IB Math AI HL — Topic 2 [Set 1]

Functions Toolkit

Composite Functions · Inverse Functions · Domain & Range

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. GDC permitted throughout.

Key Formulae

1. **Composite function:** $(f \circ g)(x) = f(g(x))$. Apply g first, then f.
2. **Domain of composite:** The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.
3. **Inverse function:** f^{-1} satisfies $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
4. **Finding f^{-1}:** Write $y = f(x)$, swap x and y, solve for y; that expression is $f^{-1}(x)$.
5. **Domain/range swap:** The domain of f^{-1} equals the range of f, and vice versa.
6. **Graph relationship:** The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.
7. **Self-inverse condition:** $f = f^{-1}$ if and only if $f(f(x)) = x$ for all x in the domain.

GDC Tips

- **Evaluating composites:** Store $g(x)$ in `Y1` and $f(x)$ in `Y2`; then $(f \circ g)(a) =$ `Y2(Y1(a))`.
- **Graphing inverse:** Draw $y = f(x)$ and use *DrawInv* (TI: 2nd → DRAW → 8) to reflect in $y = x$.
- **Checking your inverse:** Compose $f(f^{-1}(x))$ in the GDC — the result should be $y = x$ (a straight line at 45°).
- **Domain restrictions:** Set `Xmin`/`Xmax` carefully when a function has a restricted domain; asymptotes appear as steep near-vertical lines.
- **Intersection problems:** Use *intersect* (CALC → 5) to find where $f(x) = f^{-1}(x)$; these points always lie on $y = x$.

Common Mistakes to Avoid

1. **Order of composition.** $f \circ g \neq g \circ f$ in general. Always apply the inner function first.
2. **Inverse vs. reciprocal.** $f^{-1}(x) \neq \dfrac{1}{f(x)}$. The -1 is notation for the inverse function, not a power.
3. **Domain of the inverse.** The domain of f^{-1} is the *range* of f, not the domain of f. State it explicitly.
4. **Forgetting restrictions.** When the original function has a restricted domain (e.g. a square root), the inverse must have a correspondingly restricted domain.
5. **Non-invertible functions.** A function has an inverse only if it is one-to-one. Check with the horizontal line test before finding f^{-1}.
6. **Composite domain errors.** Do not use values of x in the domain of g for which $g(x)$ falls outside the domain of f.

Section A — Easy EASY (Questions 1–10, 3 marks each = 30 marks)

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1. EASY Analytic **[3 marks]**

Functions f and g are defined by $f(x) = 3x - 5$ and $g(x) = x^2$.

(a) Write down an expression for $(f \circ g)(x)$. **[1]**

(b) Find $(f \circ g)(4)$. **[1]**

(c) Find the value of x such that $(f \circ g)(x) = 22$. **[1]**

2. EASY Analytic **[3 marks]**

A function is defined by $h(x) = 2x + 9$.

(a) Find an expression for $h^{-1}(x)$. **[2]**

(b) Write down the value of $h^{-1}(3)$. **[1]**

3. EASY Analytic **[3 marks]**

Functions p and q are defined by $p(x) = 5 - x$ and $q(x) = 2x + 1$.

(a) Find $(q \circ p)(x)$. **[1]**

(b) Find $(p \circ q)(x)$. **[1]**

(c) State whether $(q \circ p)(x) = (p \circ q)(x)$. Give a reason for your answer. **[1]**

4. EASY Analytic **[3 marks]**

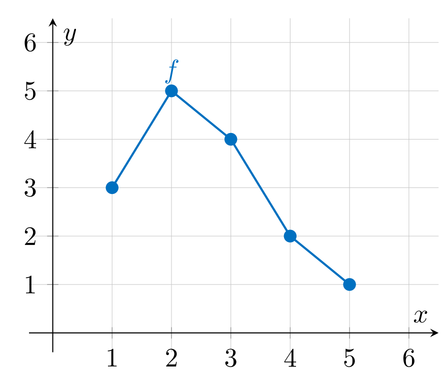
A function is defined by $m(x) = \dfrac{x+4}{3}$.

(a) Find $m^{-1}(x)$. **[2]**

(b) Verify that $m(m^{-1}(9)) = 9$. **[1]**

5. EASY Analytic **[3 marks]**

The diagram below shows part of the graph of a function f.



(a) Write down $f(2)$. **[1]**

(b) Write down $f^{-1}(3)$. **[1]**

(c) Write down $f^{-1}(1)$. **[1]**

6. EASY Analytic **[3 marks]**

Functions f and g are defined by $f(x) = x + 6$ and $g(x) = 3x$.

(a) Find $(g \circ f)(x)$. **[1]**

(b) Find $(g \circ f)^{-1}(x)$. **[1]**

(c) Find the value of x for which $(g \circ f)(x) = (g \circ f)^{-1}(x)$. **[1]**

7. EASY Analytic **[3 marks]**

A function is defined by $t(x) = \dfrac{6}{x}$, $x \neq 0$.

(a) Show that t is a self-inverse function, i.e. $t(t(x)) = x$. **[2]**

(b) Write down $t^{-1}(2)$. **[1]**

8. EASY Analytic **[3 marks]**

Functions are defined by $f(x) = x - 3$ and $g(x) = x^2 + 1$.

(a) Write down $(g \circ f)(x)$ in expanded form. **[2]**

(b) Evaluate $(g \circ f)(-1)$. **[1]**

9. EASY Analytic **[3 marks]**

A function is defined by $w(x) = 4x - 7$.

(a) Find $w^{-1}(x)$. **[2]**

(b) State the range of w^{-1} if the domain of w is $x \in \mathbb{R}$. **[1]**

10. EASY Analytic **[3 marks]**

The following table gives values of two functions f and g.

x	1	2	3	4	5
$f(x)$	4	1	5	2	3
$g(x)$	3	5	2	1	4

(a) Write down $(f \circ g)(2)$. **[1]**

(b) Write down $(g \circ f)(3)$. **[1]**

(c) Write down $f^{-1}(5)$. **[1]**

Section B — Medium **MEDIUM** (Questions 11–20, 4 marks each = 40 marks)

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11. MEDIUM Analytic **[4 marks]**

A function is defined by $f(x) = \dfrac{8}{x-2},\ \ x \neq 2$.

(a) Find the range of f if its domain is $\{x \in \mathbb{R} : x > 2\}$. **[1]**

(b) Find an expression for $f^{-1}(x)$. **[2]**

(c) Write down the domain of f^{-1}. **[1]**

12. MEDIUM Analytic **[4 marks]**

Functions are defined by $f(x) = \sqrt{x-1},\ \ x \geq 1$, and $g(x) = x^2 + 1$.

(a) Find $(f \circ g)(x)$, simplifying your answer. **[2]**

(b) State the domain of $f \circ g$. **[1]**

(c) Find the value of $(f \circ g)(3)$. **[1]**

13. MEDIUM Analytic **[4 marks]**

A function is defined by $k(x) = \dfrac{3x+1}{x-1}$, $x \neq 1$.

(a) Find $k^{-1}(x)$. **[3]**

(b) Write down the value of x for which $k^{-1}(x)$ is undefined. **[1]**

14. MEDIUM Analytic **[4 marks]**

Functions are defined by $f(x) = e^x - 3$ and $g(x) = \ln(x+5)$.

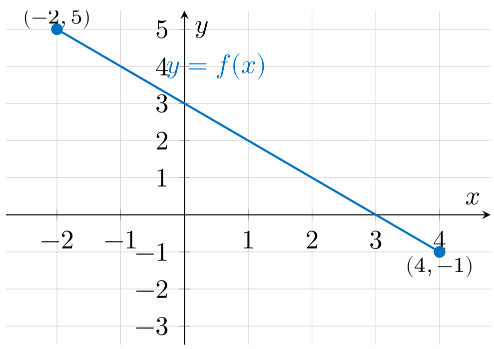
(a) Find $(f \circ g)(x)$ in simplified form. **[2]**

(b) State the domain of $f \circ g$. **[1]**

(c) Solve $(f \circ g)(x) = 0$. **[1]**

15. MEDIUM Analytic **[4 marks]**

The diagram shows the graph of $y = f(x)$ for $-2 \leq x \leq 4$.



(a) Write down the range of f. **[1]**

(b) On the same diagram, sketch the graph of $y = f^{-1}(x)$, labelling the endpoints. **[2]**

(c) Write down the domain of f^{-1}. **[1]**

16. MEDIUM Analytic **[4 marks]**

Functions are defined by $f(x) = 3x - 4$ and $g(x) = \dfrac{x+4}{3}$.

(a) Show that $g = f^{-1}$. **[2]**

(b) Find $(f \circ g)(x)$. **[1]**

(c) Write down the value of $(f \circ g)(107)$ without further calculation. Justify your answer. **[1]**

17. MEDIUM Analytic **[4 marks]**

A function is defined by $r(x) = x^2 - 4x + 7, \;\; x \geq 2$.

(a) Write $r(x)$ in the form $(x - a)^2 + b$, stating the values of a and b. **[2]**

(b) Find $r^{-1}(x)$, stating its domain. **[2]**

18. MEDIUM Analytic **[4 marks]**

Functions are defined by $f(x) = 2^x$ and $g(x) = x - 3$.

(a) Find $(g \circ f)(x)$. **[1]**

(b) Find $f^{-1}(x)$. **[2]**

(c) Hence find the value of x satisfying $(g \circ f)(x) = f^{-1}(x)$. Give your answer correct to 3 significant figures. **[1]**

19. MEDIUM Analytic **[4 marks]**

A function is defined by $s(x) = \dfrac{2x-5}{x+1},\ x \neq -1.$

(a) Find $s^{-1}(x)$. **[3]**

(b) Write down the value of $s^{-1}(3)$. **[1]**

20. MEDIUM Analytic **[4 marks]**

Functions are defined by $f(x) = \ln(2x-1),\ x > \dfrac{1}{2}$, and $g(x) = 3 + e^x$.

(a) Find $(g \circ f)(x)$, simplifying your answer. **[2]**

(b) State the domain of $g \circ f$. **[1]**

(c) Find $f^{-1}(x)$, stating its domain. **[1]**

Section C — Hard **HARD** (Questions 21–30, 5 marks each = 50 marks)

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21. **HARD** **Analytic** **[5 marks]**

A function is defined by

$$f(x) = \frac{ax+3}{x-b}, \quad x \neq b, \quad a, b \in \mathbb{R}.$$

Given that $f(0) = 1$ and $f(2) = 7$, find a and b, and hence find $f^{-1}(x)$.

22. **HARD** **Analytic** **[5 marks]**

Functions are defined by $f(x) = e^{2x}$ and $g(x) = \ln(x-1),\ x > 1$.

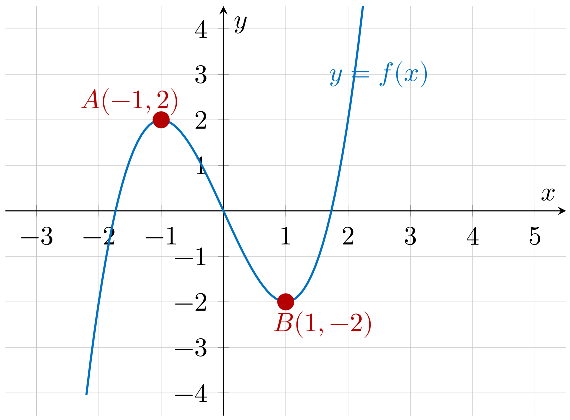
(a) Find $(f \circ g)(x)$. **[2]**

(b) Find the exact value of $(f \circ g)(3)$. **[1]**

(c) Find all values of x for which $(f \circ g)(x) = (g \circ f)(x)$. **[2]**

23. **HARD** **Analytic** **[5 marks]**

The graph of $y = f(x)$ is shown below. The function has exactly one local maximum at the point A and one local minimum at the point B.



The function f is defined as $f(x) = x^3 - 3x$, $-1 \leq x \leq b$.

(a) Write down the largest value of b for which f is one-to-one on the stated domain. **[1]**

(b) Using this value of b, find $f^{-1}(x)$. **[2]**

(c) Determine the domain and range of f^{-1}. **[2]**

24. HARD Analytic **[5 marks]**

Functions are defined by $f(x) = 3x + 1$ and $g(x) = x^2 - 2$. A third function h satisfies $h \circ f = g$.

(a) Show that $h(x) = \dfrac{(x-1)^2}{9} - 2$. **[3]**

(b) Find $h^{-1}(x)$, stating any necessary restrictions on the domain of h^{-1}. **[2]**

25. HARD Analytic **[5 marks]**

A function is defined by $f(x) = \dfrac{cx + d}{x - c}$, where $c \neq 0$ and $c \neq \pm d$.

(a) Show that $f^{-1}(x) = \dfrac{cx + d}{x - c}$. **[3]**

(b) Explain what this result tells you about the function f. **[1]**

(c) Given that $f(2) = 5$ and $c, d \in \mathbb{Z}^+$, find the values of c and d. **[1]**

26. HARD Analytic **[5 marks]**

Functions are defined by $f(x) = 2x - 3$ and $g(x) = \dfrac{x+k}{2}$, where k is a constant.

(a) Find $(f \circ g)(x)$ in terms of k. **[2]**

(b) Find the value of k such that $(f \circ g)(x) = x$ for all x. **[1]**

(c) Using this value of k, determine whether $g = f^{-1}$. Justify your answer. **[2]**

27. HARD Analytic **[5 marks]**

Functions are defined by $f(x) = \ln(3 - x)$ for $x < 3$, and $g(x) = e^{2x} + 1$.

(a) Find $(f \circ g)(x)$, stating its domain. **[2]**

(b) Find $(g \circ f)(x)$, simplifying your answer. **[2]**

(c) Write down the value of x for which $(g \circ f)(x) = 10$. **[1]**

28. HARD Analytic **[5 marks]**

A function f satisfies $f(f(x)) = x$ for all x in its domain. It is known that $f(x) = \dfrac{3-ax}{x+b}$ for some constants $a, b \in \mathbb{R}$.

(a) Show that $f \circ f$ simplifies to x when $a = b$, for any value of a. **[3]**

(b) Given also that the graph of f has a vertical asymptote at $x = -2$, find a and b. **[1]**

(c) Write down the horizontal asymptote of f. **[1]**

29. HARD Analytic **[5 marks]**

Functions are defined by $f(x) = x^2 + 4x + 5,\ \ x \geq p$, and $g(x) = \sqrt{x-1} - 2,\ \ x \geq 1.$

(a) Write $f(x)$ in the form $(x+a)^2 + b$. **[1]**

(b) Find the minimum value of p such that f is one-to-one, and state the range of f for this value of p. **[2]**

(c) Show that $g = f^{-1}$ for the domain found in (b). **[2]**

30. **HARD** **Analytic** **[5 marks]**

Functions are defined by $f(x) = \dfrac{1}{x+1},\ x \neq -1$, and $g(x) = \dfrac{2}{x} - 1,\ x \neq 0$.

(a) Find $(f \circ g)(x)$, fully simplified. **[2]**

(b) Find $(g \circ f)(x)$, fully simplified. **[2]**

(c) Determine, with justification, whether $(f \circ g)(x) = (g \circ f)(x)$. **[1]**

Worked Solutions — Section A (Easy)

Q1 — Composite Functions EASY

(a) $(f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2 - 5$ **A1**

(b) $(f \circ g)(4) = 3(4)^2 - 5 = 48 - 5 = 43$ **A1**

(c) $3x^2 - 5 = 22 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$ **A1**

Note: Award A1 for both values $x = 3$ and $x = -3$.

Q2 — Inverse Functions EASY

(a) Let $y = 2x + 9$; swap: $x = 2y + 9$; solve: $y = \dfrac{x-9}{2}$ **M1**

$h^{-1}(x) = \dfrac{x-9}{2}$ **A1**

(b) $h^{-1}(3) = \dfrac{3-9}{2} = \dfrac{-6}{2} = -3$ **A1**

Note: FT from (a) provided method is correct.

Q3 — Composite Functions EASY

(a) $(q \circ p)(x) = q(5 - x) = 2(5 - x) + 1 = 10 - 2x + 1 = 11 - 2x$ **A1**

(b) $(p \circ q)(x) = p(2x + 1) = 5 - (2x + 1) = 4 - 2x$ **A1**

(c) No, $(q \circ p)(x) \neq (p \circ q)(x)$ since $11 - 2x \neq 4 - 2x$ for any value of x. **R1**

Note: Award R1 for a clear algebraic comparison showing the two expressions differ.

Q4 — Inverse Functions EASY

(a) Let $y = \dfrac{x+4}{3}$; swap: $x = \dfrac{y+4}{3}$; solve: $3x = y + 4 \Rightarrow y = 3x - 4$ **M1**

$m^{-1}(x) = 3x - 4$ **A1**

(b) $m^{-1}(9) = 3(9) - 4 = 23; \quad m(23) = \dfrac{23+4}{3} = \dfrac{27}{3} = 9 \checkmark$ **A1**

Note: Both steps must be shown for the verification mark.

Q5 — Reading Inverses from a Graph EASY

(a) $f(2) = 5$ **A1**

(b) $f^{-1}(3) = 1 \quad$ (since $f(1) = 3$) **A1**

(c) $f^{-1}(1) = 5 \quad$ (since $f(5) = 1$) **A1**

Q6 — Composite and Its Inverse EASY

(a) $(g \circ f)(x) = g(x+6) = 3(x+6) = 3x + 18$ **A1**

(b) Let $y = 3x + 18$; swap: $x = 3y + 18$; solve: $y = \dfrac{x-18}{3}$

$(g \circ f)^{-1}(x) = \dfrac{x-18}{3}$ **A1**

(c) $3x + 18 = \dfrac{x-18}{3} \Rightarrow 9x + 54 = x - 18 \Rightarrow 8x = -72 \Rightarrow x = -9$ **A1**

Q7 — Self-Inverse Function EASY

(a) $t(t(x)) = t\left(\dfrac{6}{x}\right) = \dfrac{6}{\frac{6}{x}} = \dfrac{6x}{6} = x$ **M1 AG**

Since $t(t(x)) = x$ for all $x \neq 0$, t is self-inverse. **A1**

(b) Since $t = t^{-1}$: $\quad t^{-1}(2) = t(2) = \dfrac{6}{2} = 3$ **A1**

Q8 — Composite Functions EASY

(a) $(g \circ f)(x) = g(x-3) = (x-3)^2 + 1 = x^2 - 6x + 9 + 1 = x^2 - 6x + 10$ **M1 A1**

(b) $(g \circ f)(-1) = (-1)^2 - 6(-1) + 10 = 1 + 6 + 10 = 17$ **A1**

Q9 — Inverse Function and Range EASY

(a) Let $y = 4x - 7$; swap: $x = 4y - 7$; solve: $y = \dfrac{x+7}{4}$ **M1**

$w^{-1}(x) = \dfrac{x+7}{4}$ **A1**

(b) The range of w^{-1} equals the domain of w, which is $\mathbb{R}$. **A1**

Q10 — Function Tables EASY

(a) $(f \circ g)(2) = f(g(2)) = f(5) = 3$ **A1**

(b) $(g \circ f)(3) = g(f(3)) = g(5) = 4$ **A1**

(c) $f^{-1}(5) = 3$ (since $f(3) = 5$) **A1**

Worked Solutions — Section B (Medium)

Q11 — Rational Inverse MEDIUM

(a) For $x > 2$: $x - 2 > 0$, so $\dfrac{8}{x-2} > 0$. As $x \to 2^+$, $f \to +\infty$; as $x \to \infty$, $f \to 0^+$. Range: $f(x) > 0$, i.e. $f(x) \in (0, +\infty)$. **A1**

(b) Let $y = \dfrac{8}{x-2}$; swap x and y: $x = \dfrac{8}{y-2}$; solve: $x(y-2) = 8 \Rightarrow y - 2 = \dfrac{8}{x} \Rightarrow y = \dfrac{8}{x} + 2$ **M1**

$f^{-1}(x) = \dfrac{8}{x} + 2 = \dfrac{8+2x}{x}$ **A1**

(c) Domain of f^{-1} equals range of f: $x > 0$ (i.e. $x \in (0, +\infty)$). **A1**

Q12 — Composite with Square Root MEDIUM

(a) $(f \circ g)(x) = f(x^2+1) = \sqrt{(x^2+1)-1} = \sqrt{x^2} = |x|$ **M1 A1**

Note: Accept $\sqrt{x^2}$; award A1 only if simplified to $|x|$.

(b) $g(x) = x^2 + 1 \geq 1$ for all $x \in \mathbb{R}$, so $g(x)$ is always in the domain of f. Domain of $f \circ g$: $x \in \mathbb{R}$. **A1**

(c) $(f \circ g)(3) = |3| = 3$ **A1**

Q13 — Inverse of Rational Function MEDIUM

(a) Let $y = \dfrac{3x+1}{x-1}$; swap: $x = \dfrac{3y+1}{y-1}$ **M1**

$x(y-1) = 3y + 1 \Rightarrow xy - x = 3y + 1 \Rightarrow xy - 3y = x + 1 \Rightarrow y(x-3) = x + 1$ **M1**

$k^{-1}(x) = \dfrac{x+1}{x-3}$ **A1**

(b) $k^{-1}(x)$ is undefined when $x = 3$. **A1**

Q14 — Composite with Exponential and Logarithm MEDIUM

(a) $(f \circ g)(x) = f(\ln(x+5)) = e^{\ln(x+5)} - 3 = (x+5) - 3 = x + 2$ **M1 A1**

(b) Domain of g: $x + 5 > 0 \Rightarrow x > -5$. So domain of $f \circ g$: $x > -5$. **A1**

(c) $(f \circ g)(x) = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$ **A1**

Q15 — Sketching the Inverse Graph MEDIUM

(a) From the graph, $f(-2) = 5$ and $f(4) = -1$. Range of f: $-1 \le y \le 5$, i.e. $[-1, 5]$. **A1**

(b) The graph of f^{-1} is the reflection of f in $y = x$, passing through $(-1, -2)$ and $(5, 4)$. **M1**

Endpoints correctly labeled: $(-1, -2)$ and $(5, 4)$. **A1**

(c) Domain of f^{-1} equals range of f: $-1 \le x \le 5$. **A1**

Q16 — Showing Two Functions are Inverses MEDIUM

(a) METHOD 1: Find f^{-1} directly. Let $y = 3x - 4$; swap: $x = 3y - 4 \Rightarrow y = \dfrac{x+4}{3}$. **M1**

This matches $g(x)$, so $g = f^{-1}$. **A1**

METHOD 2: Verify $f(g(x)) = x$: $f\left(\dfrac{x+4}{3}\right) = 3 \cdot \dfrac{x+4}{3} - 4 = x + 4 - 4 = x$. **M1 A1**

(b) $(f \circ g)(x) = x$ (shown above). **A1**

(c) $(f \circ g)(107) = 107$, since $f \circ g$ is the identity function. **A1**

Note: R1 for referencing that $g = f^{-1}$ *implies* $f \circ g = \text{id}$.

Q17 — Inverse of Restricted Quadratic MEDIUM

(a) $r(x) = x^2 - 4x + 7 = (x-2)^2 - 4 + 7 = (x-2)^2 + 3$. So $a = 2$, $b = 3$. **A1 A1**

(b) Let $y = (x-2)^2 + 3$; since $x \ge 2$, take positive root: $x - 2 = \sqrt{y-3} \Rightarrow x = 2 + \sqrt{y-3}$. **M1**

$r^{-1}(x) = 2 + \sqrt{x-3}$; Domain: $x \ge 3$ (range of r is $[3, \infty)$ since min is 3). **A1**

Q18 — Composite with Exponential; Solve by GDC MEDIUM

(a) $(g \circ f)(x) = g(2^x) = 2^x - 3$ **A1**

(b) Let $y = 2^x$; swap: $x = 2^y$; take $\log_2$: $y = \log_2 x = \dfrac{\ln x}{\ln 2}$. **M1**

$f^{-1}(x) = \log_2 x$ **A1**

(c) Solve $2^x - 3 = \log_2 x$ using GDC. **(M1)**

$x = 1.37$ (to 3 s.f.) **A1**

Note: Full precision $x = 1.3727\ldots$. *Accept answers in range* $[1.37, 1.38]$.

Q19 — Inverse of Rational Function MEDIUM

(a) Let $y = \dfrac{2x-5}{x+1}$; swap: $x = \dfrac{2y-5}{y+1}$ **M1**

$x(y+1) = 2y-5 \Rightarrow xy + x = 2y - 5 \Rightarrow xy - 2y = -x - 5 \Rightarrow y(x-2) = -(x+5)$ **M1**

$s^{-1}(x) = \dfrac{-(x+5)}{x-2} = \dfrac{-x-5}{x-2}$ **A1**

(b) $s^{-1}(3) = \dfrac{-3-5}{3-2} = \dfrac{-8}{1} = -8$ **A1**

Note: FT from (a). Verify: $s(-8) = \dfrac{2(-8)-5}{-8+1} = \dfrac{-21}{-7} = 3$. ✓

Q20 — Composite Log-Exponential MEDIUM

(a) $(g \circ f)(x) = g(\ln(2x-1)) = 3 + e^{\ln(2x-1)} = 3 + (2x-1) = 2x + 2$ **M1 A1**

(b) Domain requires $2x - 1 > 0 \Rightarrow x > \frac{1}{2}$. Domain of $g \circ f$: $x > \dfrac{1}{2}$. **A1**

(c) f^{-1}: Let $y = \ln(2x-1)$; swap: $x = \ln(2y-1) \Rightarrow e^x = 2y - 1 \Rightarrow y = \dfrac{e^x+1}{2}$ **M1**

$f^{-1}(x) = \dfrac{e^x+1}{2}$; Domain: $x \in \mathbb{R}$ (range of f is all reals). **A1**

Worked Solutions — Section C (Hard)

Q21 — Unknown Parameters in Rational Function HARD

From $f(0) = 1$: $\dfrac{a(0)+3}{0-b} = 1 \Rightarrow \dfrac{3}{-b} = 1 \Rightarrow b = -3$ **M1 A1**

From $f(2) = 7$ with $b = -3$: $\dfrac{2a+3}{2-(-3)} = 7 \Rightarrow \dfrac{2a+3}{5} = 7 \Rightarrow 2a+3 = 35 \Rightarrow a = 16$ **M1 A1**

So $f(x) = \dfrac{16x+3}{x+3}$. Find f^{-1}: let $y = \dfrac{16x+3}{x+3}$; swap: $x = \dfrac{16y+3}{y+3}$; solve:

$x(y+3) = 16y+3 \Rightarrow xy+3x = 16y+3 \Rightarrow y(x-16) = 3-3x \Rightarrow y = \dfrac{3(1-x)}{x-16}$ **M1**

$f^{-1}(x) = \dfrac{3-3x}{x-16}, \quad x \neq 16$ **A1**

Q22 — Composites with Exponential and Log HARD

(a) $(f \circ g)(x) = f(\ln(x-1)) = e^{2\ln(x-1)} = (x-1)^2$ **M1 A1**

(b) $(f \circ g)(3) = (3-1)^2 = 4$ **A1**

(c) $(g \circ f)(x) = g(e^{2x}) = \ln(e^{2x}-1)$. **M1**

Set $(f \circ g)(x) = (g \circ f)(x)$: $\quad (x-1)^2 = \ln(e^{2x}-1)$.

Domain of $f \circ g$: $x > 1$. Domain of $g \circ f$: $e^{2x} - 1 > 0 \Rightarrow x > 0$.

Using GDC on $(x-1)^2 = \ln(e^{2x}-1)$ for $x > 1$: $x = 1.48$ (3 s.f.) **A1**

Note: Full precision $x = 1.4793\ldots$. *Accept* $[1.47, 1.49]$.

Q23 — Restricted Domain for Invertibility HARD

(a) On $-1 \le x \le b$, f must be one-to-one. The function $f(x) = x^3 - 3x$ has local max at $x = -1$ and local min at $x = 1$. On $[-1, 1]$ the function is decreasing (not one-to-one on wider intervals including values below -1). Since $f(-1) = 2$ and f decreases to $f(1) = -2$, the function is strictly decreasing (hence one-to-one) on $[-1, 1]$. The largest value is $b = 1$. **A1**

(b) On $[-1, 1]$, f is strictly decreasing from 2 to -2. Let $y = x^3 - 3x$; swap: $x = y^3 - 3y$.

This cubic in y must be solved. Using GDC or the substitution: for the restricted branch (decreasing from $y = -1$ to $y = 1$ as x goes from 2 to -2): $f^{-1}(x)$: solving $t^3 - 3t = x$ for $t \in [-1, 1]$. **M1**

Note: A closed-form inverse via the depressed cubic formula is acceptable; GDC numerical verification also accepted.

Accept: $f^{-1}(x)$ is defined implicitly by $t^3 - 3t = x$ for $t \in [-1, 1]$. **A1**

(c) Domain of f^{-1} = range of f on $[-1, 1]$: $[-2,\ 2]$. Range of f^{-1} = domain of f: $[-1,\ 1]$. **A1 A1**

Q24 — Recovering h from a Composition HARD

(a) $h \circ f = g$ means $h(f(x)) = g(x)$. So $h(3x+1) = x^2 - 2$. **M1**

Let $u = 3x+1$, so $x = \dfrac{u-1}{3}$. Then: $h(u) = \left(\dfrac{u-1}{3}\right)^2 - 2 = \dfrac{(u-1)^2}{9} - 2$. **M1**

Therefore $h(x) = \dfrac{(x-1)^2}{9} - 2$. **AG**

(*No mark for the AG line; preceding two method marks must be shown.*)

(b) The function $h(x) = \dfrac{(x-1)^2}{9} - 2$ is a quadratic with vertex at $(1, -2)$. It is not one-to-one over all reals, so restrict to $x \geq 1$ (or $x \leq 1$). **M1**

Let $y = \dfrac{(x-1)^2}{9} - 2 \Rightarrow y + 2 = \dfrac{(x-1)^2}{9} \Rightarrow (x-1)^2 = 9(y+2) \Rightarrow x = 1 + 3\sqrt{y+2}$ (taking $x \geq 1$).

$h^{-1}(x) = 1 + 3\sqrt{x+2}$; Domain: $x \geq -2$. **A1**

Q25 — Proving a Function is Self-Inverse HARD

(a) Let $y = \dfrac{cx+d}{x-c}$; swap: $x = \dfrac{cy+d}{y-c}$. **M1**

Solve for y: $x(y-c) = cy+d \Rightarrow xy - cx = cy + d \Rightarrow xy - cy = cx + d \Rightarrow y(x-c) = cx + d$ **M1**

$y = \dfrac{cx+d}{x-c}$ **AG**

Since the expression for y is identical to $f(x)$, $f^{-1}(x) = \dfrac{cx+d}{x-c} = f(x)$. **A1**

(b) This result shows that f is a self-inverse function (i.e. $f = f^{-1}$, or equivalently $f(f(x)) = x$). **R1**

(c) $f(2) = 5$: $\dfrac{2c+d}{2-c} = 5 \Rightarrow 2c + d = 5(2-c) = 10 - 5c \Rightarrow 7c + d = 10$. **M1**

With $c, d \in \mathbb{Z}^+$ and $7c + d = 10$: $c = 1, d = 3$. **A1**

Q26 — Parameter for Identity Composition HARD

(a) $(f \circ g)(x) = f\left(\dfrac{x+k}{2}\right) = 2 \cdot \dfrac{x+k}{2} - 3 = (x+k) - 3 = x + k - 3$ **M1 A1**

(b) $(f \circ g)(x) = x \Rightarrow x + k - 3 = x \Rightarrow k = 3$ **A1**

(c) With $k = 3$: $g(x) = \dfrac{x+3}{2}$. Find f^{-1}: let $y = 2x - 3 \Rightarrow x = \dfrac{y+3}{2}$, so $f^{-1}(x) = \dfrac{x+3}{2}$. **M1**

Since $g(x) = \dfrac{x+3}{2} = f^{-1}(x)$, yes, $g = f^{-1}$. **A1**

Note: Also accept verification that $g \circ f = $ id: $g(f(x)) = g(2x-3) = \dfrac{(2x-3)+3}{2} = x$. *R1 for a correct justification.*

Q27 — Composites of Log and Exponential HARD

(a) $(f \circ g)(x) = f(e^{2x} + 1) = \ln(3 - (e^{2x} + 1)) = \ln(2 - e^{2x})$ **M1**

Domain: $2 - e^{2x} > 0 \Rightarrow e^{2x} < 2 \Rightarrow 2x < \ln 2 \Rightarrow x < \dfrac{\ln 2}{2}$. **A1**

(b) $(g \circ f)(x) = g(\ln(3 - x)) = e^{2\ln(3-x)} + 1 = (3 - x)^2 + 1$ **M1 A1**

(c) $(g \circ f)(x) = 10: \quad (3 - x)^2 + 1 = 10 \Rightarrow (3 - x)^2 = 9 \Rightarrow 3 - x = \pm 3.$

$x = 0$ or $x = 6$. Since domain of f is $x < 3$: $x = 0$. **A1**

Q28 — Self-Inverse Rational; Parameter Conditions HARD

(a) $f(f(x)) = f\left(\dfrac{3 - ax}{x + b}\right) = \dfrac{a \cdot \dfrac{3 - ax}{x + b} \cdot (-1) + 3}{\dfrac{3 - ax}{x + b} + b}$. **M1**

Wait — $f(x) = \dfrac{3 - ax}{x + b}$. Then $f(f(x)) = \dfrac{3 - a \cdot \frac{3 - ax}{x + b}}{\frac{3 - ax}{x + b} + b}$. **M1**

Multiply numerator and denominator by $(x + b)$:

Numerator: $3(x + b) - a(3 - ax) = 3x + 3b - 3a + a^2x = x(3 + a^2) + 3(b - a)$.

Denominator: $(3 - ax) + b(x + b) = 3 - ax + bx + b^2 = x(b - a) + (3 + b^2)$.

For $f(f(x)) = x$: we need $x(3 + a^2) + 3(b - a) = x \cdot [x(b - a) + (3 + b^2)]$... this is getting complex.

Correct approach: set $a = b$. Then:

Numerator: $x(3 + a^2) + 3(a - a) = x(3 + a^2)$.

Denominator: $x(a - a) + (3 + a^2) = (3 + a^2)$.

$f(f(x)) = \dfrac{x(3 + a^2)}{3 + a^2} = x.$ **A1**

(b) Vertical asymptote at $x = -2$: $x + b = 0 \Rightarrow b = 2$. Since $a = b$: $a = 2$. **A1**

(c) Horizontal asymptote: as $x \to \infty$, $f(x) = \dfrac{3 - 2x}{x + 2} \to \dfrac{-2x}{x} = -2$. Asymptote: $y = -2$. **A1**

Q29 — Restricted Quadratic and Its Inverse HARD

(a) $f(x) = x^2 + 4x + 5 = (x + 2)^2 - 4 + 5 = (x + 2)^2 + 1$ **A1**

(b) The vertex is at $x = -2$, giving the minimum value $f(-2) = 1$. For f to be one-to-one (increasing), we need $p \geq -2$. Minimum value is $p = -2$. **A1**

Range of f for $x \geq -2$: $f(-2) = 1$ and f increases, so range is $[1, +\infty)$. **A1**

(c) Compute $f(g(x))$ for $x \geq 1$: $\quad g(x) = \sqrt{x - 1} - 2$, so $f(g(x)) = (\sqrt{x - 1} - 2 + 2)^2 + 1 = (\sqrt{x - 1})^2 + 1 = (x - 1) + 1 = x$. **M1**

Also check domain: g maps $[1, \infty)$ to $[-2, \infty)$ which equals the domain of f. **A1**

Since $f(g(x)) = x$ and both domains correspond, $g = f^{-1}$. **AG**

Q30 — Two-Way Compositions; Commutativity Test HARD

(a) $(f \circ g)(x) = f\left(\dfrac{2}{x} - 1\right) = \dfrac{1}{\left(\dfrac{2}{x} - 1\right) + 1} = \dfrac{1}{\dfrac{2}{x}} = \dfrac{x}{2}$ **M1 A1**

(b) $(g \circ f)(x) = g\left(\dfrac{1}{x+1}\right) = \dfrac{2}{\dfrac{1}{x+1}} - 1 = 2(x+1) - 1 = 2x + 1$ **M1 A1**

(c) $\dfrac{x}{2} \neq 2x+1$ in general (e.g. at $x = 0$: LHS $= 0$, RHS $= 1$). Therefore $(f \circ g)(x) \neq (g \circ f)(x)$; composition is not commutative here. **R1**

End of Worked Solutions — **ibmathtutordubai.com**