IB Math AI HL — Topic 5 [Set 1]

Further Integration

Special Functions · Reverse Chain Rule · Substitution · Definite Integrals
Negative Integrals · Area (curve & y-axis) · Area (curve & line) · Volumes of Revolution

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Scan me for help

Name:________________________ Date:______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless otherwise stated. A GDC is permitted throughout.

Key Formulae

$$\int e^{ax+b}\,dx = \tfrac{1}{a}e^{ax+b} + C \qquad \int \sin(ax+b)\,dx = -\tfrac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \tfrac{1}{a}\sin(ax+b) + C \qquad \int \frac{1}{ax+b}\,dx = \tfrac{1}{a}\ln|ax+b| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \qquad \int [f(x)]^n f'(x)\,dx = \tfrac{[f(x)]^{n+1}}{n+1} + C$$

Area (above x-axis) $= \int_a^b f(x)\,dx$ (use GDC or split at roots for negative regions)

Area (between curves) $= \int_a^b [f(x) - g(x)]\,dx$ Area (y-axis) $= \int_c^d x(y)\,dy$

$V_x = \pi \int_a^b [f(x)]^2\,dx$ (about x-axis) $V_y = \pi \int_c^d [x(y)]^2\,dy$ (about y-axis)

GDC Tips

1. **Definite integrals:** Use the $\int$ template. Always sketch first to identify the number of regions.
2. **Negative areas:** The GDC gives a signed value; split at roots and add absolute values for the geometric area.
3. **Volumes:** Enter $\pi\times$`fnInt(`$f(x)^2$`, `x`, `a`, `b`)` directly.
4. **Intersection points:** Use `CALC` $\to$ `intersect` to find limits before setting up your integral.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)

Scan me for help

1. **EASY** **Calculator** **[3 marks]**

Find each indefinite integral.

(a) $\displaystyle\int 5e^{3x}\,dx$ [1]

(b) $\displaystyle\int \cos(2x-1)\,dx$ [1]

(c) $\displaystyle\int \frac{4}{3x+5}\,dx$ [1]

2. **EASY** **Calculator** **[3 marks]**

Find each indefinite integral.

(a) $\displaystyle\int 6x(x^2+4)^2\,dx$ [2]

(b) $\displaystyle\int \frac{3x^2}{x^3-1}\,dx$ [1]

3. EASY Calculator **[3 marks]**

(a) Evaluate $\int_1^4 (2x-3)\,dx$. [2]

(b) Explain why your answer from part (a) represents the net signed area, not the total area enclosed between the line $y = 2x - 3$ and the x-axis for $1 \le x \le 4$, given that the line crosses the x-axis at $x = \frac{3}{2}$. [1]

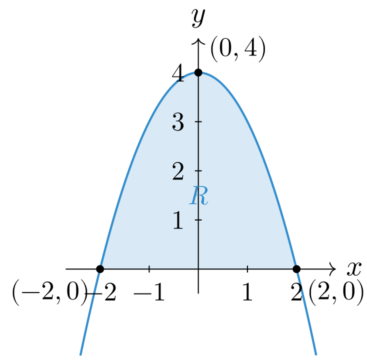
4. EASY Calculator **[3 marks]**

(a) Find $\int \frac{2x}{x^2+9}\,dx$. [2]

(b) Hence evaluate $\int_0^4 \frac{2x}{x^2+9}\,dx$, leaving your answer in the form ln k where $k \in \mathbb{Q}^+$. [1]

5. EASY Calculator **[3 marks]**

The diagram shows the graph of $y = 4 - x^2$ for $-2 \le x \le 2$.



The shaded region R is bounded by the curve and the x-axis.

(a) Write down the x-coordinates of the points where the curve meets the x-axis. [1]

(b) Find the exact area of R. [2]

6. EASY Calculator **[3 marks]**

Use integration by substitution with $u = 2x + 3$ to find

$$\int x(2x+3)^4 \, dx.$$

7. EASY Calculator **[3 marks]**

(a) Find $\int \sin(3x)\,dx$. [1]

(b) Hence evaluate $\int_0^{\pi/3} \sin(3x)\,dx$, giving your answer as an exact value. [2]

8. EASY Calculator **[3 marks]**

(a) Find $\int 8x(x^2+1)^3\,dx$. [2]

(b) Hence find $\int_0^1 8x(x^2+1)^3\,dx$. [1]

9. **EASY** **Calculator** **[3 marks]**

The region bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 9$ is rotated 2π radians about the x-axis.

(a) Write down an integral that represents the volume of the solid formed. [1]

(b) Calculate the exact volume. [2]

10. **EASY** **Calculator** **[3 marks]**

(a) Evaluate $\displaystyle\int_2^5 (x^2 - 9)\, dx$. [2]

(b) Explain why your answer to part (a) does not equal the total area enclosed between $y = x^2 - 9$ and the x-axis for $2 \leq x \leq 5$. [1]

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

Scan me for help

11. MEDIUM Calculator **[4 marks]**

(a) Show that $\dfrac{d}{dx}(x^3 - x) = 3x^2 - 1$, and hence find $\displaystyle\int \frac{6x^2 - 2}{x^3 - x}\,dx$. [2]

(b) Hence evaluate $\displaystyle\int_2^4 \frac{6x^2 - 2}{x^3 - x}\,dx$, giving your answer in the form $a \ln b$ where $a, b \in \mathbb{Z}^+$. [2]

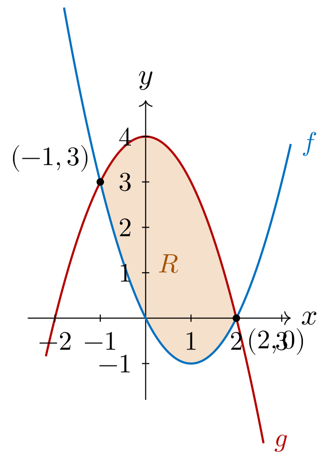
12. MEDIUM Calculator **[4 marks]**

(a) Use the substitution $u = 3x^2$ to show that $\displaystyle\int x \sin(3x^2)\,dx = -\frac{1}{6}\cos(3x^2) + C$. [2]

(b) Hence evaluate $\displaystyle\int_0^{\sqrt{\pi/6}} x \sin(3x^2)\,dx$, giving your answer as an exact value. [2]

13. MEDIUM Calculator **[4 marks]**

The diagram shows the graphs of $f(x) = x^2 - 2x$ and $g(x) = 4 - x^2$.



(a) Write down the x-coordinates of the intersection points of f and g. [1]

(b) Find the exact area of the shaded region R. [3]

14. MEDIUM Calculator **[4 marks]**

The curve C has equation $y = x^3 - 3x$ and the line ℓ has equation $y = 0$ (the x-axis).

(a) Find all x-intercepts of the curve C. [1]

(b) The curve dips below the x-axis on one of the intervals between consecutive roots. Find the total area enclosed between C and the x-axis. [3]

15. MEDIUM Calculator **[4 marks]**

The region R is bounded by the curve $y = e^x + 1$, the x-axis, the y-axis, and the line $x = 2$.

(a) Sketch the region R, labelling the y-intercept and the point on the curve at $x = 2$. [1]

(b) The region R is rotated 2π radians about the x-axis. Find the volume of the solid formed, giving your answer correct to 3 significant figures. [3]

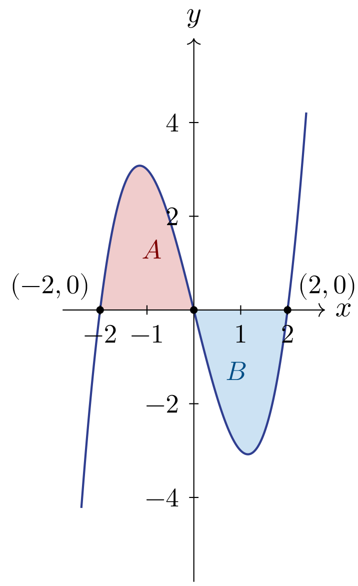
16. MEDIUM Calculator **[4 marks]**

(a) Use the substitution $u = x^2 + 4$ to find $\int \frac{x}{\sqrt{x^2+4}}\,dx$. [3]

(b) Hence evaluate $\int_0^{\sqrt{5}} \frac{x}{\sqrt{x^2+4}}\,dx$, giving your answer in exact form. [1]

17. MEDIUM Calculator **[4 marks]**

The diagram shows the graph of $y = x^3 - 4x$ with its three x-intercepts at $x = -2$, $x = 0$, and $x = 2$.



(a) Evaluate $\displaystyle\int_{-2}^{2} (x^3 - 4x)\, dx$ and explain what this implies about regions A and B. [2]

(b) Find the total shaded area $A + B$. [2]

18. MEDIUM Calculator **[4 marks]**

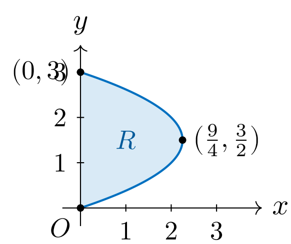
The region bounded by the curve $y = x^2$, the y-axis, and the line $y = 9$ is rotated 2π radians about the y-axis.

(a) Write x^2 in terms of y. [1]

(b) Hence find the exact volume of revolution. [3]

19. MEDIUM Calculator **[4 marks]**

The diagram shows the curve $x = 3y - y^2$ for $0 \leq y \leq 3$.



The shaded region R is bounded by the curve, the y-axis, and the line $y = 3$.

(a) Write down an integral in y that gives the area of R. [1]

(b) Calculate the exact area of R. [3]

20. MEDIUM Calculator **[4 marks]**

(a) Use the substitution $u = \cos x$ to find $\int \sin x \cos^3 x \, dx$. [3]

(b) Hence evaluate $\int_0^{\pi/2} \sin x \cos^3 x \, dx$. [1]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

Scan me for help

21. **HARD** **Calculator** **[5 marks]**

The curves $f(x) = x^2$ and $g(x) = kx$ (where $k > 0$) intersect at the origin and at a point P.

(a) Show that the x-coordinate of P is k. [1]

(b) Show that the area of the region enclosed between the two curves is $\dfrac{k^3}{6}$. [3]

(c) Find the value of k such that this enclosed area equals $\dfrac{4}{3}$. [1]

22. HARD Calculator **[5 marks]**

A container is modelled by rotating the curve $x = \sqrt{y+1}$ for $0 \leq y \leq 8$ (where x and y are in centimetres) through 2π radians about the y-axis.

(a) Find x when $y = 0$ and when $y = 8$, and describe how the radius changes with height. [2]

(b) Find the volume of the container, correct to 3 significant figures. [2]

(c) Water is poured to a depth of 5 cm. Find the volume of water, correct to 3 significant figures. [1]

23. HARD Calculator **[5 marks]**

(a) Show that $\dfrac{d}{du}\left(\dfrac{-1}{u^2}\right) = \dfrac{2}{u^3}$. [1]

(b) Use the substitution $u = x^2 - 3$ to show that

$$\int \frac{4x}{(x^2-3)^3}\,dx = \frac{-1}{(x^2-3)^2} + C, \qquad x^2 \neq 3.$$

[2]

(c) Hence evaluate $\displaystyle\int_2^3 \frac{4x}{(x^2-3)^3}\,dx$, giving your answer in the form $\dfrac{p}{q}$ where $p, q \in \mathbb{Z}$. [2]

24. HARD Calculator **[5 marks]**

The curve C has equation $y = x^3 - 6x^2 + 9x$ and the line ℓ has equation $y = mx$, where m is a constant.

(a) Show that C and ℓ always intersect at the origin. [1]

(b) For $0 < m < 9$, find the other x-coordinates of intersection in terms of m. [2]

(c) When $m = 5$, find the total area of the regions enclosed between C and ℓ. [2]

25. HARD Calculator **[5 marks]**

The region R is enclosed between the curves $y = \sqrt{x}$ and $y = x^2$ for $0 \le x \le 1$.

(a) Verify that the two curves intersect only at $x = 0$ and $x = 1$ in this interval. [1]

(b) The region R is rotated 2π radians about the x-axis. Show that the volume of the solid formed is

$$V = \pi \int_0^1 (x - x^4)\, dx.$$

[2]

(c) Hence find the exact value of V. [2]

26. HARD Calculator **[5 marks]**

(a) Use the substitution $u = 2x - 1$ to find $\int \frac{x}{(2x-1)^3}\,dx$, giving your answer in simplified form in terms of x. [4]

(b) Hence evaluate $\int_1^2 \frac{x}{(2x-1)^3}\,dx$, leaving your answer as an exact fraction. [1]

27. HARD Calculator **[5 marks]**

(a) Find the area of the region bounded by the curve $x = e^y$, the y-axis, and the lines $y = 0$ and $y = 2$, by integrating with respect to y. [3]

(b) The rectangle with vertices $O(0,0)$, $A(e^2,0)$, $B(e^2,2)$, and $D(0,2)$ contains the region in part (a). Without further integration, find the area of the region bounded by the curve $y = \ln x$, the x-axis, and the line $x = e^2$. [2]

28. HARD Calculator **[5 marks]**

(a) Find $\displaystyle\int x\, e^{x^2}\, dx$. [2]

(b) Let $f(x) = xe^{x^2}$. The region R is bounded by $y = f(x)$, the x-axis, and the line $x = 2$.

(i) Write down the integral that gives the volume when R is rotated 2π radians about the x-axis. [1]

(ii) Explain why the integrand in part (i) cannot be integrated analytically in closed form. Hence use a GDC to find the volume correct to 3 significant figures. [2]

29. HARD Calculator **[5 marks]**

The curve $y = \sin x$ and the straight line $y = \dfrac{2}{\pi}x$ both pass through the origin and intersect again at $x = \dfrac{\pi}{2}$ and at $x = \pi$ (where $\sin x = 0$ and the line gives $y = 2$, so the curve and line share the point $(0, 0)$ and meet again only at $x = \frac{\pi}{2}$ on $(0, \pi)$). In fact the two graphs meet at $x = 0$ and $x = \dfrac{\pi}{2}$ on $[0, \pi]$.

(a) Verify that $\sin\left(\dfrac{\pi}{2}\right) = \dfrac{2}{\pi} \cdot \dfrac{\pi}{2}$. [1]

(b) Find the exact area of the region enclosed between the curve and the line for $0 \le x \le \dfrac{\pi}{2}$. [3]

(c) The region in part (b) is rotated 2π radians about the x-axis. Use a GDC to find the volume of the solid formed, giving your answer correct to 3 significant figures. [1]

30. HARD Calculator **[5 marks]**

The curve C has equation $y = x^2 - 1$ for $x \geq 0$. The region R is bounded by C, the y-axis, and the horizontal line $y = k$, where $k > -1$.

(a) Write x^2 in terms of y. [1]

(b) Show that the volume generated when R is rotated 2π radians about the y-axis is

$$V = \pi \int_{-1}^{k} (y+1)\, dy.$$

[2]

(c) Given that this volume equals 18π, find the value of k. [2]

Common Mistakes to Avoid

1. **Reverse chain rule multiplier.** When integrating $\int [f(x)]^n f'(x)\,dx$, divide by $n+1$ AND by the coefficient of the inner function. Missing this reciprocal factor is the most common single error.

2. **Negative areas.** A definite integral gives a *signed* value. If the curve is below the x-axis, split at the roots and add absolute values to get the geometric area.

3. **Changing limits under substitution.** For definite integrals, always convert the limits using $u = g(x)$: the new limits are $u(a)$ and $u(b)$.

4. **Volume formula.** The formula is $V = \pi \int [f(x)]^2\,dx$ — you must *square* $f(x)$. Forgetting to square is an immediate mark loss.

5. **Volume between two curves (washer method).** Use $\pi \int ([f(x)]^2 - [g(x)]^2)\,dx$, **not** $\pi \int [f(x) - g(x)]^2\,dx$.

6. **Area with respect to the y-axis.** Rewrite the curve as $x = g(y)$ and integrate $\int g(y)\,dy$ with y-limits. Many students try to integrate with respect to x and get the wrong region.

Worked Solutions

M = method mark **A** = accuracy mark **R** = reasoning mark **AG** = answer given (no mark) FT = follow-through.

Q1 — Special Functions (Easy)

(a) $\int 5e^{3x}\,dx = \frac{5}{3}e^{3x} + C$ **A1**

(b) $\int \cos(2x-1)\,dx = \frac{1}{2}\sin(2x-1) + C$ **A1**

(c) $\int \frac{4}{3x+5}\,dx = \frac{4}{3}\ln|3x+5| + C$ **A1**

Note: $+C$ required in each part. Accept $\frac{4}{3}\ln(3x+5)+C$ for $x > -5/3$.

Q2 — Reverse Chain Rule (Easy)

(a) Recognise that $\frac{d}{dx}(x^2+4) = 2x$; write $6x = 3 \times 2x$. **M1**

$$\int 6x(x^2+4)^2\,dx = \frac{3(x^2+4)^3}{3} + C = (x^2+4)^3 + C$$

A1

(b) Recognise $\frac{d}{dx}(x^3-1) = 3x^2$:

$$\int \frac{3x^2}{x^3-1}\,dx = \ln|x^3-1| + C$$

A1

Q3 — Definite Integrals / Negative Region (Easy)

(a) $\int_1^4 (2x-3)\,dx = \left[x^2-3x\right]_1^4 = (16-12)-(1-3) = 4+2 = 6$ **M1 A1**

(b) The line $y = 2x-3$ is below the x-axis on $(1, \frac{3}{2})$, giving a negative contribution to the integral. The definite integral (signed area) is 6, but the total geometric area is larger because the small triangular region below the axis is subtracted rather than added. **R1**

Q4 — Reverse Chain Rule / ln Form (Easy)

(a) Numerator is derivative of denominator ($\frac{d}{dx}(x^2+9) = 2x$): **M1**

$$\int \frac{2x}{x^2+9}\,dx = \ln(x^2+9) + C$$

A1

(b) $\left[\ln(x^2+9)\right]_0^4 = \ln 25 - \ln 9 = \ln \frac{25}{9}$ **A1**

Note: $k = \frac{25}{9} \in \mathbb{Q}^+$. *Accept* $\ln 25 - \ln 9$.

Q5 — Area Under Curve (Easy)

(a) $x = -2$ and $x = 2$ (from $4 - x^2 = 0$) **A1**

(b) Curve is above the x-axis on $(-2, 2)$:

$$\text{Area} = \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \tfrac{x^3}{3}\right]_{-2}^{2}$$

M1

$= (8 - \frac{8}{3}) - (-8 + \frac{8}{3}) = \frac{16}{3} + \frac{16}{3} = \dfrac{32}{3}$ **A1**

Note: Exact form $\frac{32}{3}$ *required.*

Q6 — Integration by Substitution (Easy)

Let $u = 2x + 3 \Rightarrow x = \frac{u-3}{2}, \quad du = 2\,dx \Rightarrow dx = \frac{du}{2}$. **M1**

$\displaystyle\int x(2x+3)^4\,dx = \int \frac{u-3}{2} \cdot u^4 \cdot \frac{du}{2} = \frac{1}{4}\int (u^5 - 3u^4)\,du$ **A1**

$\displaystyle= \frac{1}{4}\left(\frac{u^6}{6} - \frac{3u^5}{5}\right) + C = \frac{(2x+3)^6}{24} - \frac{3(2x+3)^5}{20} + C$ **A1**

Note: Must back-substitute in terms of x.

Q7 — Trig Integration (Easy)

(a) $\displaystyle\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C$ **A1**

(b) $\displaystyle\int_0^{\pi/3} \sin(3x)\,dx = \left[-\frac{1}{3}\cos(3x)\right]_0^{\pi/3}$ **M1**

$= -\frac{1}{3}\cos\pi + \frac{1}{3}\cos 0 = -\frac{1}{3}(-1) + \frac{1}{3}(1) = \dfrac{2}{3}$ **A1**

Q8 — Reverse Chain Rule on Power (Easy)

(a) $\frac{d}{dx}(x^2+1) = 2x$; write $8x = 4 \times 2x$: **M1**

$$\int 8x(x^2+1)^3\,dx = \frac{4(x^2+1)^4}{4} + C = (x^2+1)^4 + C$$

A1

(b) $\left[(x^2+1)^4\right]_0^1 = 2^4 - 1^4 = 16 - 1 = 15$ **A1**

Q9 — Volume of Revolution about x-axis (Easy)

(a) $V = \pi\displaystyle\int_0^9 (\sqrt{x})^2\,dx = \pi\int_0^9 x\,dx$ **A1**

(b) $V = \pi\left[\frac{x^2}{2}\right]_0^9 = \pi \cdot \frac{81}{2} = \dfrac{81\pi}{2}$ **M1 A1**

Note: Exact answer required: $\frac{81\pi}{2}$.

Q10 — Negative Integrals (Easy)

(a) $\displaystyle\int_2^5 (x^2-9)\,dx = \left[\frac{x^3}{3} - 9x\right]_2^5 = \left(\frac{125}{3} - 45\right) - \left(\frac{8}{3} - 18\right) = \frac{117}{3} - 27 = 39 - 27 = 12$ **M1 A1**

(b) The curve $y = x^2 - 9$ is below the x-axis for $2 \le x < 3$ (since $x^2 < 9$ there), so the integral over $[2,3]$ is negative. This negative contribution is subtracted from the positive area over $[3,5]$ in the definite integral, giving a net signed value of 12 rather than the geometric (total) area. **R1**

Q11 — Exact Integral in ln Form (Medium)

(a) $\frac{d}{dx}(x^3 - x) = 3x^2 - 1$; note $6x^2 - 2 = 2(3x^2-1)$. **M1**

$$\int \frac{6x^2-2}{x^3-x}\,dx = 2\ln|x^3-x| + C$$

A1

(b) $\left[2\ln|x^3-x|\right]_2^4 = 2\ln(60) - 2\ln(6) = 2\ln\dfrac{60}{6} = 2\ln 10$ **M1 A1**

Note: $a = 2$, $b = 10$. *FT from (a).*

Q12 — Substitution with Trig (Medium)

(a) Let $u = 3x^2$, $du = 6x\,dx \Rightarrow x\,dx = \frac{du}{6}$. **M1**

$$\int x\sin(3x^2)\,dx = \int \sin u \cdot \frac{du}{6} = -\frac{1}{6}\cos u + C = -\frac{1}{6}\cos(3x^2) + C \quad \textbf{AG}$$

A1

(b) Limits: $x = 0 \Rightarrow u = 0$; $\quad x = \sqrt{\pi/6} \Rightarrow u = 3 \cdot \frac{\pi}{6} = \frac{\pi}{2}$. **M1**

$$\left[-\frac{1}{6}\cos u\right]_0^{\pi/2} = -\frac{1}{6}\cos\frac{\pi}{2} + \frac{1}{6}\cos 0 = 0 + \frac{1}{6} = \frac{1}{6}$$

A1

Q13 — Area Between Two Curves (Medium)

(a) Solve $x^2 - 2x = 4 - x^2$: $2x^2 - 2x - 4 = 0$, $x^2 - x - 2 = 0$, $(x-2)(x+1) = 0$ **(M1)**
$x = -1$ and $x = 2$ **A1**

(b) $g(x) - f(x) = (4 - x^2) - (x^2 - 2x) = -2x^2 + 2x + 4$. Since $g > f$ on $(-1, 2)$: **M1**

$$\int_{-1}^{2}(-2x^2 + 2x + 4)\,dx = \left[-\tfrac{2x^3}{3} + x^2 + 4x\right]_{-1}^{2}$$

$= (-\frac{16}{3} + 4 + 8) - (\frac{2}{3} + 1 - 4) = \frac{20}{3} - (-\frac{7}{3}) = \frac{27}{3} = 9$ **A1 A1**
Note: First A1 for correct integrand; second A1 for exact answer 9.

Q14 — Area Enclosed Between Curve and x-axis (Medium)

(a) $x^3 - 3x = x(x^2 - 3) = 0 \Rightarrow x = 0,\ x = \sqrt{3},\ x = -\sqrt{3}$ **A1**

(b) On $(-\sqrt{3}, 0)$: $x^3 - 3x > 0$ (above x-axis). On $(0, \sqrt{3})$: $x^3 - 3x < 0$ (below). **M1**

$\int_{-\sqrt{3}}^{0}(x^3 - 3x)\,dx = \left[\frac{x^4}{4} - \frac{3x^2}{2}\right]_{-\sqrt{3}}^{0} = 0 - (\frac{9}{4} - \frac{9}{2}) = \frac{9}{4}$

By symmetry (odd function): $\left|\int_0^{\sqrt{3}}(x^3 - 3x)\,dx\right| = \frac{9}{4}$ **A1**

Total area $= \frac{9}{4} + \frac{9}{4} = \dfrac{9}{2}$ **A1**

Note: Accept GDC numerical answer. FT from (a).

Q15 — Volume of Revolution: Composite Region (Medium)

(a) Sketch: $y = e^x + 1$ is always above the x-axis. At $x = 0$: $y = 2$; at $x = 2$: $y = e^2 + 1 \approx 8.39$. Label $(0, 2)$ and $(2,\, e^2 + 1)$. **A1**

(b) $V = \pi \displaystyle\int_0^2 (e^x + 1)^2\,dx = \pi \int_0^2 (e^{2x} + 2e^x + 1)\,dx$ **M1**

$= \pi\left[\frac{e^{2x}}{2} + 2e^x + x\right]_0^2 = \pi\left[\left(\frac{e^4}{2} + 2e^2 + 2\right) - \left(\frac{1}{2} + 2\right)\right]$ **A1**

$= \pi\left(\frac{e^4}{2} + 2e^2 - \frac{1}{2}\right) = \pi(27.299 + 14.778 - 0.5) = 131$ (3 sf) **A1**

Note: Full precision: $130.58\ldots$; *accept* 131.

Q16 — Substitution (Rational/Root) (Medium)

(a) Let $u = x^2 + 4$, $du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$. **M1**

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \int \frac{du/2}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{x^2+4} + C$$

A1 A1

(b) Limits: $x = 0 \Rightarrow \sqrt{4} = 2$; $\quad x = \sqrt{5} \Rightarrow \sqrt{9} = 3$. $\quad$ Result: $3 - 2 = 1$ **A1**

Q17 — Negative Integrals / Symmetry (Medium)

(a) $\displaystyle\int_{-2}^{2} (x^3 - 4x)\,dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{2} = (4-8) - (4-8) = 0$ **M1 A1**

This implies $A = B$: the positive region A above the axis (left lobe) exactly equals the negative region B below (right lobe), so they cancel in the signed integral.

(b) By odd symmetry: $A = \displaystyle\int_{-2}^{0} (x^3 - 4x)\,dx = \int_0^2 (4x - x^3)\,dx$ **M1**

$= \left[2x^2 - \frac{x^4}{4}\right]_0^2 = 8 - 4 = 4$; $\quad$ Total area $A + B = 2 \times 4 = 4$ **A1**

Q18 — Volume About y-axis (Medium)

(a) $x^2 = y$ **A1**

(b) $V = \pi \int_0^9 x^2\,dy = \pi \int_0^9 y\,dy$ **M1 A1**

$= \pi \left[\frac{y^2}{2}\right]_0^9 = \pi \cdot \frac{81}{2} = \dfrac{81\pi}{2}$ **A1**

Q19 — Area with Respect to y-axis (Medium)

(a) Area of $R = \int_0^3 (3y - y^2)\,dy$ **A1**

(b) $\int_0^3 (3y - y^2)\,dy = \left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3$ **M1 A1**

$= (\frac{27}{2} - 9) - 0 = \frac{27}{2} - \frac{18}{2} = \dfrac{9}{2}$ **A1**

Note: Accept 4.5.

Q20 — Substitution (Trig) (Medium)

(a) Let $u = \cos x$, $du = -\sin x\,dx$: **M1**

$$\int \sin x \cos^3 x\,dx = \int u^3(-du) = -\frac{u^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

A1 A1

(b) $\left[-\frac{\cos^4 x}{4}\right]_0^{\pi/2} = -\frac{0}{4} + \frac{1}{4} = \dfrac{1}{4}$ **A1**

Q21 — Enclosed Area with Parameter (Hard)

(a) Intersect when $x^2 = kx \Rightarrow x(x-k) = 0$. Since $k > 0$, the non-origin intersection is at $x = k$. **AG R1**

(b) For $0 \le x \le k$: $g(x) - f(x) = kx - x^2 \ge 0$. **M1**

$$\text{Area} = \int_0^k (kx - x^2)\,dx = \left[\frac{kx^2}{2} - \frac{x^3}{3}\right]_0^k = \frac{k^3}{2} - \frac{k^3}{3} = \frac{k^3}{6} \quad \textbf{AG}$$

A1 A1

(c) $\dfrac{k^3}{6} = \dfrac{4}{3} \Rightarrow k^3 = 8 \Rightarrow k = 2$ **A1**

Q22 — Volume About y-axis (Container Model) (Hard)

(a) At $y = 0$: $x = \sqrt{1} = 1$ cm. At $y = 8$: $x = \sqrt{9} = 3$ cm. The radius increases from 1 cm at the base to 3 cm at the top; the container widens with height. **A1 A1**

(b) $x^2 = y + 1$, so: **M1**

$$V = \pi \int_0^8 (y+1)\,dy = \pi\left[\tfrac{y^2}{2} + y\right]_0^8 = \pi(32+8) = 40\pi \approx 126 \text{ cm}^3$$

A1

(c) $V_{\text{water}} = \pi \displaystyle\int_0^5 (y+1)\,dy = \pi\left[\tfrac{y^2}{2} + y\right]_0^5 = \pi(\tfrac{25}{2} + 5) = \tfrac{35\pi}{2} \approx 55.0 \text{ cm}^3$ **A1**

Q23 — Substitution with Rational Power (Hard)

(a) $\dfrac{d}{du}\left(\dfrac{-1}{u^2}\right) = \dfrac{d}{du}(-u^{-2}) = 2u^{-3} = \dfrac{2}{u^3}$ **A1**

(b) Let $u = x^2 - 3$, $du = 2x\,dx$: **M1**

$$\int \frac{4x}{(x^2-3)^3}\,dx = \int \frac{2\,du}{u^3} = 2\int u^{-3}\,du = 2\cdot\frac{u^{-2}}{-2} + C = \frac{-1}{u^2} + C = \frac{-1}{(x^2-3)^2} + C \quad \textbf{AG}$$

A1

(c) $\left[\dfrac{-1}{(x^2-3)^2}\right]_2^3 = \dfrac{-1}{(9-3)^2} - \dfrac{-1}{(4-3)^2} = \dfrac{-1}{36} + 1 = \dfrac{35}{36}$ **M1 A1**

Q24 — Area with Parameter (Hard)

(a) At $x = 0$: $y_C = 0$ and $y_\ell = 0$. Both pass through the origin. **AG** **R1**

(b) $x^3 - 6x^2 + 9x = mx \Rightarrow x^3 - 6x^2 + (9-m)x = 0 \Rightarrow x[x^2 - 6x + (9-m)] = 0$ **M1**

Quadratic: $x = \dfrac{6 \pm \sqrt{36 - 4(9-m)}}{2} = 3 \pm \sqrt{m}$ **A1**

(c) When $m = 5$: intersections at $x = 0$, $x = 3 - \sqrt{5} \approx 0.764$, $x = 3 + \sqrt{5} \approx 5.236$. **(M1)**

$C - \ell = x^3 - 6x^2 + 4x$. Using GDC to find $\int_0^{3-\sqrt{5}} (x^3 - 6x^2 + 4x)\,dx$ and $\left|\int_{3-\sqrt{5}}^{3+\sqrt{5}} (x^3 - 6x^2 + 4x)\,dx\right|$:

Total area $\approx 1.45 + 32.0 = 33.5$ (3 sf) **A1**

Note: Exact total area = $4\sqrt{5}(5 - \sqrt{5}/3)$*... Accept GDC numerical answer* ≈ 33.5.

Q25 — Volume Between Two Curves (Hard)

(a) $\sqrt{x} = x^2 \Rightarrow x = x^4 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$. No other solutions in $[0, 1]$. **AG** **R1**

(b) For $0 < x < 1$: $\sqrt{x} > x^2$, so washer with outer radius $\sqrt{x}$, inner radius x^2: **M1**

$$V = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2]\, dx = \pi \int_0^1 (x - x^4)\, dx \quad \textbf{AG}$$

A1

(c) $V = \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi(\frac{1}{2} - \frac{1}{5}) = \pi \cdot \frac{3}{10} = \dfrac{3\pi}{10}$ **A1 A1**

Q26 — Substitution with Rational Function (Hard)

(a) Let $u = 2x - 1$, $x = \frac{u+1}{2}$, $dx = \frac{du}{2}$: **M1**

$$\int \frac{x}{(2x-1)^3}\, dx = \int \frac{(u+1)/2}{u^3} \cdot \frac{du}{2} = \frac{1}{4} \int \frac{u+1}{u^3}\, du = \frac{1}{4} \int (u^{-2} + u^{-3})\, du$$

A1

$= \dfrac{1}{4}(-u^{-1} - \frac{1}{2}u^{-2}) + C$ **A1**

$= \dfrac{-1}{4(2x-1)} - \dfrac{1}{8(2x-1)^2} + C$ **A1**

(b) $\left[\dfrac{-1}{4(2x-1)} - \dfrac{1}{8(2x-1)^2}\right]_1^2$:

At $x = 2$: $-\frac{1}{12} - \frac{1}{72} = -\frac{7}{72}$; at $x = 1$: $-\frac{1}{4} - \frac{1}{8} = -\frac{3}{8}$ **(M1)**

$= -\frac{7}{72} - (-\frac{27}{72}) = \dfrac{20}{72} = \dfrac{5}{18}$ **A1**

Q27 — Area w.r.t. y-axis and Complementary Region (Hard)

(a) $\displaystyle\int_0^2 e^y\, dy = \left[e^y\right]_0^2 = e^2 - 1$ **M1 A1 A1**

(b) The rectangle O-A-B-D has dimensions e^2 (width) $\times$ 2 (height), so area $= 2e^2$. **M1**

The region under $y = \ln x$ (above x-axis, left of $x = e^2$) $=$ Rectangle $-$ region from (a):

$$\text{Area} = 2e^2 - (e^2 - 1) = e^2 + 1$$

A1

Q28 — Reverse Chain Rule and Volume (Hard)

(a) $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$, so $xe^{x^2} = \frac{1}{2}\frac{d}{dx}(e^{x^2})$: **M1**

$$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

A1

(b)(i) $V = \pi\int_0^2 (xe^{x^2})^2\,dx = \pi\int_0^2 x^2e^{2x^2}\,dx$ **A1**

(b)(ii) The integrand $x^2e^{2x^2}$ contains e^{x^2} whose antiderivative is the Gaussian error function (non-elementary); no closed-form antiderivative exists. **R1**

Using GDC: $V = \pi \times 57.98\ldots \approx 182$ (3 sf) **A1**

Note: Accept 182–183.

Q29 — Trig and Linear Area + Volume (Hard)

(a) $\sin\left(\frac{\pi}{2}\right) = 1$ and $\frac{2}{\pi}\cdot\frac{\pi}{2} = 1$. Both equal 1, confirming intersection. **AG** **A1**

(b) On $(0, \frac{\pi}{2})$: $\sin x > \frac{2}{\pi}x$ (the sine curve is above the line). **M1**

$$\text{Area} = \int_0^{\pi/2}\left(\sin x - \frac{2x}{\pi}\right)dx = \left[-\cos x - \frac{x^2}{\pi}\right]_0^{\pi/2}$$

A1

$= \left(0 - \frac{\pi}{4}\right) - (-1 - 0) = 1 - \frac{\pi}{4}$ **A1**

(c) $V = \pi\int_0^{\pi/2}\left(\sin^2 x - \frac{4x^2}{\pi^2}\right)dx \approx 0.452$ (3 sf) **A1**

Note: GDC numerical answer; accept 0.452.

Q30 — Volume About y-axis with Unknown Constant (Hard)

(a) $y = x^2 - 1 \Rightarrow x^2 = y + 1$ **A1**

(b) The region R (for $x \geq 0$) is bounded on the right by $x = \sqrt{y+1}$, on the left by the y-axis ($x = 0$), between $y = -1$ and $y = k$. **M1**

$$V = \pi\int_{-1}^{k}(\sqrt{y+1})^2\,dy = \pi\int_{-1}^{k}(y+1)\,dy \quad \textbf{AG}$$

A1

(c) $\pi\left[\frac{(y+1)^2}{2}\right]_{-1}^{k} = \pi \cdot \frac{(k+1)^2}{2} = 18\pi$ **M1**

$(k+1)^2 = 36 \Rightarrow k+1 = 6$ (taking positive root since $k > -1$) $\Rightarrow k = 5$ **A1**