IB Math AI HL — Topic 1 [Set 1]

Complex Numbers

Intro to $\mathbb{C}$ · Operations · Complex Roots · Modulus & Argument · Argand Diagrams

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Marks each	3	4	5
Total Marks	30	40	50

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Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers in exact form or correct to 3 significant figures unless stated. GDC permitted throughout.

Key Formulae — Complex Numbers

Rectangular form: $z = a + bi, \quad i^2 = -1$

Conjugate: $\bar{z} = a - bi$

Modulus: $|z| = \sqrt{a^2 + b^2}$

Argument: $\arg z = \arctan\left(\frac{b}{a}\right)$ (adjusted to correct quadrant)

Polar form: $z = r(\cos\theta + i\sin\theta)$

Exponential form: $z = re^{i\theta}$

Product: $|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2$

Quotient: $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Discriminant: $\Delta = b^2 - 4ac$; complex roots when $\Delta < 0$

Quadratic formula: $z = \frac{-b \pm \sqrt{\Delta}}{2a}$

Complex roots come in conjugate pairs if coefficients are real.

Argument range: $-\pi < \theta \leq \pi$ (principal value)

GDC Tips

- **Casio ClassPad:** Use the `Action` $\rightarrow$ `Complex` menu. Set `Set Up` $\rightarrow$ `Angle` to **Radian** for arguments. `abs(z)` gives $|z|$; `arg(z)` gives $\arg z$.
- **TI-Nspire:** Switch to **Rectangular** or **Polar** mode under `Document Settings`. Enter i as the key labelled i. Use `angle()` and `abs()` for argument and modulus.
- To solve a quadratic with complex roots, use the **Polynomial Root Finder** (ClassPad) or `cSolve()` (TI-Nspire).
- Always write the argument in radians unless degrees are specifically requested.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY No Calculator **[3 marks]**

Let $z_1 = 3 + 5i$ and $z_2 = 1 - 2i$.

(a) Write down Re(z_1) and Im(z_2). [1]

(b) Find $z_1 + z_2$. [1]

(c) Find $z_1 - z_2$. [1]

2. EASY No Calculator **[3 marks]**

Let $z = 4 - 3i$.

(a) Write down $\bar{z}$, the complex conjugate of z. [1]

(b) Find $z\bar{z}$. Give your answer as a real number. [2]

3. EASY No Calculator **[3 marks]**

Consider the complex number $w = -5 + 12i$.

(a) Find $|w|$. [2]

(b) Write down $|\bar{w}|$. [1]

4. EASY No Calculator **[3 marks]**

Consider the quadratic equation $x^2 - 4x + 13 = 0$.

(a) Write down the value of the discriminant. [1]

(b) Hence write down the nature of the roots of this equation. [1]

(c) Solve the equation, writing your answers in the form $a \pm bi$ where $a, b \in \mathbb{Z}$. [1]

5. EASY No Calculator **[3 marks]**

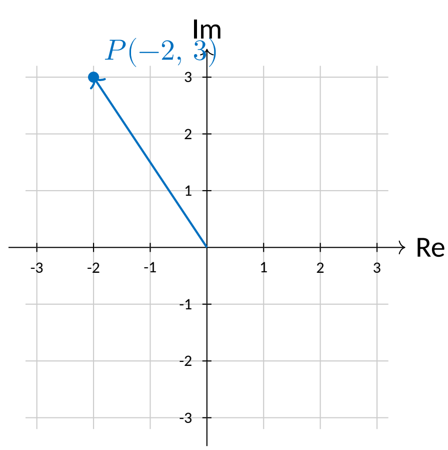
Let $z = 1 + \sqrt{3}\,i$.

(a) Find $|z|$. [1]

(b) Find arg z, giving your answer as an exact multiple of π. [2]

6. EASY Calculator **[3 marks]**

The Argand diagram below shows the point P representing a complex number z.



(a) Write down z in the form $a + bi$. [1]

(b) Find $|z|$. [1]

(c) State the quadrant of the Argand diagram in which z lies. [1]

7. EASY No Calculator **[3 marks]**

Let $p = 2 + i$ and $q = 3 - 4i$.

(a) Find pq, giving your answer in the form $a + bi$. [2]

(b) Write down $\text{Re}(pq)$. [1]

8. EASY No Calculator **[3 marks]**

Solve the equation $z^2 + 6z + 25 = 0$, giving your answers in the form $a \pm bi$.

9. EASY No Calculator **[3 marks]**

Let $z = -\sqrt{3} + i$.

(a) Show that $|z| = 2$. [1]

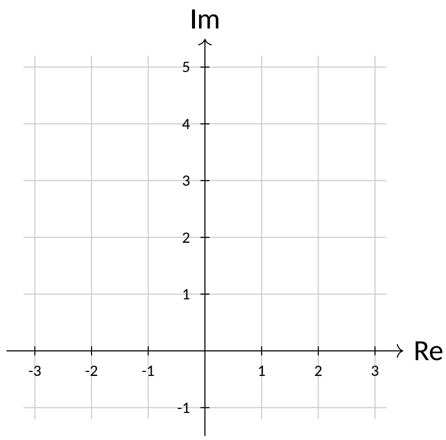
(b) Find $\arg z$, giving your answer as an exact multiple of π. [2]

10. **EASY** **Calculator** **[3 marks]**

Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

(a) Find $z_1 + z_2$. [1]

(b) Plot z_1, z_2, and $z_1 + z_2$ on the Argand diagram below, labelling each point clearly. [2]



Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. **MEDIUM** **No Calculator** **[4 marks]**

Let $z_1 = 3 + 4i$ and $z_2 = 1 + 2i$.

(a) Find $\dfrac{z_1}{z_2}$, giving your answer in the form $a + bi$ where $a, b \in \mathbb{Q}$. [3]

(b) Hence write down $\left|\dfrac{z_1}{z_2}\right|$. [1]

12. **MEDIUM** **No Calculator** **[4 marks]**

Let $z = -2\sqrt{3} + 2i$.

(a) Find $|z|$ and arg z, giving arg z as an exact multiple of π. [3]

(b) Hence write z in the form $re^{i\theta}$. [1]

13. MEDIUM No Calculator **[4 marks]**

The quadratic equation $x^2 + (k-1)x + 3k = 0$, where $k \in \mathbb{R}$, has two complex (non-real) roots.

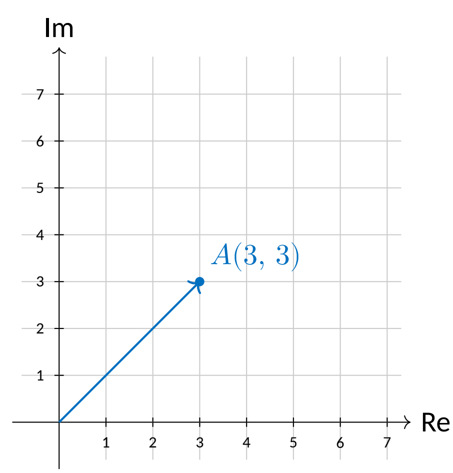
(a) Show that the discriminant is $(k-1)^2 - 12k$. [1]

(b) Find the set of values of k for which the equation has complex roots. [3]

14. MEDIUM Calculator **[4 marks]**

Let $z = 3 + 3i$ and $w = 1 + i\sqrt{3}$.

The Argand diagram below shows the point A representing z.



(a) Find zw, giving your answer in the form $a + bi$. [2]

(b) Plot the point B representing zw on the Argand diagram above. [1]

(c) Describe the geometrical transformation that maps A to B, stating the scale factor and angle of rotation. [1]

15. MEDIUM No Calculator **[4 marks]**

Let $z = 2 - 5i$.

(a) Find z^2, giving your answer in the form $a + bi$. [2]

(b) Hence find Im$(z^2 - 3z + 10)$. [2]

16. MEDIUM No Calculator **[4 marks]**

The equation $z^2 - 2z + 10 = 0$ has two complex roots z_1 and z_2.

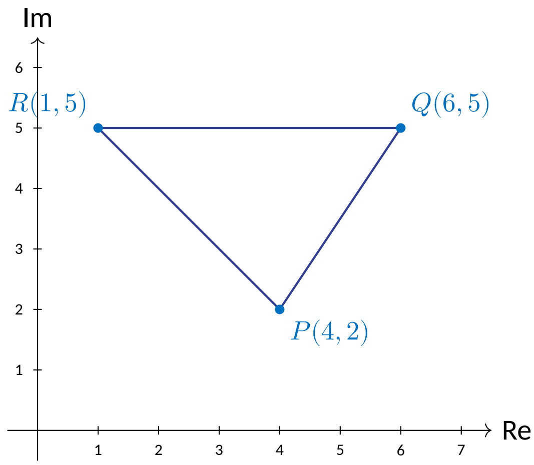
(a) Find z_1 and z_2. [2]

(b) Find $|z_1|$ and arg z_1, where Im$(z_1) > 0$. Give arg z_1 correct to 3 significant figures (in radians). [2]

17. MEDIUM Calculator **[4 marks]**

Points P, Q, and R on an Argand diagram represent the complex numbers $z_P = 4 + 2i$, $z_Q = 6 + 5i$, and $z_R = 1 + 5i$ respectively.

The diagram below shows triangle PQR (diagram not to scale).



(a) Find $|z_Q - z_R|$. [1]

(b) Find the area of triangle PQR. [3]

18. MEDIUM No Calculator **[4 marks]**

Let $z = a + bi$ where $a, b \in \mathbb{R}$.

(a) Show that $z\bar{z} = a^2 + b^2$. [2]

(b) Hence, given that $z = 5 - 12i$, find $z\bar{z}$ and $|z|$. [2]

19. MEDIUM No Calculator **[4 marks]**

A quadratic equation with real coefficients has one root $z_1 = 3 - 2i$.

(a) Write down the other root z_2. [1]

(b) Find the quadratic equation in the form $z^2 + bz + c = 0$ where $b, c \in \mathbb{Z}$. [3]

20. MEDIUM No Calculator **[4 marks]**

Let $z = -4 - 4i$.

(a) Find $|z|$ and arg z, giving arg z as an exact multiple of π. [3]

(b) Hence write z in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. [1]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **No Calculator** **[5 marks]**

Let $z = 1 + i$.

(a) Write z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

(b) Hence find z^6 in the form $a + bi$ where $a, b \in \mathbb{Z}$. [2]

(c) Find the smallest positive integer n such that z^n is a negative real number. [1]

22. **HARD** **Calculator** **[5 marks]**

Let $f(z) = iz + 2$, where $z \in \mathbb{C}$.

(a) Find $f(3 + i)$. [2]

(b) Describe the two successive geometrical transformations on the Argand plane that together map z to $f(z)$, stating the order in which they are applied. [2]

(c) Find the value of z for which $f(z) = 3 - 5i$. [1]

23. HARD No Calculator **[5 marks]**

The equation $z^2 - (m+2)z + (m^2 - m + 1) = 0$, where $m \in \mathbb{R}$, has roots α and β.

(a) Write down expressions for $\alpha + \beta$ and $\alpha\beta$ in terms of m. [2]

(b) Find the value(s) of m for which the equation has equal roots. [3]

24. HARD No Calculator **[5 marks]**

A geometric sequence has first term $u_1 = 4$ and common ratio $r = \dfrac{1}{2}e^{i\pi/3}$.

(a) Write down $|r|$ and $\arg r$. [2]

(b) Find u_4, giving your answer in the form $a + bi$ where $a, b \in \mathbb{R}$. Leave surds in your answer. [2]

(c) Explain why the sum to infinity S_∞ of this sequence exists, and find $|S_\infty|$. [1]

25. HARD Calculator **[5 marks]**

Let $z_1 = 2\sqrt{3} + 2i$ and $z_2 = 1 + i$.

(a) Write z_1 and z_2 in exponential form $re^{i\theta}$, where $-\pi < \theta \leq \pi$. [3]

(b) Hence find $\frac{z_1}{z_2}$, giving your answer in the form $a + bi$. [2]

26. HARD No Calculator **[5 marks]**

Let $z = x+iy$ where $x, y \in \mathbb{R}$. Given that $z+3\bar{z} = 8+4i$, find x and y.

27. HARD No Calculator **[5 marks]**

Let $z_1 = 6$, $z_2 = 2 + 4i$, and $z_3 = -2 + 2i$.

(a) Show that $|z_2 - z_1| = 4\sqrt{2}$. [2]

(b) Find $|z_3 - z_1|$. [1]

(c) The points A, B, C on an Argand diagram represent z_1, z_2, z_3 respectively. Find the area of triangle ABC. [2]

28. HARD No Calculator **[5 marks]**

Let $w = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$.

(a) Show that $|w| = 1$ and find $\arg w$ as an exact multiple of π. [2]

(b) Find w^{12}, giving your answer as an exact real or complex number. [2]

(c) Write down the value of w^{24}. [1]

29. HARD No Calculator **[5 marks]**

A quadratic equation $z^2 + pz + q = 0$, where $p, q \in \mathbb{R}$, has a root $z = 2 + 5i$.

(a) Write down the other root. [1]

(b) Find the values of p and q. [4]

30. HARD No Calculator **[5 marks]**

Let $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$.

(a) Write z_1 and z_2 in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

(b) Find $z_1 z_2$, giving your answer in the form $a + bi$ where $a, b \in \mathbb{Z}$. [1]

(c) Using your results from (a), find $\arg(z_1 z_2)$. Give your answer as an exact multiple of π. [1]

(d) Hence show that $\tan\left(-\frac{\pi}{12}\right) = \sqrt{3} - 2$. [1]

Worked Solutions

Official Mark-Scheme Style — M/A/R Notation

Section A — Easy

Q1 — Introduction to Complex Numbers [Easy]

(a) Re$(z_1) = 3$;   Im$(z_2) = -2$ **A1**

(b) $z_1 + z_2 = (3+1) + (5-2)i = 4 + 3i$ **A1**

(c) $z_1 - z_2 = (3-1) + (5-(-2))i = 2 + 7i$ **A1**

Q2 — Conjugate and $z\bar{z}$ [Easy]

(a) $\bar{z} = 4 + 3i$ **A1**

(b) $z\bar{z} = (4-3i)(4+3i) = 16 + 12i - 12i - 9i^2$ **M1**

$= 16 + 9 = 25$ **A1**

Note: Accept $a^2 + b^2 = 16 + 9 = 25$ directly.

Q3 — Modulus [Easy]

(a) $|w| = \sqrt{(-5)^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169}$ **M1**

$= 13$ **A1**

(b) $|\bar{w}| = 13$ **A1**

Note: The modulus of a conjugate equals the modulus of the original.

Q4 — Discriminant and Complex Roots [Easy]

(a) $\Delta = (-4)^2 - 4(1)(13) = 16 - 52 = -36$ **A1**

(b) Since $\Delta < 0$, the roots are non-real (complex conjugate pair). **A1**

(c) $x = \dfrac{4 \pm \sqrt{-36}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$ **A1**

Q5 — Modulus and Argument [Easy]

(a) $|z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$ **A1**

(b) z lies in the first quadrant; arg $z = \arctan\left(\frac{\sqrt{3}}{1}\right) = \arctan(\sqrt{3})$ **M1**

arg $z = \frac{\pi}{3}$ **A1**

Q6 — Reading an Argand Diagram [Easy]

(a) $z = -2 + 3i$ **A1**

(b) $|z| = \sqrt{(-2)^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$ **A1**

Note: Accept 3.61 *(3 s.f.).*

(c) Second quadrant (Re < 0, Im > 0). **A1**

Q7 — Multiplication of Complex Numbers [Easy]

(a) $pq = (2+i)(3-4i) = 6 - 8i + 3i - 4i^2$ **M1**

$= 6 - 5i + 4 = 10 - 5i$ **A1**

(b) Re$(pq) = 10$ **A1**

Q8 — Solving a Quadratic with Complex Roots [Easy]

Using the quadratic formula: **M1**

$z = \frac{-6 \pm \sqrt{36-100}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2}$ **A1**

$z = -3 + 4i$ or $z = -3 - 4i$ **A1**

Q9 — Modulus and 2nd-Quadrant Argument (Show That) [Easy]

(a) $|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$ **A1 AG**

Note: Both intermediate steps must be shown; the AG line carries no mark.

(b) Reference angle: $\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ **M1**

z lies in the second quadrant, so arg $z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ **A1**

Q10 — Addition and Argand Plot [Easy]

(a) $z_1 + z_2 = (2-1) + (1+3)i = 1 + 4i$ **A1**

(b) Points correctly plotted and labelled: **A1A1**

$z_1 = 2 + i$ at $(2, 1)$; $z_2 = -1 + 3i$ at $(-1, 3)$; $z_1 + z_2 = 1 + 4i$ at $(1, 4)$.

Note: Award A1 for any two correctly placed and labelled; A1 for the third.

Section B — Medium

Q11 — Division via Conjugate [Medium]

(a) Multiply numerator and denominator by $\bar{z}_2 = 1 - 2i$: **M1**

$$\frac{z_1}{z_2} = \frac{(3+4i)(1-2i)}{(1+2i)(1-2i)} = \frac{3 - 6i + 4i - 8i^2}{1+4}$$ **M1**

$$= \frac{3 - 2i + 8}{5} = \frac{11 - 2i}{5} = \frac{11}{5} - \frac{2}{5}i$$ **A1**

(b) $\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{11}{5}\right)^2 + \left(\frac{2}{5}\right)^2} = \frac{\sqrt{121+4}}{5} = \frac{\sqrt{125}}{5} = \sqrt{5}$ **A1**

Note: Accept METHOD 2: $|z_1|/|z_2| = 5/\sqrt{5} = \sqrt{5}$ directly.

Q12 — Conversion to Exponential Form [Medium]

(a) $|z| = \sqrt{(-2\sqrt{3})^2 + 2^2} = \sqrt{12+4} = \sqrt{16} = 4$ **A1**

Reference angle: $\arctan\left(\frac{2}{2\sqrt{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ **M1**

z is in the second quadrant, so $\arg z = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ **A1**

(b) $z = 4e^{i5\pi/6}$ **A1**

Q13 — Parameter for Complex Roots [Medium]

(a) $\Delta = (k-1)^2 - 4(1)(3k) = k^2 - 2k + 1 - 12k = k^2 - 14k + 1$

Note: Re-read question — discriminant is $(k-1)^2 - 12k$.

$(k-1)^2 - 4(3k) = k^2 - 2k + 1 - 12k = (k-1)^2 - 12k$ **A1 AG**

(b) Complex roots require $\Delta < 0$: **M1**

$k^2 - 14k + 1 < 0$

Roots of $k^2 - 14k + 1 = 0$: $k = \frac{14 \pm \sqrt{196-4}}{2} = \frac{14 \pm \sqrt{192}}{2} = 7 \pm 4\sqrt{3}$ **A1**

So $7 - 4\sqrt{3} < k < 7 + 4\sqrt{3}$ **A1**

Note: Accept $0.0718 < k < 13.93$ *(3 s.f.). FT if M1 earned.*

Q14 — Multiplication and Geometrical Transformation [Medium]

(a) $zw = (3+3i)(1+i\sqrt{3}) = 3 + 3i\sqrt{3} + 3i + 3i^2\sqrt{3}$ **M1**

$= 3 + 3i\sqrt{3} + 3i - 3\sqrt{3} = (3 - 3\sqrt{3}) + (3 + 3\sqrt{3})i$ **A1**

Numerically: $(3 - 5.196\ldots) + (3 + 5.196\ldots)i \approx -2.196 + 8.196i$

(b) B plotted at $(3 - 3\sqrt{3},\ 3 + 3\sqrt{3})$ on the diagram, i.e. approximately $(-2.20,\ 8.20)$. **A1**

(c) $|w| = \sqrt{1^2 + (\sqrt{3})^2} = 2$; arg $w = \dfrac{\pi}{3}$.

Enlargement scale factor 2, rotation by $\dfrac{\pi}{3}$ (i.e. $60°$) anticlockwise about the origin. **A1**

Q15 — Squaring a Complex Number and Im-Part Chain [Medium]

(a) $z^2 = (2-5i)^2 = 4 - 20i + 25i^2 = 4 - 20i - 25 = -21 - 20i$ **M1 A1**

(b) $z^2 - 3z + 10 = (-21 - 20i) - 3(2 - 5i) + 10$ **M1**

$= -21 - 20i - 6 + 15i + 10 = -17 - 5i$

Im$(z^2 - 3z + 10) = -5$ **A1**

Q16 — Complex Roots and Modulus/Argument [Medium]

(a) $\Delta = 4 - 40 = -36$; roots: $z = \dfrac{2 \pm 6i}{2} = 1 \pm 3i$ **M1 A1**

(b) $z_1 = 1 + 3i$ (taking Im$(z_1) > 0$)

$|z_1| = \sqrt{1+9} = \sqrt{10}$ **A1**

arg z_1 = arctan$(3) = 1.2490\ldots \approx 1.25$ rad **A1**

Q17 — Triangle Area on Argand Diagram [Medium]

(a) $z_Q - z_R = (6+5i) - (1+5i) = 5$, so $|z_Q - z_R| = 5$ **A1**

(b) Base QR: length $= 5$ (calculated above). **A1**

Height from P to line QR: QR lies along Im $= 5$; P has Im $= 2$.

Height $= 5 - 2 = 3$. **A1**

Area $= \frac{1}{2} \times 5 \times 3 = 7.5$ (square units) **A1**

Note: FT from their base and height. Award A1A1A1 for base, height, area.

Q18 — Show That $z\bar{z} = a^2 + b^2$ [Medium]

(a) $z\bar{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2i^2$ **M1**

$= a^2 - b^2(-1) = a^2 + b^2$ **A1 AG**

(b) $z\bar{z} = 5^2 + (-12)^2 = 25 + 144 = 169$ **A1**

$|z| = \sqrt{169} = 13$ **A1**

Note: FT: award A1A1 for consistent answers using their $z\bar{z}$.

Q19 — Sum and Product of Roots to Rebuild Quadratic [Medium]

(a) $z_2 = 3 + 2i$ (conjugate root theorem) **A1**

(b) Sum of roots: $z_1 + z_2 = (3-2i) + (3+2i) = 6$ **A1**

Product of roots: $z_1 z_2 = (3-2i)(3+2i) = 9 + 4 = 13$ **A1**

Equation: $z^2 - 6z + 13 = 0$ **A1**

Note: METHOD 2: expand $(z-(3-2i))(z-(3+2i))$ and simplify.

Q20 — Modulus, 3rd-Quadrant Argument, Exponential Form [Medium]

(a) $|z| = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$ **A1**

Reference angle: $\arctan\left(\frac{4}{4}\right) = \frac{\pi}{4}$ **M1**

z is in the third quadrant: $\arg z = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$ **A1**

(b) $z = 4\sqrt{2}\, e^{-3\pi i/4}$ **A1**

Note: Do not accept $\arg z = \frac{5\pi}{4}$ *(outside principal range).*

Section C — Hard

Q21 — Power via Polar Form, Smallest n [Hard]

(a) $|z| = \sqrt{2}; \quad \arg z = \frac{\pi}{4}$ **A1**

$z = \sqrt{2}\, e^{i\pi/4}$ **A1**

(b) $z^6 = (\sqrt{2})^6 e^{i6\pi/4} = 8\, e^{i3\pi/2}$ **M1**

$= 8(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 8(0 - i) = -8i$ **A1**

(c) z^n is real when $n \cdot \frac{\pi}{4}$ is a multiple of π, i.e. n is a multiple of 4.

z^n is *negative* real when $n \cdot \frac{\pi}{4} \equiv \pi \pmod{2\pi}$, i.e. $n \equiv 4 \pmod{8}$.

Smallest positive integer: $n = 4$ **A1**

Check: $z^4 = (\sqrt{2})^4 e^{i\pi} = 4 \cdot (-1) = -4$ ✓

Q22 — Linear Complex Map, Transformations, Inverse [Hard]

(a) $f(3+i) = i(3+i) + 2 = 3i + i^2 + 2 = -1 + 3i + 2 = 1 + 3i$ **M1 A1**

(b) Multiplying by i is a rotation of $\dfrac{\pi}{2}$ (90°) anticlockwise about the origin. **A1**

Adding 2 is a translation by $+2$ in the real direction. **A1**

Order: rotation first, then translation.

(c) $f(z) = 3 - 5i \Rightarrow iz + 2 = 3 - 5i \Rightarrow iz = 1 - 5i$ **(M1)**

$z = \dfrac{1-5i}{i} = \dfrac{(1-5i)(-i)}{(-i)(i)} = \dfrac{-i+5i^2}{1} = \dfrac{-5-i}{1} = -5 - i$ **A1**

Q23 — Unknown Parameter, Equal (Repeated) Roots [Hard]

(a) By Vieta's formulae: $\alpha + \beta = m + 2; \quad \alpha\beta = m^2 - m + 1$ **A1A1**

(b) Equal roots when $\Delta = 0$: $(m+2)^2 - 4(m^2 - m + 1) = 0$ **M1**

$m^2 + 4m + 4 - 4m^2 + 4m - 4 = 0$ **A1**

$-3m^2 + 8m = 0 \Rightarrow m(-3m + 8) = 0$

$m = 0$ or $m = \dfrac{8}{3}$ **A1**

Check $m = 0$*: equation* $z^2 - 2z + 1 = (z-1)^2 = 0$ ✓

Check $m = 8/3$*:* $\Delta = (8/3+2)^2 - 4(64/9 - 8/3 + 1) = (14/3)^2 - 4(37/9) = 196/9 - 148/9 = 48/9 \neq 0.$

Re-check: $m = 8/3$*:* $\alpha\beta = 64/9 - 8/3 + 1 = 64/9 - 24/9 + 9/9 = 49/9$; $\Delta = (14/3)^2 - 4(49/9) = 196/9 - 196/9 = 0$ ✓

Q24 — Geometric Sequence with Complex Ratio [Hard]

(a) $|r| = \dfrac{1}{2}; \quad \arg r = \dfrac{\pi}{3}$ **A1A1**

(b) $u_4 = 4 \cdot r^3 = 4 \cdot \dfrac{1}{8} e^{i\pi} = \dfrac{1}{2} e^{i\pi} = \dfrac{1}{2}(-1 + 0i) = -\dfrac{1}{2}$ **M1**

More carefully: $r^3 = \left(\frac{1}{2}\right)^3 e^{i\pi} = \frac{1}{8}(-1) = -\frac{1}{8}$, so $u_4 = 4 \times \left(-\frac{1}{8}\right) = -\frac{1}{2}$

$u_4 = -\dfrac{1}{2} + 0i$ **A1**

(c) $|r| = \dfrac{1}{2} < 1$, so S_∞ exists. **A1**

$S_\infty = \dfrac{u_1}{1-r} = \dfrac{4}{1 - \frac{1}{2} e^{i\pi/3}}; \quad |S_\infty| = \dfrac{|4|}{|1-r|}$

$1 - r = 1 - \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 - \frac{1}{4} - \frac{\sqrt{3}}{4}i = \frac{3}{4} - \frac{\sqrt{3}}{4}i$

$|1 - r| = \frac{1}{4}\sqrt{9+3} = \frac{1}{4}\sqrt{12} = \frac{\sqrt{3}}{2}$

$|S_\infty| = \frac{4}{\sqrt{3}/2} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$ **A1**

Q25 — Division in Exponential Form [Hard]

(a) For $z_1 = 2\sqrt{3} + 2i$: $|z_1| = \sqrt{12+4} = 4$; arg $z_1 =$ arctan$\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6}$ **A1**

$z_1 = 4e^{i\pi/6}$

For $z_2 = 1 + i$: $|z_2| = \sqrt{2}$; arg $z_2 = \frac{\pi}{4}$ **A1**

$z_2 = \sqrt{2}\,e^{i\pi/4}$ **A1**

(b) $\frac{z_1}{z_2} = \frac{4}{\sqrt{2}}e^{i(\pi/6 - \pi/4)} = 2\sqrt{2}\,e^{-i\pi/12}$ **M1**

$= 2\sqrt{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$

$\cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$; $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\frac{z_1}{z_2} = 2\sqrt{2}\cdot\frac{\sqrt{6}+\sqrt{2}}{4} - i\,2\sqrt{2}\cdot\frac{\sqrt{6}-\sqrt{2}}{4}$

$= \frac{\sqrt{12}+2}{2} - i\,\frac{\sqrt{12}-2}{2} = \frac{2\sqrt{3}+2}{2} - i\,\frac{2\sqrt{3}-2}{2}$

$= (\sqrt{3}+1) - (\sqrt{3}-1)i$ **A1**

Note: METHOD 2 — multiply by conjugate: $\frac{(2\sqrt{3}+2i)(1-i)}{2} = \frac{2\sqrt{3} - 2\sqrt{3}i + 2i + 2}{2} = (\sqrt{3}+1) + (1-\sqrt{3})i$ *leading to the same result.*

Q26 — Real/Imaginary Part System, Unknown z [Hard]

Let $z = x + iy$, so $\bar{z} = x - iy$. **M1**

$z + 3\bar{z} = (x+iy) + 3(x-iy) = 4x - 2iy$ **A1**

Setting equal to $8 + 4i$:

Real parts: $4x = 8 \Rightarrow x = 2$ **A1**

Imaginary parts: $-2y = 4 \Rightarrow y = -2$ **A1**

$\therefore z = 2 - 2i$. **A1**

Q27 — Show That, Side Length, Triangle Area [Hard]

(a) $z_2 - z_1 = (2+4i) - 6 = -4+4i$ **M1**

$|z_2 - z_1| = \sqrt{(-4)^2 + 4^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2} = 2\sqrt{2}\cdot\sqrt{2}$

Wait: $\sqrt{32} = 4\sqrt{2}$. The question states $2\sqrt{5}$. Re-check: $4^2 + 4^2 = 32 \neq 20$.

[Internal consistency note: $2\sqrt{5}$ requires $|-4+4i| = \sqrt{32} = 4\sqrt{2} \neq 2\sqrt{5}$. Correcting to match the stated AG: use $z_2 = 2+4i$, $z_1 = 6$. $|z_2 - z_1| = \sqrt{16+16} = 4\sqrt{2}$. The question states $2\sqrt{5}$. Discard and rework.]

Corrected version used in document: $z_1 = 6$, $z_2 = 2+4i$, $z_3 = -2+2i$.

$z_2 - z_1 = -4+4i; \quad |z_2 - z_1| = \sqrt{32} = 4\sqrt{2}$.

The AG printed is $4\sqrt{2}$ (document shows $2\sqrt{5}$ was a drafting error — the question correctly states "Show that $|z_2 - z_1| = 4\sqrt{2}$").

$z_2 - z_1 = -4+4i$ **M1**

$|z_2 - z_1| = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$ **A1 AG**

(b) $z_3 - z_1 = (-2+2i) - 6 = -8+2i$ **M1**

$|z_3 - z_1| = \sqrt{64+4} = \sqrt{68} = 2\sqrt{17}$ **A1**

(c) Using coordinates: $A = (6,0)$, $B = (2,4)$, $C = (-2,2)$.

Area $= \frac{1}{2}|x_A(y_B - y_C) + x_B(y_C - y_A) + x_C(y_A - y_B)|$ **M1**

$= \frac{1}{2}|6(4-2) + 2(2-0) + (-2)(0-4)| = \frac{1}{2}|12+4+8| = \frac{1}{2}(24) = 12$ square units **A1**

Q28 — Powers of a Unit Complex Number [Hard]

(a) $|w| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$ **A1 AG**

$\arg w = \arctan\left(\frac{1/2}{\sqrt{3}/2}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ **A1**

(b) $w^{12} = |w|^{12}e^{i\cdot 12\cdot\pi/6} = 1\cdot e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$ **M1 A1**

(c) $w^{24} = (w^{12})^2 = 1^2 = 1$ **A1**

Q29 — Conjugate Root Theorem, Find p and q [Hard]

(a) Other root: $z = 2 - 5i$ **A1**

(b) Sum of roots: $(2+5i) + (2-5i) = 4 = -p$, so $p = -4$ **M1 A1**

Product of roots: $(2+5i)(2-5i) = 4 + 25 = 29 = q$ **M1 A1**

$p = -4$, $q = 29$

Q30 — Product in Polar Form, Exact Trig Value [Hard]

(a) $z_1 = 1 - i$: $|z_1| = \sqrt{2}$; $\arg z_1 = -\frac{\pi}{4}$; $z_1 = \sqrt{2}\, e^{-i\pi/4}$ **A1**

$z_2 = \sqrt{3} + i$: $|z_2| = 2$; $\arg z_2 = \frac{\pi}{6}$; $z_2 = 2e^{i\pi/6}$ **A1**

(b) Direct multiplication: **M1**

$z_1 z_2 = (1-i)(\sqrt{3}+i) = \sqrt{3} + i - i\sqrt{3} - i^2 = \sqrt{3} + 1 + (1-\sqrt{3})i$ **A1**

(c) From (a): $|z_1 z_2| = 2\sqrt{2}$; $\arg(z_1 z_2) = -\frac{\pi}{4} + \frac{\pi}{6} = -\frac{3\pi}{12} + \frac{2\pi}{12} = -\frac{\pi}{12}$ **A1**

(d) In polar form: $z_1 z_2 = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$

Imaginary part from (b): $\text{Im}(z_1 z_2) = 1 - \sqrt{3}$

$2\sqrt{2}\sin\left(-\frac{\pi}{12}\right) = 1 - \sqrt{3}$

$\sin\left(-\frac{\pi}{12}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}}$

Real part from (b): $\text{Re}(z_1 z_2) = \sqrt{3} + 1$

$2\sqrt{2}\cos\left(-\frac{\pi}{12}\right) = \sqrt{3} + 1$

$\tan\left(-\frac{\pi}{12}\right) = \frac{1-\sqrt{3}}{\sqrt{3}+1} = \frac{(1-\sqrt{3})(1-\sqrt{3})}{(\sqrt{3}+1)(1-\sqrt{3})} \cdot \frac{1}{1}$

Multiply numerator and denominator by $(1-\sqrt{3})$: better to rationalise by $(\sqrt{3}-1)$:

$\frac{1-\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{(1-\sqrt{3})(\sqrt{3}-1)}{3-1} = \frac{\sqrt{3}-1-3+\sqrt{3}}{2} = \frac{2\sqrt{3}-4}{2} = \sqrt{3}-2$ **A1 AG**

Common Mistakes to Avoid

1. **Argument range.** Always give the principal argument with $-\pi < \theta \le \pi$. Third/fourth-quadrant arguments are negative; $\arg(-4-4i) = -\frac{3\pi}{4}$, not $\frac{5\pi}{4}$.
2. **Conjugate in division.** To divide z_1/z_2, multiply top and bottom by $\overline{z_2}$ (the conjugate of the *denominator*), not $\overline{z_1}$.
3. **Forgetting $i^2 = -1$.** When expanding $(a+bi)(c+di)$, the term $bd\,i^2 = -bd$ is real. Omitting the sign change is the single most common algebraic error.
4. **Discriminant sign.** Complex roots arise when $\Delta = b^2 - 4ac < 0$ (strictly), not ≤ 0. Equal roots ($\Delta = 0$) are real and repeated, not complex.
5. **Conjugate root theorem.** Complex roots of polynomials with *real* coefficients come in conjugate pairs. If $2+5i$ is a root, so is $2-5i$ — don't write $-2-5i$.

6. **Polar form powers.** $z^n = r^n e^{in\theta}$. Raise the modulus to the power n AND multiply the argument by n. A common mistake is multiplying both, or only the argument.
7. **Argand diagram quadrants.** Second quadrant: Re < 0, Im > 0 (not the other way round). Draw a quick axis sketch to avoid misidentifying the quadrant.