict $s$ when $t = 8$. Comment on the reliability of this prediction. [2]

**28.** HARD Calculator **[5 marks]**

Two pricing strategies for a product give profit $P_A$ and $P_B$ (in \$) per day as functions of the quantity sold $x$:

$$P_A(x) = -2x^2 + (k+10)x - 12, \qquad P_B(x) = 3x - 2.$$

Strategy A has a maximum daily profit of \$20.

(a) Find the value of $k$. [2]

(b) Find the value of $x$ at which $P_A$ is maximised. [1]

(c) Find the values of $x$ at which $P_A(x) = P_B(x)$, giving your answers correct to 3 significant figures. [2]

**29.** HARD Calculator **[5 marks]**

Four data points are collected to fit a linear model $y = mx + c$:

$$(2,\,5), \quad (4,\,9), \quad (6,\,p), \quad (8,\,17).$$

(a) Show that the mean of the $x$-values is 5 and find the mean of the $y$-values in terms of $p$. [1]

(b) Show that the gradient of the regression line is $m = \dfrac{27+p}{20}$. [2]

(c) Find the value of $p$ for which the regression gradient is exactly 3. [1]

(d) Write down the equation of the regression line when $p$ takes this value. [1]

**30.** HARD Calculator **[5 marks]**

A company models its weekly revenue (in $ 000) by $R(t) = -2t^2 + (k+10)t - 12$ and its weekly costs by $C(t) = 3t - 2$, where $t$ is weeks after launch and $k$ is a positive constant.

It is found that the *maximum* revenue equals $20 000.

(a) Find the value of $k$. [2]

(b) Find the week in which revenue is maximised. [1]

(c) Find the values of $t$ for which revenue exceeds cost, giving your answers correct to 3 significant figures. [2]

## Worked Solutions

**Q1 — Linear Model** **EASY** [3 marks

] **(a)** **M1** Setting up simultaneous equations: $14 = m(1) + c$ and $23 = m(4) + c$.
Subtracting: $3m = 9 \Rightarrow m = 3$. **A1**
$c = 14 - 3 = 11$. $C(w) = 3w + 11$. **A1**

**(b)** $C(7) = 3(7) + 11 = 21 + 11 = 32$ **A1**

*Note: Award M1A0A0 if both equations set up correctly but arithmetic error.*

**Q2 — Quadratic Model** **EASY** [3 marks

] **(a)** Vertex: $(2, 11)$ **A1**

**(b)** Initial height $= h(0) = -2(0)^2 + 8(0) + 3 = 3$ m **A1**

**(c)** Maximum height $= y$-coordinate of vertex $= 11$ m **A1**

*Note: For (c), award A1 FT from their vertex in (a). Do not accept 11 m if vertex stated incorrectly.*

**Q3 — Inverse Variation** **EASY** [3 marks

] **(a)** $V = \dfrac{k}{T}$; when $T = 4$, $V = 150$: $k = 150 \times 4 = 600$ **A1**

**(b)** $V = \dfrac{600}{T}$ **A1**

**(c)** $V = \dfrac{600}{10} = 60$ visitors **A1**

*Note: Award A1 FT in (c) from their $k$ in (a).*

**Q4 — Direct Variation** **EASY** [3 marks

] **(a)** $d = kv^2$; when $v = 30$, $d = 45$: $k = \dfrac{45}{900} = 0.05$ **A1**

**(b)** $d = 0.05 \times 50^2 = 0.05 \times 2500 = 125$ m **A1**

**(c)** $80 = 0.05v^2 \Rightarrow v^2 = 1600 \Rightarrow v = 40\ \text{km h}^{-1}$ **A1**

*Note: In (c), reject $v = -40$ as speed must be positive. FT from their $k$.*

**Q5 — Sinusoidal Model EASY [3 marks**

] **(a)** Amplitude $= 3$ (m) **A1**

**(b)** Period $= \dfrac{2\pi}{\pi/6} = 12$ hours **A1**

**(c)** Midline $= 5$ m **A1**

**Q6 — Exponential Growth EASY [3 marks**

] **(a)** $P(10) = 200\,e^{0.05\times 10} = 200\,e^{0.5} = 200 \times 1.6487\ldots \approx 330$ bacteria **A1**

**(b) M1** $200\,e^{0.05t} = 500 \Rightarrow e^{0.05t} = 2.5$
$0.05t = \ln(2.5) = 0.9162\ldots$
$t = \dfrac{\ln 2.5}{0.05} = 18.3258\ldots \approx 18.3$ hours **A1**

*Note: (a) accept 329 or 330. For (b) award M1 for correct equation formed; A1 for 18.3.*

**Q7 — Linear Cost Model EASY [3 marks**

] **(a)** Fixed charge $= F(0) = 120$ AED **A1**

**(b)** The fare increases by 7.5 AED for each additional kilometre travelled. **R1**

**(c) M1** $120 + 7.5n = 270 \Rightarrow 7.5n = 150 \Rightarrow n = 20$ km **A1**

**Q8 — Quadratic Zeros EASY [3 marks**

] **(a)** $-x^2 + 6x - 5 = 0 \Rightarrow x^2 - 6x + 5 = 0 \Rightarrow (x-1)(x-5) = 0$
$x = 1$ and $x = 5$ **A1**

**(b)** Axis of symmetry: $x = -\dfrac{6}{2(-1)} = 3.$ **M1**

$A(3) = -(9) + 18 - 5 = 4$ m$^2$. Maximum area is $\mathbf{4\,m^2}$ at $x = 3$. **A1**

**Q9 — Exponential Decay EASY [3 marks**

] **(a)** Initial mass $= A(0) = 800$ g **A1**

**(b)** $A(60) = 800\left(\dfrac{1}{2}\right)^{60/30} = 800 \times \left(\dfrac{1}{4}\right) = 200$ g **A1**

**(c)** $800\left(\dfrac{1}{2}\right)^{t/30} = 100 \Rightarrow \left(\dfrac{1}{2}\right)^{t/30} = \dfrac{1}{8} = \left(\dfrac{1}{2}\right)^{3} \Rightarrow \dfrac{t}{30} = 3 \Rightarrow t = 90$ years **A1**

**Q10 — Sinusoidal Ferris Wheel** EASY [3 marks

] **(a)** Maximum height $= 4 + 7 = 11$ m **A1**

**(b)** Minimum height $= -4 + 7 = 3$ m **A1**

**(c)** $h(2) = 4\sin\left(\frac{\pi \times 2}{4}\right) + 7 = 4\sin\left(\frac{\pi}{2}\right) + 7 = 4(1) + 7 = 11$ m **A1**

**Q11 — Linear Plumber Model** MEDIUM [4 marks

] **(a)** **M1** Gradient $m = \frac{390 - 190}{7 - 2} = \frac{200}{5} = 40$. **A1**

$C$-intercept: $190 = 40(2) + c \Rightarrow c = 110$. **A1**

**(b)** $C(h) = 40h + 110$ **A1**

**(c)** The gradient 40 represents the hourly rate: the plumber charges \$40 for each additional hour worked. **R1**

*Note: Award R1 for a contextually correct interpretation referencing \$40 per hour.*

**Q12 — Quadratic Arch** MEDIUM [4 marks

] **(a)** Using vertex form: $h(x) = a(x - 3)^2 + 23$.
**M1** Substituting $h(0) = 5$: $a(9) + 23 = 5 \Rightarrow 9a = -18 \Rightarrow a = -2$.
$h(x) = -2(x - 3)^2 + 23 = -2(x^2 - 6x + 9) + 23 = -2x^2 + 12x - 18 + 23$
$h(x) = -2x^2 + 12x + 5$ **AG**

**(b)** Maximum height is the $y$-value of the vertex $= 23$ m. **A1**

**(c)** $-2x^2 + 12x + 5 = 5 \Rightarrow -2x^2 + 12x = 0 \Rightarrow -2x(x - 6) = 0$
$x = 0$ or $x = 6$. The other end is at $x = 6.00$ m. **A1**

*Note: A1 in (c) for $x = 6$ or $x = 6.00$; award full marks for either exact or decimal. Do not accept $x = 0$ as "the other end".*

**Q13 — Direct Variation with Cube** MEDIUM [4 marks

] **(a)** $V = kr^3$; $270 = k(27) \Rightarrow k = 10$ **A1**

**(b)** $V = 10(5)^3 = 10 \times 125 = 1250$ cm$^3$ **A1**

**(c)** **M1** Since $V \propto r^3$, doubling $r$ multiplies $V$ by $2^3 = 8$.
Ratio $= 8 : 1$. **R1A1**

*Note: Award M1R1 for the power-of-2 argument; A1 for stating the ratio is 8 (or 8:1).*

### Q14 — Rational/Inverse Variation Model MEDIUM [4 marks

] **(a)** Horizontal asymptote: $y = 0$. It represents the fuel efficiency approaching zero as speed increases without bound. **A1**

**(b)** $f(0) = \dfrac{60}{0+2} = 30$ litres per 100 km. This is the fuel consumption when the vehicle is stationary (or at very low speed); the model has limited physical meaning here. **A1**

**(c)** Let $v = \dfrac{60}{f+2}$; swap: $f = \dfrac{60}{v+2} \Rightarrow v + 2 = \dfrac{60}{f} \Rightarrow v = \dfrac{60}{f} - 2$.

$f^{-1}(v) = \dfrac{60}{v} - 2$ **A1**

**(d)** $f^{-1}(12) = \dfrac{60}{12} - 2 = 5 - 2 = 3 \text{ km h}^{-1}$ **A1**

### Q15 — Sinusoidal Inequality MEDIUM [4 marks

] **(a)** Maximum $T = 8 + 20 = 28\,°\text{C}$ **A1**

**(b)** **M1** $8\sin(\pi t/6) + 20 = 24 \Rightarrow \sin(\pi t/6) = 0.5$
$\dfrac{\pi t}{6} = \dfrac{\pi}{6} \Rightarrow t = 1, \quad \dfrac{\pi t}{6} = \dfrac{5\pi}{6} \Rightarrow t = 5$ **A1A1**

**(c)** $T > 24$ for $1 < t < 5$, a duration of **4** months. **A1**

*Note: For (b) award A1A1 for both $t = 1$ and $t = 5$; award A1A0 for one correct value.*

### Q16 — Cubic Model from Table MEDIUM [4 marks

] **(a)** $D(t) = t^3 + pt + q$.
$D(0) = q = 2$. **A1**
$D(1) = 1 + p + q = 5 \Rightarrow 1 + p + 2 = 5 \Rightarrow p = 2$. **A1**

**Verification:** $D(2) = 8 + 4 + 2 = 14\checkmark \quad D(3) = 27 + 6 + 2 = 35\checkmark$

**(b)** $D(t) = t^3 + 2t + 2$ **A1**

**(c)** **M1** Solving $t^3 + 2t + 2 = 80$ using GDC: $t = 3.97\ldots \approx \mathbf{3.97}$ minutes **A1**

*Note: Accept answers in range 3.96–3.98 from GDC. Award M1 for correct equation set up.*

### Q17 — Sinusoidal Temperature Fit MEDIUM [4 marks

] **(a)** Amplitude $= \dfrac{32-8}{2} = 12\,°\text{C}$; midline $= \dfrac{32+8}{2} = 20\,°\text{C}$ **A1A1**

**(b)** Period $= 2 \times (14 - 2) = 24$ months **A1**

**(c)** $b = \dfrac{2\pi}{24} = \dfrac{\pi}{12}$; maximum at $t = 2$, so $c = 2$; $d = 20$.

$T(t) = 12\cos\left(\frac{\pi}{12}(t-2)\right) + 20$ **A1**

*Note: Accept* $T(t) = 12\cos\left(\frac{\pi(t-2)}{12}\right) + 20$. *FT from their amplitude and period.*

**Q18 — Exponential Population** MEDIUM [4 marks

] **(a)** $P(t) = 3200 \times (1.06)^t$ **A1**

**(b)** $P(10) = 3200 \times (1.06)^{10} = 3200 \times 1.7908\ldots = 5730.27\ldots \approx \mathbf{5730}$ people **A1**

**(c)** **M1** $3200 \times (1.06)^t = 8000 \Rightarrow (1.06)^t = 2.5$

$t = \frac{\ln 2.5}{\ln 1.06} = \frac{0.9162\ldots}{0.05827\ldots} = 15.7252\ldots \approx \mathbf{15.7}$ years **A1**

*Note: Award M1 for* $(1.06)^t = 2.5$ *or* $t\ln(1.06) = \ln(2.5)$*; A1 for 15.7.*

**Q19 — Piecewise Parking Model** MEDIUM [4 marks

] **(a)** At $t = 3$: upper branch gives $C = 2(3) = 6$; lower branch gives $C = 6 + 1.5(3-3) = 6$.
Both branches give $C = 6$, so $C$ is continuous at $t = 3$. **A1**

**(b)** $t = 5 > 3$, so use lower branch: $C(5) = 6 + 1.5(5-3) = 6 + 3 = \$9$ **A1**

**(c)** $C = 12 > 6$, so lower branch applies.
**M1** $6 + 1.5(t-3) = 12 \Rightarrow 1.5(t-3) = 6 \Rightarrow t - 3 = 4 \Rightarrow t = 7$ hours **A1**

**Q20 — Model Comparison** MEDIUM [4 marks

] **(a)** $N = 2w^2$: checking all five points:
$2(1)^2 = 2\checkmark \quad 2(4) = 8\checkmark \quad 2(9) = 18\checkmark \quad 2(16) = 32\checkmark \quad 2(25) = 50\checkmark$ **A1**

**(b)** First differences: $8-2=6,\ 18-8=10,\ 32-18=14,\ 50-32=18$.
Second differences: $10-6=4,\ 14-10=4,\ 18-14=4$. **A1**
The second differences are constant, which indicates a **quadratic** model. **R1**

**(c)** For an exponential model, the ratios of consecutive $N$-values would be constant; here the ratios $8/2 = 4$, $18/8 = 2.25$, $32/18 = 1.78$ are not constant. The quadratic model gives a perfect fit and the constant second differences confirm this structure. **R1**

**Q21 — Linear–Quadratic Synthesis** HARD [5 marks

] **(a)** **M1** $R(x) = C(x): -x^2 + 32x = 50 + 12x \Rightarrow -x^2 + 20x - 50 = 0 \Rightarrow x^2 - 20x + 50 = 0.$

$x = \frac{20 \pm \sqrt{400-200}}{2} = \frac{20 \pm \sqrt{200}}{2} = 10 \pm 5\sqrt{2}$ **A1A1**

**(b)** $P(x) = R(x) - C(x) = (-x^2 + 32x) - (50 + 12x) = -x^2 + 20x - 50$ **A1**

**(c) M1** Vertex: $x = -\dfrac{20}{2(-1)} = 10$.

$P(10) = -(100) + 200 - 50 = \$\mathbf{50}$ maximum profit, achieved when $x = 10$ items are sold. **A1**

## Q22 — Sinusoidal Unknown Parameters HARD [5 marks

] **(a)** $h(\pi) = a\sin(b\pi) + 5 = 5 \Rightarrow a\sin(b\pi) = 0$.
Since $a > 0$, we need $\sin(b\pi) = 0 \Rightarrow b\pi = n\pi$ for some integer $n$.
**M1** Considering $h(\pi/2) = 9$: $a\sin(b\pi/2) + 5 = 9 \Rightarrow a\sin(b\pi/2) = 4 \neq 0$.
So $b\pi/2 \neq k\pi$, meaning $b$ is odd. The smallest positive odd integer value gives $b = 1$. **AG**

**(b)** With $b = 1$: $a\sin(\pi/2) = 4 \Rightarrow a \cdot 1 = 4 \Rightarrow a = 4$. **A1**

**(c)** Period $= \dfrac{2\pi}{b} = \dfrac{2\pi}{1} = 2\pi$ seconds. **A1**

**(d)** One limitation: the model assumes constant amplitude forever; in reality the spring loses energy (damping) so amplitude decreases over time. **R1**

## Q23 — Exponential Unknown Base HARD [5 marks

] **(a)** $P(2) = P_0 b^2 = 36$ and $P(5) = P_0 b^5 = 972$.

**M1** Dividing: $\dfrac{P_0 b^5}{P_0 b^2} = b^3 = \dfrac{972}{36} = 27 \Rightarrow b = 3$. **AG**

**(b)** $P_0 \cdot 9 = 36 \Rightarrow P_0 = 4$ **A1**

**(c)** $4 \cdot 3^t = 2916 \Rightarrow 3^t = 729 = 3^6 \Rightarrow t = 6$ days. **A1**

**(d)** The model assumes unlimited food/space; it is valid only while resources are not limiting (no crowding or competition). **R1**

## Q24 — Cubic Model with Unknown Parameters HARD [5 marks

] **(a)** $D(0) = 0 - 0 + 0 + q = q = 2 \Rightarrow q = 2$. **A1**
$D(1) = 1 - 4 + p + 2 = 5 \Rightarrow p - 1 = 5 \Rightarrow p = 6$. **A1**

**Note:** Using $D(2)$: $8 - 16 + 2p + 2 = 6 \Rightarrow 2p = 12 \Rightarrow p = 6$ ✓(consistent).

**(b) M1** Equations: $q = 2$ and $p - 1 = 5$.
$p = 6, \quad q = 2$. **A1A1**

**(c)** $D(4) = 64 - 64 + 6(4) + 2 = 0 + 24 + 2 = 26\,°\text{C}$. **A1**

**Q25 — Sinusoidal Inequality HARD [5 marks**

] **(a)** Maximum $d = 5 + 8 = 13$ m.
**M1** $\sin(\pi t/4) = 1 \Rightarrow \pi t/4 = \pi/2 \Rightarrow t = 2$ hours after noon.
First maximum at $t = 2$ hours after noon. **A1A1**

**(b)** $5\sin(\pi t/4) + 8 \geq 10.5 \Rightarrow \sin(\pi t/4) \geq 0.5$.
**M1** $\sin\theta = 0.5 \Rightarrow \theta = \pi/6$ or $5\pi/6$:
$\frac{\pi t}{4} = \frac{\pi}{6} \Rightarrow t = \frac{2}{3}$; $\quad \frac{\pi t}{4} = \frac{5\pi}{6} \Rightarrow t = \frac{10}{3}$.
Second period: $t = \frac{2}{3} + 8 = \frac{26}{3}$; $\quad t = \frac{10}{3} + 8 = \frac{34}{3}$.
Solution: $\frac{2}{3} \leq t \leq \frac{10}{3}$ and $\frac{26}{3} \leq t \leq \frac{34}{3}$ **A1A1**

**(c)** Each interval has length $\frac{10}{3} - \frac{2}{3} = \frac{8}{3}$ hours; total $= 2 \times \frac{8}{3} = \frac{16}{3} \approx 5.33$ hours. **A1**

**Q26 — Inverse Square Law HARD [5 marks**

] **(a)** $I = \frac{k}{r^2}$; when $r = 2$, $I = 25$: $k = 25 \times 4 = 100$. **M1A1**

$I(r) = \frac{100}{r^2}$ **A1**

**(b)** $I(5) = \frac{100}{25} = 4$ lux **A1**

**(c)** $\frac{100}{r^2} \geq 1 \Rightarrow r^2 \leq 100 \Rightarrow r \leq 10$ m. Maximum distance $= 10$ m. **A1**

**(d)** New $r = 2 \times 1.5 = 3$ m; $I(3) = \frac{100}{9} = 11.11\ldots$ lux.
**M1** Percentage change $= \frac{100/9 - 25}{25} \times 100 = \frac{(100 - 225)/9}{25} \times 100 = \frac{-125}{225} \times 100 = -55.6\%$.
Intensity **decreases by 55.6%**. **A1**

**Q27 — Strategy for Modelling HARD [5 marks**

] **(a)** First differences: $2, 6, 10, 14, 18$.
Second differences: $4, 4, 4, 4$ (constant). **A1**

**(b)** Constant second differences indicate a **quadratic model** is most appropriate. **R1**

**(c)** $s = at^2 + bt + c$. Since $s(0) = 0$: $c = 0$. Second difference $= 2a = 4 \Rightarrow a = 2$.
$s(1) = 2 + b = 2 \Rightarrow b = 0$. $\quad s(t) = 2t^2$ **A1**

**(d)** $s(8) = 2(64) = 128$ m. **A1**
This prediction is an **extrapolation** beyond the observed range ($0 \leq t \leq 5$); reliability is limited since the vehicle may change speed or behaviour after $t = 5$. **R1**

**Q28 — Quadratic Synthesis with Unknown Parameter HARD [5 marks**

] **(a)** $P_A(x) = -2x^2 + (k+10)x - 12$. Vertex at $x = \dfrac{k+10}{4}$.

**M1** Max $P_A = \dfrac{(k+10)^2}{8} - 12 = 20 \Rightarrow (k+10)^2 = 256 \Rightarrow k + 10 = 16$ (taking positive root, since $k > 0$).
$k = 6$ **A1A1**

**(b)** $P_A(x) = -2x^2 + 16x - 12$. Vertex at $x = \dfrac{16}{4} = 4$. **A1**

**(c) M1** $-2x^2 + 16x - 12 = 3x - 2 \Rightarrow -2x^2 + 13x - 10 = 0 \Rightarrow 2x^2 - 13x + 10 = 0$.
$x = \dfrac{13 \pm \sqrt{169 - 80}}{4} = \dfrac{13 \pm \sqrt{89}}{4}$
$x \approx \dfrac{13 + 9.434}{4} \approx 5.61$ or $x \approx \dfrac{13 - 9.434}{4} \approx 0.892$ **A1**

**Q29 — Linear Regression with Unknown Parameter HARD [5 marks**

] **(a)** $\bar{x} = \dfrac{2+4+6+8}{4} = \dfrac{20}{4} = 5$. **A1**
$\bar{y} = \dfrac{5+9+p+17}{4} = \dfrac{31+p}{4}$ **A1**

**(b)** $\sum x_i y_i = 2(5) + 4(9) + 6p + 8(17) = 10 + 36 + 6p + 136 = 182 + 6p$.
$\sum x_i^2 = 4 + 16 + 36 + 64 = 120$.

**M1** $m = \dfrac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2} = \dfrac{(182+6p) - 4(5) \cdot \frac{31+p}{4}}{120 - 4(25)}$
$= \dfrac{(182+6p) - 5(31+p)}{120 - 100} = \dfrac{182 + 6p - 155 - 5p}{20} = \dfrac{27+p}{20}$ **AG**

**(c)** $\dfrac{27+p}{20} = 3 \Rightarrow 27 + p = 60 \Rightarrow p = 33$ **A1**

**(d)** $\bar{y} = \dfrac{31+33}{4} = 16$; intercept $c = \bar{y} - m\bar{x} = 16 - 3(5) = 1$.
Regression line: $y = 3x + 1$ **A1**

**Q30 — Revenue–Cost Synthesis HARD [5 marks**

] **(a)** $R(t) = -2t^2 + (k+10)t - 12$. Vertex at $t = \dfrac{k+10}{4}$.

**M1** Max $R = \dfrac{(k+10)^2}{8} - 12 = 20 \Rightarrow (k+10)^2 = 256 \Rightarrow k + 10 = 16$.
$k = 6$ **A1A1**

**(b)** Vertex: $t = \dfrac{16}{4} = 4$. Revenue is maximised in **week 4**. **A1**

**(c)** $R(t) > C(t) : -2t^2 + 16t - 12 > 3t - 2 \Rightarrow -2t^2 + 13t - 10 > 0 \Rightarrow 2t^2 - 13t + 10 < 0$.

**M1** Roots: $t = \dfrac{13 \pm \sqrt{89}}{4}$; $t \approx 0.892$ and $t \approx 5.61$.

Revenue exceeds cost for $0.892 < t < 5.61$ (i.e. weeks 1 through 5, approximately). **A1**

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## End of Worksheet

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