IB Math AI HL — Topic 5: Calculus

Kinematics

Displacement · Velocity · Acceleration · Calculus for Kinematics

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:______________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated. GDC permitted throughout.

Key Formulae — Kinematics

Definitions (motion along a line):

$v(t) = \dfrac{ds}{dt} = \dot{s}$ (velocity = rate of change of displacement)

$a(t) = \dfrac{dv}{dt} = \ddot{s}$ (acceleration = rate of change of velocity)

$s(t) = \int v(t)\,dt + C$ (displacement from integrating velocity)

Particle at rest: $v(t) = 0$

Direction changes: sign of $v(t)$ changes

Distance vs Displacement:

Total distance $= \displaystyle\int_a^b |v(t)|\,dt$

Net displacement $= \displaystyle\int_a^b v(t)\,dt$

Greatest speed: find $|v(t)|$ at $v'(t) = 0$ and endpoints

Constant acceleration: $v = u + at$, $\quad s = ut + \frac{1}{2}at^2$

$v^2 = u^2 + 2as$, $\quad s = \frac{1}{2}(u+v)t$

GDC Tips

1. **Finding zeros of $v(t)$:** Graph $v(t)$ and use `zero` (TI) or `x-intercept` (Casio) to find when the particle is at rest.
2. **Total distance:** On Casio fx-CG50/TI-Nspire, integrate $|v(t)|$ using `abs()` in the function entry, e.g. `abs(3t^2—6t)`.
3. **Finding max/min speed:** Graph $|v(t)|$ or use `minimum/maximum` on the derivative screen.
4. **Numerical derivatives:** Use `d/dx` at a point to evaluate acceleration from a velocity function quickly.
5. **Definite integrals for displacement:** $\int_a^b v\,dt$ gives *signed* displacement — note this may be negative even though the particle has moved.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. **EASY** **Calculator** **[3 marks]**

A drone moves along a straight horizontal track. Its velocity, v metres per second, at time t seconds is given by

$$v(t) = t^2 - 6t + 8, \quad t \geq 0.$$

(a) Write down the initial velocity of the drone. [1]

(b) Find the times when the drone is momentarily at rest. [2]

2. **EASY** **Calculator** **[3 marks]**

A boat moves along a canal. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = 12 - 4t, \quad 0 \leq t \leq 5.$$

(a) Find the acceleration of the boat. [1]

(b) Find the time when the boat is momentarily at rest. [1]

(c) Find the displacement of the boat from its starting position when $t = 3$. [1]

3. EASY Calculator **[3 marks]**

A cyclist moves along a straight path. Her displacement, s metres from a fixed point O, at time t seconds is given by

$$s(t) = 3t^2 - t + 5, \quad t \geq 0.$$

(a) Find an expression for the velocity of the cyclist at time t. **[1]**

(b) Find the velocity when $t = 4$. **[1]**

(c) Find the acceleration of the cyclist. **[1]**

4. EASY Calculator **[3 marks]**

A remote-controlled car moves along a straight track. Its velocity, v m s^{-1}, is given by $v(t) = 6t - 2$, where t is time in seconds. The car starts at a position 3 m to the right of a marker M.

(a) Write down the position of the car at time $t = 0$. **[1]**

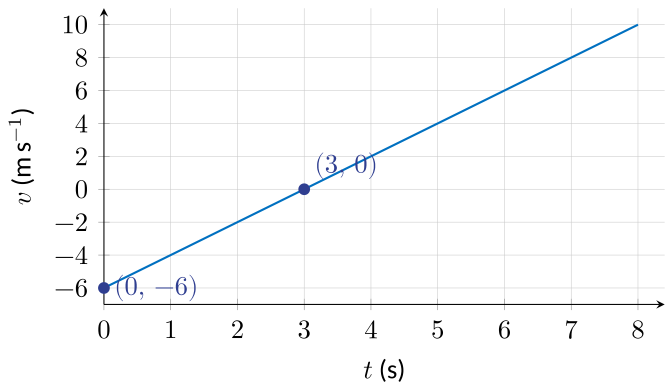
(b) Find the displacement of the car from M at time $t = 4$ seconds. **[2]**

5. EASY Calculator **[3 marks]**

A jogger runs along a straight road. Her velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = 2t - 6, \quad 0 \le t \le 8.$$

The diagram below shows the velocity–time graph of the jogger.



(a) Write down the time when the jogger changes direction. **[1]**

(b) Find the total distance travelled by the jogger in the interval $0 \le t \le 8$. **[2]**

6. EASY Calculator **[3 marks]**

A particle moves along the x-axis. Its acceleration, a m s^{-2}, at time t seconds is given by $a(t) = 6t - 4$. At $t = 0$, the particle has velocity 5 m s^{-1}.

(a) Find an expression for the velocity of the particle at time t. **[2]**

(b) Find the velocity of the particle when $t = 2$. **[1]**

7. EASY Calculator **[3 marks]**

A ball is rolled along a straight groove. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = -3t^2 + 12t - 5, \quad 0 \leq t \leq 5.$$

(a) Find the acceleration of the ball when $t = 1$. [2]

(b) Write down the value of t at which the velocity is greatest in $0 \leq t \leq 5$. [1]

8. EASY Calculator **[3 marks]**

A swimmer moves along a straight lane. His velocity, v m s^{-1}, at time t seconds satisfies $v(t) = 4\cos(2t)$, where t is measured in radians.

(a) Write down the initial velocity of the swimmer. [1]

(b) Find the displacement of the swimmer from his starting position in the interval $0 \leq t \leq \frac{\pi}{4}$. [2]

9. EASY Calculator **[3 marks]**

A trolley moves along a straight rail. Its velocity, v m s^{-1}, at time t seconds is given by

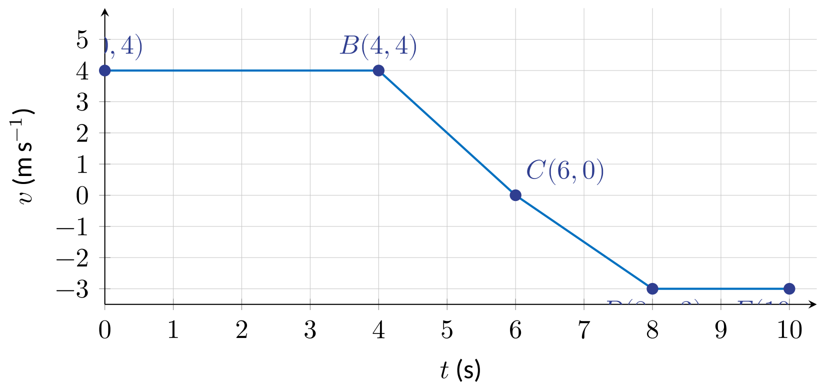
$$v(t) = \sqrt{t+1}, \quad 0 \leq t \leq 8.$$

(a) Write down the velocity of the trolley when $t = 0$. [1]

(b) Find the displacement of the trolley in the interval $0 \leq t \leq 8$. [2]

10. EASY Calculator **[3 marks]**

The diagram below shows the velocity–time graph of a particle moving along a straight line for $0 \le t \le 10$.



(a) Write down the initial velocity of the particle. **[1]**

(b) Write down the time when the particle changes direction. **[1]**

(c) Write down the greatest speed of the particle in $0 \le t \le 10$. **[1]**

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. **MEDIUM** **Calculator** **[4 marks]**

A particle P moves along the x-axis. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = t^2 - 5t + 4, \quad 0 \leq t \leq 5.$$

(a) Find the times when P is momentarily at rest. **[2]**

(b) Find the total distance travelled by P in the interval $0 \leq t \leq 5$. **[2]**

12. **MEDIUM** **Calculator** **[4 marks]**

A particle moves along a straight line. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = -2t^2 + 8t - 3, \quad 0 \leq t \leq 5.$$

(a) Find the greatest velocity of the particle for $0 \leq t \leq 5$. **[2]**

(b) The particle starts at the origin. Find the displacement of the particle from the origin when $t = 5$. **[2]**

13. MEDIUM Calculator **[4 marks]**

A platform oscillates along a horizontal rail. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = 3\sin(2t), \quad 0 \leq t \leq 2\pi.$$

(a) Find the first two positive values of t for which the platform is momentarily at rest. [2]

(b) Find the total distance travelled by the platform for $0 \leq t \leq \pi$. [2]

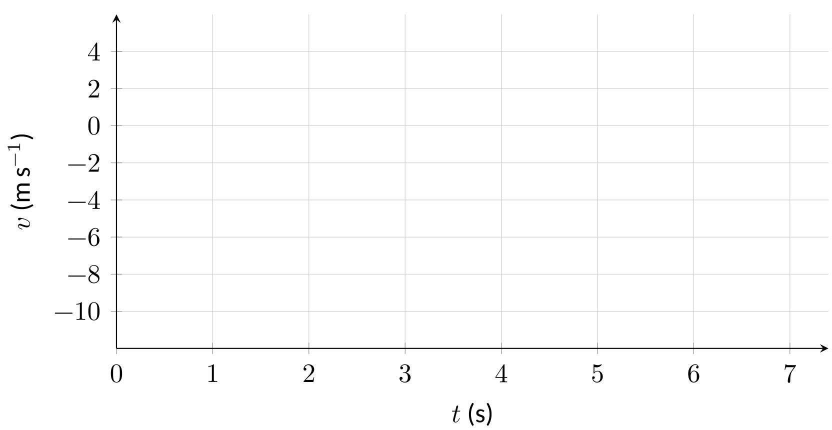
14. MEDIUM Calculator **[4 marks]**

A train moves along a straight track. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = -t^2 + 6t - 5, \quad 0 \leq t \leq 7.$$

(a) On the axes below, sketch the velocity–time graph for $0 \leq t \leq 7$, labelling any intercepts with the axes and the turning point. [3]

(b) Write down the times during which the train moves in the negative direction. [1]



15. MEDIUM Calculator **[4 marks]**

A particle moves along the y-axis. Its acceleration, a m s^{-2}, at time t seconds is given by $a(t) = 6 - 4t$. At $t = 0$, the particle is at the origin with velocity 2 m s^{-1}.

(a) Find an expression for the velocity $v(t)$ of the particle at time t. **[2]**

(b) Find the displacement of the particle from the origin when $t = 3$. **[2]**

16. MEDIUM Calculator **[4 marks]**

A model rocket moves vertically upwards after launch. Its velocity, v m s^{-1}, at time t seconds after launch is given by

$$v(t) = 30t - 5t^2, \quad 0 \le t \le 7.$$

(a) Find the acceleration of the rocket when $t = 2$. **[2]**

(b) Find the maximum height reached by the rocket above its launch point. **[2]**

17. MEDIUM Calculator **[4 marks]**

A delivery robot moves along a straight corridor. The table below gives its velocity v m s^{-1} at selected times.

t (s)	0	2	4	6	8	10
v (m s^{-1})	0	3	5	5	2	0

(a) Use the trapezoid rule with five strips of width 2 to estimate the displacement of the robot over $0 \leq t \leq 10$. **[3]**

(b) State whether the robot moves in one direction only. Justify your answer. **[1]**

18. MEDIUM Calculator **[4 marks]**

A particle moves along a straight line. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = e^{0.5t} - 2, \quad 0 \leq t \leq 6.$$

(a) Find the time when the particle is momentarily at rest. Give your answer exact. **[2]**

(b) Find the acceleration of the particle when $t = 4$. **[2]**

19. MEDIUM Calculator **[4 marks]**

A car moves along a straight road. Starting from a point O at $t = 0$, its velocity, v m s^{-1}, at time t seconds is given by

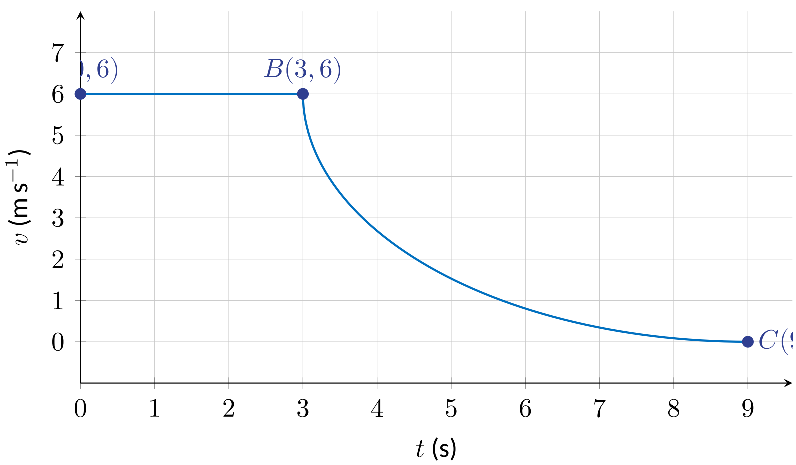
$$v(t) = 3t^2 - 12t + 9, \quad t \geq 0.$$

(a) Find the times when the car is momentarily at rest. **[2]**

(b) Find the displacement of the car from O at $t = 4$. **[2]**

20. MEDIUM Calculator **[4 marks]**

The diagram shows the velocity–time graph of a particle Q moving along a straight line for $0 \leq t \leq 9$. The graph consists of a straight line from $t = 0$ to $t = 3$, then a quarter circle of radius 6 from $t = 3$ to $t = 9$.



(a) Write down the acceleration of Q for $0 \leq t \leq 3$. **[1]**

(b) Find the total distance travelled by Q for $0 \leq t \leq 9$. **[3]**

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **Calculator** **[5 marks]**

A particle moves along the x-axis. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = 2t^3 - 9t^2 + 12t - 4, \quad 0 \leq t \leq 4.$$

(a) Find all values of t in $0 < t < 4$ for which the particle is momentarily at rest. **[2]**

(b) Find the net displacement of the particle from its position at $t = 0$ to $t = 4$. **[1]**

(c) Find the total distance travelled by the particle for $0 \leq t \leq 4$. **[2]**

22. HARD Calculator **[5 marks]**

A particle moves along a straight line. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = t^2 - kt + 6, \quad t \geq 0,$$

where k is a positive constant. The particle is momentarily at rest at $t = 2$.

(a) Show that $k = 5$. [2]

(b) Find the acceleration of the particle when $t = 2$. [1]

(c) The particle starts at a point A. Find the position of the particle relative to A when $t = 3$. [2]

23. HARD Calculator **[5 marks]**

A buoy bobs along a straight channel. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = 5\sin(t) - 5\cos(t), \quad 0 \leq t \leq 2\pi.$$

(a) Show that $v(t)$ can be written in the form $R\sin(t + \varphi)$, stating the values of R and φ. [2]

(b) Hence find the maximum speed of the buoy. [1]

(c) Find the total distance travelled by the buoy for $0 \leq t \leq 2\pi$. [2]

24. HARD Calculator **[5 marks]**

A skydiver falls vertically. Modelling the acceleration (in the downward direction) as $a = 10 - 0.2v$, where v m s^{-1} is the downward velocity and a m s^{-2} is the acceleration, with $v = 0$ when $t = 0$:

(a) Write down a differential equation for v in terms of t. **[1]**

(b) Solve the differential equation to find $v(t)$. **[3]**

(c) Hence find the terminal velocity of the skydiver. **[1]**

25. HARD Calculator **[5 marks]**

A particle moves along the x-axis. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = 3e^{-0.5t}, \quad t \geq 0.$$

At $t = 0$ the particle is at $x = 2$.

(a) Find an expression for the displacement $x(t)$ of the particle. **[2]**

(b) Show that the particle never reaches $x = 8$. **[2]**

(c) State the value that $x(t)$ approaches as $t \to \infty$. **[1]**

26. HARD Calculator **[5 marks]**

A particle moves along a straight line. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = \begin{cases} 3t^2 - 12 & 0 \leq t < 3 \\ kt - 3 & 3 \leq t \leq 7 \end{cases}$$

where k is a constant.

(a) Find the value of k such that $v(t)$ is continuous at $t = 3$. **[2]**

(b) Using this value of k, find the total distance travelled by the particle for $0 \leq t \leq 7$. **[3]**

27. HARD Calculator **[5 marks]**

A particle moves in the xy-plane. Its position vector at time t seconds is given by

$$\mathbf{r}(t) = (t^3 - 6t)\,\mathbf{i} + (3t^2)\,\mathbf{j}, \quad t \geq 0.$$

Displacement is measured in metres.

(a) Find an expression for the velocity vector $\mathbf{v}(t)$ of the particle. **[1]**

(b) Find the speed of the particle when $t = 2$. **[2]**

(c) Find the acceleration vector $\mathbf{a}(t)$ and evaluate $|\mathbf{a}|$ when $t = 1$. **[2]**

28. HARD Calculator **[5 marks]**

A particle moves along the x-axis. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = 2t^3 - 15t^2 + 24t, \quad 0 \leq t \leq 6.$$

(a) Find the values of t in $0 < t < 6$ at which the acceleration of the particle is zero. [2]

(b) Hence write down the values of t in $0 < t < 6$ at which the velocity is a local maximum or local minimum, and state which is which. [1]

(c) Find the total distance travelled by the particle for $0 \leq t \leq 6$. [2]

29. HARD Calculator **[5 marks]**

A particle starts from rest at a point O and moves along the positive x-axis. Its acceleration, a m s^{-2}, at position x metres from O satisfies

$$a = v\frac{dv}{dx} = 4x - x^2,$$

where v m s^{-1} is the particle's velocity.

(a) Show that $v^2 = 4x^2 - \dfrac{2x^3}{3}$. [3]

(b) Find the speed of the particle when $x = 3$. [1]

(c) Find the value of $x > 0$ at which the particle next comes to rest, and hence find the total distance it travels from O until it first returns to rest. [1]

30. HARD Calculator **[5 marks]**

A particle P moves along the x-axis. Its velocity, v m s^{-1}, at time t seconds is given by

$$v(t) = t^2 - 4t + m,$$

where m is a constant. The net displacement of P from $t = 0$ to $t = 6$ is 24 m.

(a) Show that $m = 4$. **[2]**

(b) Using $m = 4$, find all times in $0 < t < 6$ at which P is momentarily at rest. **[1]**

(c) Find the total distance travelled by P for $0 \leq t \leq 6$. **[2]**

Worked Solutions — Section A (Easy)

Q1 — Velocity polynomial: initial velocity and times at rest EASY

(a) $v(0) = 0 - 0 + 8 = 8$ m s^{-1} **A1**

(b) Set $v(t) = 0$: $t^2 - 6t + 8 = 0$ **M1**

$(t-2)(t-4) = 0 \implies t = 2$ and $t = 4$ **A1**

Q2 — Linear velocity: acceleration, rest, displacement EASY

(a) $a(t) = \dfrac{dv}{dt} = -4$ m s^{-2} **A1**

(b) $12 - 4t = 0 \implies t = 3$ s **A1**

(c) $s = \displaystyle\int_0^3 (12 - 4t)\,dt = \left[12t - 2t^2\right]_0^3 = 36 - 18 = 18$ m **A1**

Q3 — Displacement given: find velocity and acceleration EASY

(a) $v(t) = s'(t) = 6t - 1$ **A1**

(b) $v(4) = 6(4) - 1 = 23$ m s^{-1} **A1**

(c) $a(t) = v'(t) = 6$ m s^{-2} **A1**

Q4 — Integrate velocity, initial condition EASY

(a) Position at $t = 0$ is 3 m to the right of M. **A1**

(b) $s(t) = \displaystyle\int (6t - 2)\,dt = 3t^2 - 2t + C$ **M1**

$s(0) = 3 \implies C = 3$, so $s(t) = 3t^2 - 2t + 3$

$s(4) = 3(16) - 8 + 3 = 48 - 8 + 3 = 43$ m **A1**

Note: Displacement from M is 43 m.

Q5 — Linear v(t): direction change, total distance EASY

(a) $v(t) = 0 \implies 2t - 6 = 0 \implies t = 3$ s **A1**

(b) Total distance $= \displaystyle\int_0^8 |2t - 6|\,dt$ **M1**

$$= \int_0^3 (6-2t)\,dt + \int_3^8 (2t-6)\,dt$$

$$= \Big[6t - t^2\Big]_0^3 + \Big[t^2 - 6t\Big]_3^8 = 9 + (64 - 48 - 9 + 18) = 9 + 25 = 34 \text{ m}$$ **A1**

Q6 — Integrate acceleration, initial velocity given EASY

(a) $v(t) = \int (6t - 4)\,dt = 3t^2 - 4t + C$ **M1**

$v(0) = 5 \implies C = 5$, so $v(t) = 3t^2 - 4t + 5$ **A1**

(b) $v(2) = 3(4) - 8 + 5 = 12 - 8 + 5 = 9 \text{ m s}^{-1}$ **A1**

Q7 — Quadratic v: acceleration at t=1, time of greatest velocity EASY

(a) $a(t) = v'(t) = -6t + 12$ **M1**

$a(1) = -6 + 12 = 6 \text{ m s}^{-2}$ **A1**

(b) Maximum of $v(t)$ occurs at $v'(t) = 0 \implies t = 2$ s **A1**

Note: $v(2) = -12 + 24 - 5 = 7$ *m s*$^{-1}$.

Q8 — Cosine velocity: initial velocity, displacement by integration EASY

(a) $v(0) = 4\cos(0) = 4 \text{ m s}^{-1}$ **A1**

(b) Displacement $= \int_0^{\pi/4} 4\cos(2t)\,dt = \Big[2\sin(2t)\Big]_0^{\pi/4}$ **M1**

$= 2\sin\left(\frac{\pi}{2}\right) - 2\sin(0) = 2(1) - 0 = 2$ m **A1**

Q9 — Square-root velocity: initial value, displacement EASY

(a) $v(0) = \sqrt{0+1} = 1 \text{ m s}^{-1}$ **A1**

(b) Displacement $= \int_0^8 \sqrt{t+1}\,dt = \Big[\frac{2}{3}(t+1)^{3/2}\Big]_0^8$ **M1**

$= \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3}(27) - \frac{2}{3}(1) = 18 - \frac{2}{3} = \dfrac{52}{3} \approx 17.3$ m **A1**

Q10 — Reading a velocity–time graph **EASY**

(a) $v(0) = 4$ m s^{-1} **A1**

(b) The particle changes direction when $v = 0$: at $t = 6$ s **A1**

(c) Greatest speed $= 4$ m s^{-1} (at $0 \leq t \leq 4$) **A1**

Note: Speed = $|v|$; maximum of $|v|$ over $[0, 10]$ is 4 m s^{-1}.

Worked Solutions — Section B (Medium)

Q11 — Quadratic v: rest times and total distance MEDIUM

(a) $v(t) = 0$: $t^2 - 5t + 4 = 0 \implies (t-1)(t-4) = 0$ **M1**

$t = 1$ s and $t = 4$ s **A1**

(b) Total distance $= \displaystyle\int_0^5 |t^2 - 5t + 4|\,dt$ **M1**

$$= \int_0^1 (t^2 - 5t + 4)\,dt + \int_1^4 -(t^2 - 5t + 4)\,dt + \int_4^5 (t^2 - 5t + 4)\,dt$$

$= \frac{5}{6} + \frac{9}{2} + \frac{11}{6} = \frac{5+27+11}{6} = \dfrac{43}{6} \approx 7.17$ m **A1**

Note: $\int_0^1 = \left[\frac{t^3}{3} - \frac{5t^2}{2} + 4t\right]_0^1 = \frac{1}{3} - \frac{5}{2} + 4 = \frac{11}{6}$; $\int_1^4$ *gives* $\frac{27}{6} = \frac{9}{2}$; $\int_4^5 = \frac{11}{6}$.

Q12 — Quadratic v: greatest velocity, displacement MEDIUM

(a) $a(t) = v'(t) = -4t + 8 = 0 \implies t = 2$ **M1**

$v(2) = -8 + 16 - 3 = 5\ \text{m s}^{-1}$ (greatest in interior; check endpoints: $v(0) = -3$, $v(5) = -13$) **A1**

(b) $s(5) = \displaystyle\int_0^5 (-2t^2 + 8t - 3)\,dt = \left[-\frac{2t^3}{3} + 4t^2 - 3t\right]_0^5$ **M1**

$= -\dfrac{250}{3} + 100 - 15 = 85 - \dfrac{250}{3} = \dfrac{255 - 250}{3} = \dfrac{5}{3} \approx 1.67$ m **A1**

Q13 — Sinusoidal v: rest times, total distance MEDIUM

(a) $v(t) = 0$: $3\sin(2t) = 0 \implies 2t = \pi, 2\pi, \ldots$ **M1**

First two: $t = \dfrac{\pi}{2}$ and $t = \pi$ **A1**

(b) Total distance for $0 \le t \le \pi$: $v(t)$ changes sign at $t = \dfrac{\pi}{2}$ **M1**

$$= \int_0^{\pi/2} 3\sin(2t)\,dt + \int_{\pi/2}^{\pi} (-3\sin(2t))\,dt$$

$$= \left[-\tfrac{3}{2}\cos(2t)\right]_0^{\pi/2} + \left[\tfrac{3}{2}\cos(2t)\right]_{\pi/2}^{\pi}$$

$= \left(-\frac{3}{2}(-1) + \frac{3}{2}\right) + \left(\frac{3}{2}(-1) - \frac{3}{2}(-1)\right) = 3 + 3 = 6$ m **A1**

Note: Each half-period contributes 3 m.

Q14 — Sketch v–t graph and identify negative direction MEDIUM

$v(t) = -(t-1)(t-5)$; roots at $t = 1, 5$; vertex at $t = 3$: $v(3) = 4$; $v(0) = -5$; $v(7) = -8$.

(a) Marks awarded for:
Shape: downward-opening parabola **A1**
t-intercepts at $(1, 0)$ and $(5, 0)$, v-intercept at $(0, -5)$, labelled **A1**
Turning point labelled $(3, 4)$ **A1**

(b) Train moves in the negative direction when $v < 0$: for $0 \leq t < 1$ and $5 < t \leq 7$. **A1**

Q15 — Integrate a(t) twice with initial conditions MEDIUM

(a) $v(t) = \displaystyle\int (6 - 4t)\, dt = 6t - 2t^2 + C$ **M1**

$v(0) = 2 \implies C = 2$, so $v(t) = 6t - 2t^2 + 2$ **A1**

(b) $s(t) = \displaystyle\int_0^3 (6t - 2t^2 + 2)\, dt = \left[3t^2 - \tfrac{2t^3}{3} + 2t\right]_0^3$ **M1**

$= 27 - 18 + 6 = 15$ m **A1**

Q16 — Model rocket: acceleration and max height MEDIUM

(a) $a(t) = v'(t) = 30 - 10t$; $a(2) = 30 - 20 = 10$ m s^{-2} **M1 A1**

(b) Max height when $v(t) = 0$: $30t - 5t^2 = 0 \implies 5t(6 - t) = 0 \implies t = 6$ s **M1**

$h(6) = \displaystyle\int_0^6 (30t - 5t^2)\, dt = \left[15t^2 - \tfrac{5t^3}{3}\right]_0^6 = 540 - 360 = 180$ m **A1**

Q17 — Trapezoid rule for displacement from table MEDIUM

(a) Trapezoid rule: $h = 2$, $n = 5$

Estimate $= \dfrac{2}{2}[(v_0 + v_{10}) + 2(v_2 + v_4 + v_6 + v_8)]$ **M1**

$= 1 \cdot [(0 + 0) + 2(3 + 5 + 5 + 2)] = 2(15) = 30$ m **A1A1**

Note: Award A1 for correct formula structure, A1 for answer.

(b) All values of v in the table are ≥ 0, so the robot moves in the positive direction only (or remains stationary at endpoints). **R1**

Q18 — Exponential velocity: time at rest (exact) and acceleration MEDIUM

(a) $e^{0.5t} - 2 = 0 \implies e^{0.5t} = 2 \implies 0.5t = \ln 2 \implies t = 2\ln 2$ **M1 A1**

(b) $a(t) = v'(t) = 0.5e^{0.5t}$; $a(4) = 0.5e^2 = 0.5(7.389\ldots) = 3.694\ldots \approx 3.69$ m s^{-2} **M1 A1**

Q19 — Cubic v: rest times and displacement MEDIUM

(a) $v(t) = 3t^2 - 12t + 9 = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$ **M1**

$v = 0$ at $t = 1$ s and $t = 3$ s **A1**

(b) $s(4) = \displaystyle\int_0^4 (3t^2 - 12t + 9)\,dt = \Big[t^3 - 6t^2 + 9t\Big]_0^4$ **M1**

$= 64 - 96 + 36 = 4$ m **A1**

Q20 — v–t graph: straight line + quarter circle, distance MEDIUM

(a) Straight line from $(0, 6)$ to $(3, 6)$: slope $= 0$, so $a = 0$ m s^{-2}. **A1**

(b) Distance for $0 \le t \le 3$: area of rectangle $= 3 \times 6 = 18$ m **A1**

Distance for $3 \le t \le 9$: area under quarter circle of radius 6

$= \dfrac{1}{4}\pi r^2 = \dfrac{1}{4}\pi(6)^2 = 9\pi \approx 28.3$ m **M1 A1**

Total distance $= 18 + 9\pi \approx 46.3$ m **A1**

Note: Accept $18 + 9\pi$ as exact answer. Area under v–t curve gives distance since $v \ge 0$ throughout.

Worked Solutions — Section C (Hard)

Q21 — Cubic v: rest times, net displacement, total distance HARD

Factor: $v(t) = 2t^3 - 9t^2 + 12t - 4 = 2\left(t - \frac{1}{2}\right)(t-2)^2$

(a) $v(t) = 0 \implies t = \dfrac{1}{2}$ s and $t = 2$ s **M1 A1**

(b) Let $F(t) = \frac{t^4}{2} - 3t^3 + 6t^2 - 4t$.

Net displacement $= F(4) - F(0) = (128 - 192 + 96 - 16) - 0 = 16$ m **A1**

(c) Since $(t-2)^2 \geq 0$ always, sign(v) = sign$(t - \frac{1}{2})$:

$v < 0$ for $0 \leq t < \frac{1}{2}$; $\quad v \geq 0$ for $\frac{1}{2} \leq t \leq 4$ **M1**

$F\left(\frac{1}{2}\right) = \frac{1}{32} - \frac{3}{8} + \frac{3}{2} - 2 = -\frac{27}{32}$

Total distance $= \left|F\left(\frac{1}{2}\right) - F(0)\right| + \left|F(4) - F\left(\frac{1}{2}\right)\right| = \frac{27}{32} + \left(16 + \frac{27}{32}\right) = 16 + \frac{27}{16} = \dfrac{283}{16} \approx 17.7$ m **A1**

Note: Accept GDC result $\int_0^{0.5} |v|\,dt + \int_{0.5}^{4} |v|\,dt \approx 17.7$ m.

Q22 — Unknown parameter k: show k=5, then a and displacement HARD

(a) At $t = 2$, $v(2) = 0$: **M1**

$4 - 2k + 6 = 0 \implies 2k = 10 \implies k = 5$ **A1 AG**

(b) $a(t) = v'(t) = 2t - 5$; $a(2) = 4 - 5 = -1$ m s^{-2} **A1**

(c) With $k = 5$: $s(3) = \displaystyle\int_0^3 (t^2 - 5t + 6)\,dt = \left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_0^3$ **M1**

$= 9 - \frac{45}{2} + 18 = 27 - 22.5 = 4.5$ m from A **A1**

Q23 — Trig velocity: R-form, max speed, total distance HARD

(a) $5\sin t - 5\cos t = R\sin(t + \varphi)$ where $R = \sqrt{5^2 + 5^2} = 5\sqrt{2}$ **M1**

$\tan\varphi = \dfrac{-5}{5} = -1 \implies \varphi = -\dfrac{\pi}{4}$ (since sin coeff positive, cos coeff negative)

So $v(t) = 5\sqrt{2}\,\sin\left(t - \dfrac{\pi}{4}\right)$ **A1**

(b) Maximum speed $= R = 5\sqrt{2} \approx 7.07$ m s^{-1} **A1**

(c) $v(t) = 0$ when $t - \dfrac{\pi}{4} = 0, \pi \implies t = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

Total distance $= \displaystyle\int_0^{2\pi} |v(t)|\,dt = 4 \times 5\sqrt{2} = 20\sqrt{2} \approx 28.3$ m **M1 A1**

Note: Over a full period the integral of $|R\,\sin(\cdot)|$ is $4R$.

Q24 — ODE $dv/dt = 10 - 0.2v$: separate, solve, terminal velocity **HARD**

(a) $\dfrac{dv}{dt} = 10 - 0.2v$ **A1**

(b) Separate variables: $\displaystyle\int \frac{dv}{10 - 0.2v} = \int dt$ **M1**

$-5\ln|10 - 0.2v| = t + C$

At $t = 0$, $v = 0$: $-5\ln 10 = C$

$-5\ln|10 - 0.2v| = t - 5\ln 10 \implies \ln\left(\dfrac{10 - 0.2v}{10}\right) = -\dfrac{t}{5}$ **A1**

$10 - 0.2v = 10e^{-t/5} \implies v(t) = 50\left(1 - e^{-t/5}\right)$ **A1**

(c) As $t \to \infty$, $e^{-t/5} \to 0$, so terminal velocity $= 50\ \text{m s}^{-1}$ **A1**

Q25 — Exponential v: find x(t), show never reaches 8, limit **HARD**

(a) $x(t) = \displaystyle\int 3e^{-0.5t}\,dt + C = -6e^{-0.5t} + C$ **M1**

$x(0) = 2 \implies -6 + C = 2 \implies C = 8$, so $x(t) = 8 - 6e^{-0.5t}$ **A1**

(b) $x(t) = 8$ requires $8 - 6e^{-0.5t} = 8 \implies e^{-0.5t} = 0$, which has no finite solution. **M1**

Since $e^{-0.5t} > 0$ for all t, we have $x(t) < 8$ for all $t \geq 0$. **R1 AG**

(c) As $t \to \infty$: $x \to 8$ m **A1**

Q26 — Piecewise v: find k for continuity, then total distance **HARD**

(a) Continuity at $t = 3$: $\lim_{t\to 3^-} v = 3(9) - 12 = 15$; $\lim_{t\to 3^+} v = 3k - 3$ **M1**

$3k - 3 = 15 \implies k = 6$ **A1**

(b) With $k = 6$: $v(t) = 3t^2 - 12$ for $0 \leq t < 3$; $v(t) = 6t - 3$ for $3 \leq t \leq 7$.

On $[0, 3]$: $v = 0$ when $3t^2 = 12 \implies t = 2$. $v < 0$ on $(0, 2)$, $v > 0$ on $(2, 3)$. **M1**

On $[3, 7]$: $6t - 3 > 0$ for all $t > 0.5$, so $v > 0$ throughout.

Distance $= \displaystyle\int_0^2 (12 - 3t^2)\,dt + \int_2^3 (3t^2 - 12)\,dt + \int_3^7 (6t - 3)\,dt$

$= \left[12t - t^3\right]_0^2 + \left[t^3 - 12t\right]_2^3 + \left[3t^2 - 3t\right]_3^7$

$= (24 - 8) + (27 - 36 - 8 + 24) + (147 - 21 - 27 + 9)$

$= 16 + 7 + 108 = 131$ m **A1A1**

Note: Check: $\int_2^3 (3t^2 - 12) = [t^3 - 12t]_2^3 = (27 - 36) - (8 - 24) = -9 + 16 = 7$. $\int_3^7 (6t - 3) = [3t^2 - 3t]_3^7 = (147 - 21) - (27 - 9) = 126 - 18 = 108$.

Q27 — Vector kinematics: velocity, speed, acceleration HARD

(a) $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt} = (3t^2 - 6)\,\mathbf{i} + 6t\,\mathbf{j}$ **A1**

(b) At $t = 2$: $\mathbf{v}(2) = (12 - 6)\,\mathbf{i} + 12\,\mathbf{j} = 6\,\mathbf{i} + 12\,\mathbf{j}$ **M1**

Speed $= |\mathbf{v}(2)| = \sqrt{6^2 + 12^2} = \sqrt{36 + 144} = \sqrt{180} = 6\sqrt{5} \approx 13.4$ m s^{-1} **A1**

(c) $\mathbf{a}(t) = \dfrac{d\mathbf{v}}{dt} = 6t\,\mathbf{i} + 6\,\mathbf{j}$ **M1**

$|\mathbf{a}(1)| = |6\,\mathbf{i} + 6\,\mathbf{j}| = \sqrt{36 + 36} = 6\sqrt{2} \approx 8.49$ m s^{-2} **A1**

Q28 — Cubic v: zeros of acceleration identify max/min, total distance HARD

(a) $a(t) = v'(t) = 6t^2 - 30t + 24 = 6(t^2 - 5t + 4) = 6(t - 1)(t - 4)$ **M1**

$a = 0$ at $t = 1$ and $t = 4$ **A1**

(b) $v(1) = 2 - 15 + 24 = 11$ m s^{-1} (local maximum since a changes from $+$ to $-$)

$v(4) = 128 - 240 + 96 = -16$ m s^{-1} (local minimum since a changes from $-$ to $+$) **A1**

(c) Zeros of v: $2t^3 - 15t^2 + 24t = t(2t^2 - 15t + 24)$

$t = 0$; $t = \dfrac{15 \pm \sqrt{225 - 192}}{4} = \dfrac{15 \pm \sqrt{33}}{4}$: $t \approx 2.32$ and $t \approx 5.18$ **M1**

Total distance $= \displaystyle\int_0^6 |v(t)|\,dt \approx 17.4 + 22.1 + 8.44 \approx 47.9$ m

(Using GDC: $\int_0^{2.32} |v|\,dt + \int_{2.32}^{5.18} |v|\,dt + \int_{5.18}^{6} |v|\,dt \approx 17.4 + 22.2 + 8.40 \approx 48.0$ m) **A1**

Note: Accept 48.0 *m from GDC or any value in* $[47.9, 48.1]$.

Q29 — $a = v\,dv/dx$ form: show v^2 formula, speed, next rest HARD

(a) $v\,\dfrac{dv}{dx} = 4x - x^2$ **M1**

$\displaystyle\int v\,dv = \int (4x - x^2)\,dx$ **M1**

$\dfrac{v^2}{2} = 2x^2 - \dfrac{x^3}{3} + C$

At $x = 0$, $v = 0$ (starts from rest): $C = 0$

$v^2 = 4x^2 - \dfrac{2x^3}{3}$ **A1 AG**

(b) At $x = 3$: $v^2 = 4(9) - \dfrac{2(27)}{3} = 36 - 18 = 18$

Speed $= \sqrt{18} = 3\sqrt{2} \approx 4.24$ m s^{-1} **A1**

(c) Next rest: $v^2 = 0 \implies 4x^2 - \dfrac{2x^3}{3} = 0 \implies x^2\left(4 - \dfrac{2x}{3}\right) = 0 \implies x = 6$ (since $x > 0$) **A1**

Note: The total distance from O until it returns to rest is $x = 6$ m (travels to $x = 6$ without reversing since $v^2 \geq 0$ throughout $(0, 6)$).

Q30 — Unknown m: show m=4 from displacement, rest times, total distance HARD

(a) Net displacement $= \displaystyle\int_0^6 (t^2 - 4t + m)\,dt = \left[\tfrac{t^3}{3} - 2t^2 + mt\right]_0^6$ **M1**

$= 72 - 72 + 6m = 6m$

$6m = 24 \implies m = 4$ **A1 AG**

(b) $v(t) = t^2 - 4t + 4 = (t-2)^2 = 0 \implies t = 2$ (double root) **A1**

(c) $(t-2)^2 \geq 0$ for all t, so $v \geq 0$ throughout $[0, 6]$ and the particle never moves backwards. **M1**

Total distance $= \displaystyle\int_0^6 (t-2)^2\,dt = \left[\tfrac{(t-2)^3}{3}\right]_0^6 = \frac{64}{3} - \frac{-8}{3} = \frac{72}{3} = 24$ m **A1**

Common Mistakes to Avoid

1. **Displacement $\neq$ Distance.** $\int_a^b v\,dt$ gives net (signed) displacement. For total distance you must integrate $|v(t)|$, splitting the integral at every sign change.
2. **Forgetting the constant of integration.** When finding $s(t) = \int v\,dt$, always apply the initial condition (e.g. $s(0) = 0$ or $s(0) = x_0$) to determine C before evaluating.
3. **Double roots and direction changes.** If $v(t_0) = 0$ is a double root, the particle touches zero velocity but does *not* change direction — v does not change sign there. Never assume $v = 0$ means the particle reverses.
4. **Speed vs Velocity.** Speed is $|v(t)|$, which is always non-negative. The *maximum speed* may occur at an endpoint or at a local extremum of $|v|$ — check both.
5. **Units in kinematic chains.** If v is in m s^{-1} and t in seconds, then $\int v\,dt$ gives metres. Keep track of units throughout, especially in ODE questions.

6. **Terminal velocity.** When $a \to 0$ as $t \to \infty$, terminal velocity is found by setting $a = 0$ in the equation of motion, *not* by differentiating $v(t)$.