IB Math AI HL — Topic 4 [Set 1]

Probability Distributions & Expected Values

Discrete Probability Distributions · Probability Tables · Expected Value · Variance · Fair Games

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated. GDC permitted throughout.

Key Formulae

- **Probability axioms:** $0 \le P(X=x) \le 1$ for all x; $\sum_{\text{all } x} P(X=x) = 1$
- **Expected value:** $E(X) = \mu = \sum_{\text{all } x} x \cdot P(X=x)$
- **Variance:** $\text{Var}(X) = \sigma^2 = \sum_{\text{all } x} x^2 \cdot P(X=x) - \mu^2 = E(X^2) - [E(X)]^2$
- **Standard deviation:** $\sigma = \sqrt{\text{Var}(X)}$
- **Linear transformations:** $E(aX+b) = aE(X)+b$; $\text{Var}(aX+b) = a^2\,\text{Var}(X)$
- **Fair game:** A game is fair when $E(\text{net gain}) = 0$, i.e. expected gain equals cost to play.
- **Long-run average:** Over n independent plays, total expected outcome $= n \cdot E(X)$.

GDC Tips

- To compute $E(X)$: enter x-values in `L1`, probabilities in `L2`, then use `1-Var Stats L1, L2`. The value $\bar{x}$ gives $E(X)$.
- To compute $\text{Var}(X)$: square σ_x from `1-Var Stats` output.
- When solving for an unknown probability k: set up the equation $\sum P = 1$, solve algebraically, then verify on GDC.
- Use `Table` mode to evaluate $x \cdot P(x)$ and $x^2 \cdot P(x)$ efficiently for variance calculations.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)

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1. **EASY** **GDC** **[3 marks]**

The random variable X represents the number of goals scored by a football team in a randomly chosen match. The probability distribution of X is shown in the table below.

x	0	1	2	3	4
$P(X = x)$	0.15	0.30	k	0.20	0.05

(a) Write down the value of k. [1]

(b) Find $P(X \geq 2)$. [1]

(c) Find the expected number of goals scored. [1]

2. EASY GDC **[3 marks]**

A spinner has four sectors labelled 1, 2, 3 and 4. The probability that the spinner lands on each sector is given below.

Sector	1	2	3	4
Probability	0.10	0.25	0.40	0.25

Let X be the number shown after one spin.

(a) Verify that the table represents a valid probability distribution. [1]

(b) Find $E(X)$. [1]

(c) Find $P(X > 2)$. [1]

3. EASY GDC **[3 marks]**

The discrete random variable W has the following probability distribution:

w	-1	0	3
$P(W = w)$	0.4	0.3	0.3

(a) Find $E(W)$. [1]

(b) Find $E(3W - 2)$. [1]

(c) State whether $E(W) > 0$, $E(W) = 0$, or $E(W) < 0$, and interpret this in context. [1]

4. EASY GDC **[3 marks]**

A bag contains 3 red, 5 blue, and 2 green counters. One counter is chosen at random. Let X be the prize amount (in dollars) awarded: $4 for red, $1 for blue, and $0 for green.

(a) Write down the probability distribution of X in a table. [1]

(b) Find $E(X)$. [1]

(c) A player pays $2 to play. Determine whether the game is fair for the player. [1]

5. EASY GDC **[3 marks]**

The discrete random variable Y takes only the values 0, 1, and 2. It is known that $P(Y = 0) = 0.3$ and $P(Y = 1) = 0.5$.

(a) Write down $P(Y = 2)$. [1]

(b) Find $E(Y)$. [1]

(c) Find $P(Y < 2)$. [1]

6. EASY GDC **[3 marks]**

The table shows the probability distribution of Z, the number of text messages sent by a student in a one-hour study session.

z	0	1	2	3
$P(Z = z)$	0.50	0.30	0.15	0.05

(a) Write down the modal value of Z. [1]
(b) Find $E(Z)$. [1]
(c) Find $P(Z \leq 1)$. [1]

7. EASY GDC **[3 marks]**

A discrete random variable X has $E(X) = 2.4$ and Var$(X) = 1.6$.

(a) Write down $E(5X)$. [1]
(b) Find Var$(5X)$. [1]
(c) Find the standard deviation of X. [1]

8. EASY GDC **[3 marks]**

A discrete random variable T takes the values 10, 20 and 30, each with equal probability.

(a) Write down $P(T = 10)$. [1]
(b) Find $E(T)$. [1]
(c) Find $E(T^2) - [E(T)]^2$. [1]

9. EASY GDC **[3 marks]**

The probability distribution of X, the number of defective items found in a randomly selected box of components, is given below.

x	0	1	2	3
$P(X = x)$	0.60	0.25	0.10	0.05

(a) Find the probability that a box contains at most one defective item. [1]

(b) Find $E(X)$. [1]

(c) A factory inspector checks 200 such boxes. Find the expected total number of defective items found. [1]

10. EASY GDC **[3 marks]**

A discrete random variable X has $E(X) = 5$ and Var$(X) = 4$.

(a) Find $E(2X + 3)$. [1]

(b) Find Var$(2X + 3)$. [1]

(c) Find the standard deviation of $2X + 3$. [1]

Section B — Medium **MEDIUM** (10 questions $\times$ 4 marks = 40 marks)

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11. MEDIUM GDC **[4 marks]**

The probability distribution of X, the score on a biased six-sided die, is given in the table below, where a and b are constants.

x	1	2	3	4	5	6
$P(X=x)$	a	a	0.15	0.20	b	0.10

It is known that $E(X) = 3.35$.

(a) Write down two equations in a and b. [2]

(b) Find the values of a and b. [1]

(c) Find $P(X \leq 3)$. [1]

12. MEDIUM GDC **[4 marks]**

A game at a school fair involves rolling a standard fair die. A player wins $6 if the die shows a 6, wins $2 if the die shows a 4 or 5, and loses $1 if the die shows 1, 2, or 3. Let X be the net gain (in dollars) for one play.

(a) Write down the probability distribution of X. [2]

(b) Find $E(X)$ and comment on whether the game favours the player. [2]

13. MEDIUM GDC **[4 marks]**

The discrete random variable X has the probability distribution shown below.

x	1	2	3	4	5
$P(X = x)$	$2k$	$3k$	$4k$	$3k$	$2k$

(a) Show that $k = \frac{1}{14}$. [1]

(b) Find $E(X)$. [1]

(c) Find Var(X). [2]

14. MEDIUM GDC **[4 marks]**

Each morning, Layla records whether she cycles to school (C) or takes the bus (B). The probability that she cycles on any given morning is 0.6. She cycles or takes the bus independently each day.

Over a two-day period, let X be the number of mornings she cycles.

(a) Complete the probability distribution table for X. [2]

x	0	1	2
$P(X = x)$			

(b) Find $E(X)$. [1]

(c) Find the probability that Layla cycles on at least one morning. [1]

15. MEDIUM GDC **[4 marks]**

The probability distribution of Y, the daily profit (in hundreds of dollars) made by a market stall, is given below.

y	-2	0	1	3	5
$P(Y = y)$	0.10	0.20	0.35	0.25	0.10

(a) Find $E(Y)$. [2]

(b) Find Var(Y). [2]

16. MEDIUM GDC **[4 marks]**

A carnival game has a wheel with five sectors, each equally likely to be landed on. Two sectors pay out \$5, one sector pays out \$10, and two sectors pay nothing. A player pays c dollars to play one game. Let X be the net gain for the player after one spin.

(a) Write down $E(X)$ in terms of c. [2]

(b) Find the value of c that makes the game fair. [1]

(c) If $c = 4$, find the expected net gain for the player over 25 games. [1]

17. MEDIUM GDC **[4 marks]**

The discrete random variable V has the following probability distribution, where p is a constant.

v	0	1	2	4
$P(V=v)$	0.3	p	0.2	0.1

(a) Find the value of p. [1]

(b) Find $E(V)$ and $E(V^2)$. [2]

(c) Hence find Var(V). [1]

18. MEDIUM GDC **[4 marks]**

Two fair coins are flipped simultaneously. Let X be the number of heads obtained.

(a) Write down the sample space and find the probability distribution of X. [2]

(b) Find $E(X)$ and Var(X). [2]

19. MEDIUM GDC **[4 marks]**

A restaurant offers a choice of three desserts: chocolate cake (C), fruit salad (F), and ice cream (I). Over a large number of observations, the probabilities of customers choosing each dessert are: $P(C) = 0.45$, $P(F) = 0.30$, $P(I) = 0.25$. The prices charged (in dollars) are $7 for C, $4 for F, and $5 for I.

Let R be the revenue (in dollars) from one randomly selected dessert order.

(a) Write down the probability distribution of R. [1]

(b) Find $E(R)$. [1]

(c) In a sitting of 80 customers, all of whom order a dessert, find the expected total dessert revenue. [1]

(d) Find Var(R). [1]

20. MEDIUM GDC **[4 marks]**

The probability distribution of X is shown below, where $m > 3$ is a positive constant.

x	1	2	m	8
$P(X = x)$	0.2	0.3	0.3	0.2

It is given that $E(X) = 4$.

(a) Find the value of m. [2]

(b) Find Var(X). [2]

Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)

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21. **HARD** **GDC** **[5 marks]**

The probability distribution of X is given below, where a and b are constants with $a > 0$ and $b > 0$.

x	1	2	3	5
$P(X = x)$	$a + b$	$2a$	$a - b$	0.2

It is also given that $E(X) = 2.6$.

(a) Write down two equations in a and b. [2]

(b) Find the values of a and b. [2]

(c) Find $P(X \geq 2)$. [1]

22. HARD GDC **[5 marks]**

A shop sells handmade candles. Let X be the number of candles sold per day. The probability distribution of X is given below.

x	0	1	2	3	4
$P(X = x)$	0.05	0.20	0.40	0.25	0.10

Each candle is sold for \$18. The shop has a fixed daily overhead cost of \$12.
Let P be the daily profit (in dollars).

(a) Write down P as a function of X. [1]

(b) Find $E(P)$. [2]

(c) Find the probability that the shop makes a profit on a given day (i.e. $P > 0$). [1]

(d) Find Var(P). [1]

23. HARD GDC **[5 marks]**

A biased coin has probability p of landing heads, where $0 < p < 1$. The coin is tossed twice. Let X be the number of heads.

(a) Show that $E(X) = 2p$. [2]

(b) Show that Var$(X) = 2p(1 - p)$. [2]

(c) Given that Var$(X) = 0.32$, find the possible values of p. [1]

24. HARD GDC **[5 marks]**

A discrete random variable X takes values 0, 1, 2 and 3. It is known that:

- $P(X = 0) = P(X = 3) = t$
- $P(X = 1) = 2t$
- $P(X = 2) = 1 - 4t$

(a) Show that $E(X) = 2 - 3t$. [2]

(b) Find the value of t such that $E(X) = 1.4$. [1]

(c) For this value of t, find Var(X). [2]

25. HARD GDC **[5 marks]**

A game uses a fair die with faces numbered 1 to 6. A player pays $\$d$ to roll the die once. They win $10 if the die shows a prime number, and $0 otherwise. Let X be the net gain (winnings minus cost) for one play.

(a) Write down the probability distribution of X in terms of d. [2]

(b) Find $E(X)$ in terms of d. [1]

(c) The game organiser wants the game to favour the house. Find the range of values of d for which this is the case. [1]

(d) For the game to be fair, find the exact value of d. [1]

26. HARD GDC **[5 marks]**

The discrete random variable X has $E(X) = 3$ and $E(X^2) = 11$.

A new random variable Y is defined by $Y = 2X - 5$.

(a) Find Var(X). [1]

(b) Find $E(Y)$. [1]

(c) Find Var(Y). [1]

(d) Find $E(Y^2)$. [2]

27. HARD GDC **[5 marks]**

A quality-control inspector samples one item per hour from a production line. The number of faulty items X found in an 8-hour shift has the probability distribution shown below.

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.168	0.336	0.294	0.147	0.043	0.009	0.002	0.001	0.000

(a) Find $E(X)$. [2]

(b) Find Var(X). [2]

(c) If each faulty item costs the company $35 to repair, find the expected repair cost per shift. [1]

28. HARD GDC **[5 marks]**

Two independent discrete random variables X and Y satisfy:

$$E(X) = 4, \quad \text{Var}(X) = 3, \quad E(Y) = 2, \quad \text{Var}(Y) = 5.$$

A new random variable W is defined by $W = 3X - 2Y + 1$.

(a) Find $E(W)$. [2]

(b) Find $\text{Var}(W)$. [2]

(c) Find the standard deviation of W. [1]

29. HARD GDC **[5 marks]**

The random variable X takes values in the set $\{1, 2, 3, n\}$ where $n > 3$ is a positive integer. The probability distribution is:

x	1	2	3	n
$P(X = x)$	0.3	0.4	0.2	0.1

(a) Write down $E(X)$ in terms of n. [1]

(b) Given that $\text{Var}(X) = 2.84$, find the value of n. [4]

30. HARD GDC **[5 marks]**

A fundraising raffle sells 500 tickets at \$3 each. There are three prizes: one prize of \$200, two prizes of \$50, and five prizes of \$20. All other tickets win nothing. Farida buys one ticket.

Let X be the net gain (in dollars) for Farida from her single ticket.

(a) Write down the probability distribution of X. [2]

(b) Find $E(X)$. [2]

(c) Determine whether the raffle is a fair game for Farida. Justify your answer. [1]

Common Mistakes to Avoid

1. **Forgetting $\sum P = 1$.** Always verify the probabilities sum to exactly 1 before computing $E(X)$. A missing or wrong k value cascades through the whole question.
2. **$E(X^2) \neq [E(X)]^2$.** These are different quantities. Var$(X) = E(X^2) - [E(X)]^2 \geq 0$, never negative.
3. **Variance of a linear transform.** Var$(aX + b) = a^2$ Var(X). The constant b does *not* affect the variance — shifting a distribution does not change its spread.
4. **Combining variances of independent variables.** Var$(aX \pm bY) = a^2$ Var$(X) + b^2$ Var(Y) — note the $+$ sign even when subtracting Y.
5. **Fair-game condition.** A game is fair when $E(\text{net gain}) = 0$. Remember that "net gain" = winnings − cost to play.
6. **Probability vs probability distribution.** Writing a single number instead of a complete table loses the method marks for constructing the distribution.

Worked Solutions

Official Mark-Scheme Style — M / A / R notation

Q1 – Discrete Probability Distribution EASY

(a) $\sum P = 1$: $0.15 + 0.30 + k + 0.20 + 0.05 = 1 \implies k = 0.30$ **A1**

(b) $P(X \geq 2) = 0.30 + 0.20 + 0.05 = 0.55$ **A1**

(c) $E(X) = 0(0.15) + 1(0.30) + 2(0.30) + 3(0.20) + 4(0.05)$
$= 0 + 0.30 + 0.60 + 0.60 + 0.20 = 1.70$ **A1**

Q2 – Valid Distribution and Expected Value EASY

(a) $0.10 + 0.25 + 0.40 + 0.25 = 1.00$ and each probability $\in [0, 1]$; valid distribution **A1**

(b) $E(X) = 1(0.10) + 2(0.25) + 3(0.40) + 4(0.25)$
$= 0.10 + 0.50 + 1.20 + 1.00 = 2.80$ **A1**

(c) $P(X > 2) = P(X = 3) + P(X = 4) = 0.40 + 0.25 = 0.65$ **A1**

Q3 – Linear Transformation of EX EASY

(a) $E(W) = (-1)(0.4) + (0)(0.3) + (3)(0.3) = -0.4 + 0 + 0.9 = 0.5$ **A1**

(b) $E(3W - 2) = 3E(W) - 2 = 3(0.5) - 2 = -0.5$ **A1**

(c) $E(W) = 0.5 > 0$; on average the outcome is positive (e.g. positive average score/gain) **A1**

Q4 – Expected Value and Fair Game EASY

(a) Probability distribution of X:

x	0	1	4
$P(X = x)$	$\frac{2}{10} = 0.2$	$\frac{5}{10} = 0.5$	$\frac{3}{10} = 0.3$

A1

(b) $E(X) = 0(0.2) + 1(0.5) + 4(0.3) = 0 + 0.5 + 1.2 = 1.70$ **A1**

(c) Net gain $= E(X) -$ cost $= 1.70 - 2.00 = -0.30 < 0$; the game is **not** fair for the player (player loses \$0.30 on average per game). **A1**

Q5 – Simple DRV from Given Probabilities EASY

(a) $P(Y = 2) = 1 - 0.3 - 0.5 = 0.2$ **A1**

(b) $E(Y) = 0(0.3) + 1(0.5) + 2(0.2) = 0 + 0.5 + 0.4 = 0.9$ **A1**

(c) $P(Y < 2) = P(Y = 0) + P(Y = 1) = 0.3 + 0.5 = 0.8$ **A1**

Q6 – Mode and Expected Value from Table EASY

(a) Modal value of Z is **0** (highest probability 0.50). **A1**

(b) $E(Z) = 0(0.50) + 1(0.30) + 2(0.15) + 3(0.05) = 0 + 0.30 + 0.30 + 0.15 = 0.75$ **A1**

(c) $P(Z \leq 1) = P(Z = 0) + P(Z = 1) = 0.50 + 0.30 = 0.80$ **A1**

Q7 – Linear Transformation: E and Var EASY

(a) $E(5X) = 5E(X) = 5 \times 2.4 = 12$ **A1**

(b) $\text{Var}(5X) = 5^2 \times \text{Var}(X) = 25 \times 1.6 = 40$ **A1**

(c) $\sigma(X) = \sqrt{1.6} = 1.26\ldots \approx 1.26$ **A1**

Q8 – Equal-Probability DRV and Variance EASY

(a) $P(T = 10) = \frac{1}{3} \approx 0.333$ **A1**

(b) $E(T) = \frac{1}{3}(10 + 20 + 30) = \frac{60}{3} = 20$ **A1**

(c) $E(T^2) = \frac{1}{3}(100 + 400 + 900) = \frac{1400}{3}$
$E(T^2) - [E(T)]^2 = \frac{1400}{3} - 400 = \frac{1400-1200}{3} = \frac{200}{3} \approx 66.7$ **A1**
Note: This is Var(T).

Q9 – Defective Items: Probability and Long-run Total EASY

(a) $P(X \leq 1) = P(X = 0) + P(X = 1) = 0.60 + 0.25 = 0.85$ **A1**

(b) $E(X) = 0(0.60) + 1(0.25) + 2(0.10) + 3(0.05) = 0 + 0.25 + 0.20 + 0.15 = 0.60$ **A1**

(c) Expected total $= 200 \times E(X) = 200 \times 0.60 = 120$ defective items **A1**

Q10 – Linear Transform: E and Var given EASY

(a) $E(2X + 3) = 2E(X) + 3 = 2(5) + 3 = 13$ **A1**

(b) $\text{Var}(2X + 3) = 2^2 \cdot \text{Var}(X) = 4 \times 4 = 16$ **A1**

(c) $\text{SD}(2X+3) = \sqrt{16} = 4$ **A1**

Q11 – Find Parameters from Sum and EX MEDIUM

(a) Sum of probabilities $= 1$:

$$2a + b + 0.15 + 0.20 + 0.10 = 1 \implies 2a + b = 0.55 \quad \textbf{A1}$$

Expected value $= 3.35$:

$$1(a) + 2(a) + 3(0.15) + 4(0.20) + 5(b) + 6(0.10) = 3.35$$

$$3a + 5b + 0.45 + 0.80 + 0.60 = 3.35 \implies 3a + 5b = 1.50 \quad \textbf{A1}$$

(b) From $2a + b = 0.55 \Rightarrow b = 0.55 - 2a$. Substituting:

$$3a + 5(0.55 - 2a) = 1.50 \implies 3a + 2.75 - 10a = 1.50 \implies -7a = -1.25 \implies a = \frac{5}{28} \approx 0.179$$

$$b = 0.55 - 2\left(\tfrac{5}{28}\right) = 0.55 - \tfrac{10}{28} = \tfrac{15.4-10}{28} = \tfrac{5.4}{28}$$

$a = 0.179$ (3 s.f.), $\quad b = 0.193$ (3 s.f.) **A1**

Note: Accept exact fractions $a = 5/28,\ b = 27/140$.

(c) $P(X \le 3) = a + a + 0.15 = 2(0.179) + 0.15 = 0.507$ **A1**

Note: FT from their a.

Q12 – Game: Distribution and Fair Test MEDIUM

(a) Net gain values and probabilities:

- $X = 6$: show 6; $P = \frac{1}{6}$
- $X = 2$: show 4 or 5; $P = \frac{2}{6} = \frac{1}{3}$
- $X = -1$: show 1, 2, or 3; $P = \frac{3}{6} = \frac{1}{2}$

A1 A1

(b) $E(X) = 6\cdot\frac{1}{6} + 2\cdot\frac{1}{3} + (-1)\cdot\frac{1}{2} = 1 + \frac{2}{3} - \frac{1}{2} = 1 + 0.667 - 0.500 = 1.167\ldots \approx \$1.17$

$E(X) > 0$; the game **favours the player** (expected profit of $\approx \$1.17$ per play). **A1 A1**

Q13 – Proportional Probabilities: Show k, EX, VarX MEDIUM

(a) $\sum P = (2k + 3k + 4k + 3k + 2k) = 14k = 1 \implies k = \dfrac{1}{14}$ **AG**

(b) $E(X) = \dfrac{1}{14}[1(2) + 2(3) + 3(4) + 4(3) + 5(2)] = \dfrac{2 + 6 + 12 + 12 + 10}{14} = \dfrac{42}{14} = 3$ **A1**

(c) $E(X^2) = \dfrac{1}{14}[1^2(2) + 2^2(3) + 3^2(4) + 4^2(3) + 5^2(2)]$

$= \frac{1}{14}(2+12+36+48+50) = \frac{148}{14} = \frac{74}{7}$

Var$(X) = E(X^2) - [E(X)]^2 = \frac{74}{7} - 9 = \frac{74-63}{7} = \frac{11}{7} \approx 1.57$ **M1 A1**

Q14 – Two-Day Distribution from Single-Day Probability MEDIUM

(a) $P(C) = 0.6$, $P(B) = 0.4$, independent days:

$$P(X=0) = 0.4 \times 0.4 = 0.16$$

$$P(X=1) = 2 \times 0.6 \times 0.4 = 0.48$$

$$P(X=2) = 0.6 \times 0.6 = 0.36$$

A1 A1

(b) $E(X) = 0(0.16) + 1(0.48) + 2(0.36) = 0 + 0.48 + 0.72 = 1.20$ **A1**

(c) $P(X \geq 1) = 1 - P(X=0) = 1 - 0.16 = 0.84$ **A1**

Q15 – EY and VarY from Profit Distribution MEDIUM

(a) $E(Y) = (-2)(0.10) + 0(0.20) + 1(0.35) + 3(0.25) + 5(0.10)$
$= -0.20 + 0 + 0.35 + 0.75 + 0.50 = 1.40$ (hundreds of dollars) **M1 A1**

(b) $E(Y^2) = (-2)^2(0.10) + 0^2(0.20) + 1^2(0.35) + 3^2(0.25) + 5^2(0.10)$
$= 0.40 + 0 + 0.35 + 2.25 + 2.50 = 5.50$
Var$(Y) = 5.50 - (1.40)^2 = 5.50 - 1.96 = 3.54$ **M1 A1**

Q16 – Fair-Game Cost Parameter MEDIUM

Payouts: \$5 with $P = \frac{2}{5}$; \$10 with $P = \frac{1}{5}$; \$0 with $P = \frac{2}{5}$.

(a) $E(\text{winnings}) = 5 \cdot \frac{2}{5} + 10 \cdot \frac{1}{5} + 0 \cdot \frac{2}{5} = 2 + 2 + 0 = 4$
Net gain X = winnings $- c$, so $E(X) = 4 - c$ **A1 A1**

(b) Fair game: $E(X) = 0 \implies 4 - c = 0 \implies c = \$4$ **A1**

(c) If $c = 4$: $E(X) = 4 - 4 = 0$; expected net gain over 25 games $= 25 \times 0 = \$0$ **A1**

Q17 – EV and VarV MEDIUM

(a) $p = 1 - 0.3 - 0.2 - 0.1 = 0.4$ **A1**

(b) $E(V) = 0(0.3) + 1(0.4) + 2(0.2) + 4(0.1) = 0 + 0.4 + 0.4 + 0.4 = 1.20$ **A1**
$E(V^2) = 0^2(0.3) + 1^2(0.4) + 2^2(0.2) + 4^2(0.1) = 0 + 0.4 + 0.8 + 1.6 = 2.80$ **A1**
(c) $\text{Var}(V) = E(V^2) - [E(V)]^2 = 2.80 - (1.20)^2 = 2.80 - 1.44 = 1.36$ **A1**

Q18 – Two Coins: Distribution, EX and VarX MEDIUM

(a) Sample space: $\{HH, HT, TH, TT\}$ — 4 equally likely outcomes.
$P(X = 0) = \frac{1}{4}$; $P(X = 1) = \frac{2}{4} = \frac{1}{2}$; $P(X = 2) = \frac{1}{4}$ **A1 A1**
(b) $E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 0 + 0.5 + 0.5 = 1$ **A1**
$E(X^2) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 0 + 0.5 + 1.0 = 1.5$
$\text{Var}(X) = 1.5 - 1^2 = 0.5$ **A1**

Q19 – Revenue Distribution, ER and VarR MEDIUM

(a) Probability distribution of R:

R (\$)	4	5	7
$P(R)$	0.30	0.25	0.45

A1
(b) $E(R) = 4(0.30) + 5(0.25) + 7(0.45) = 1.20 + 1.25 + 3.15 = 5.60$ **A1**
(c) Expected total revenue $= 80 \times 5.60 = \$448$ **A1**
(d) $E(R^2) = 16(0.30) + 25(0.25) + 49(0.45) = 4.80 + 6.25 + 22.05 = 33.10$
$\text{Var}(R) = 33.10 - (5.60)^2 = 33.10 - 31.36 = 1.74$ **A1**

Q20 – Find Parameter m Then VarX MEDIUM

(a) Setting $E(X) = 4$:
$$1(0.2) + 2(0.3) + m(0.3) + 8(0.2) = 4$$
$$0.2 + 0.6 + 0.3m + 1.6 = 4 \implies 0.3m = 1.6$$
M1
$m = \dfrac{16}{3} \approx 5.33$ **A1**
(b) $E(X^2) = 1^2(0.2) + 2^2(0.3) + \left(\frac{16}{3}\right)^2(0.3) + 8^2(0.2)$
$= 0.2 + 1.2 + \dfrac{256}{9}(0.3) + 12.8 = 14.2 + \dfrac{76.8}{9} = 14.2 + 8.5\overline{3} = 22.7\overline{3}$
$\text{Var}(X) = 22.7\overline{3} - 4^2 = 22.7\overline{3} - 16 = 6.7\overline{3} \approx 6.73$ **M1 A1**

Q21 – Two-Parameter Distribution: Solve System HARD

(a) Sum of probabilities $= 1$:

$$(a+b)+2a+(a-b)+0.2=1 \implies 4a=0.8 \implies a=0.2 \quad \textbf{A1}$$

Expected value $= 2.6$:

$$1(a+b)+2(2a)+3(a-b)+5(0.2)=2.6$$

$$a+b+4a+3a-3b+1.0=2.6 \implies 8a-2b=1.6 \quad \textbf{A1}$$

(b) From (a): $a=0.2$; substitute into $8(0.2)-2b=1.6$:
$1.6-2b=1.6 \implies b=0$ **A1**
Verify: $P(X=1)=0.2,\ P(X=2)=0.4,\ P(X=3)=0.2,\ P(X=5)=0.2$; *sum* $=1$ ✓

(c) $P(X\geq 2)=0.4+0.2+0.2=0.8$ **A1**

Note: Award A1 FT from their a and b provided all probabilities are non-negative.

Q22 – Profit as Linear Function of X HARD

(a) $P=18X-12$ **A1**

(b) $E(X)=0(0.05)+1(0.20)+2(0.40)+3(0.25)+4(0.10)$
$=0+0.20+0.80+0.75+0.40=2.15$ **A1**
$E(P)=18\times 2.15-12=38.70-12=\$26.70$ **A1**

(c) $P>0 \iff 18X-12>0 \iff X>\frac{12}{18}=\frac{2}{3}$, so $X\geq 1$:
$P(X\geq 1)=1-P(X=0)=1-0.05=0.95$ **A1**

(d) Var$(P)=18^2\cdot$ Var$(X)=324\cdot$ Var(X)
$E(X^2)=0+0.20+1.60+2.25+1.60=5.65$; Var$(X)=5.65-2.15^2=5.65-4.6225=1.0275$
Var$(P)=324\times 1.0275=332.91\ldots\approx 333$ **A1**

Q23 – Biased Coin: Show EX and VarX in terms of p HARD

Distribution of X (two tosses, $P(\text{H})=p$):

$$P(X=0)=(1-p)^2,\quad P(X=1)=2p(1-p),\quad P(X=2)=p^2$$

(a) $E(X)=0(1-p)^2+1\cdot 2p(1-p)+2p^2$
$=2p(1-p)+2p^2=2p-2p^2+2p^2=2p$ **AG** **M1 A1**

(b) $E(X^2)=0+1\cdot 2p(1-p)+4p^2=2p-2p^2+4p^2=2p+2p^2$
Var$(X)=(2p+2p^2)-(2p)^2=2p+2p^2-4p^2=2p-2p^2=2p(1-p)$ **AG** **M1 A1**

(c) $2p(1-p)=0.32 \implies 2p-2p^2=0.32 \implies p^2-p+0.16=0$
$(p-0.2)(p-0.8)=0 \implies p=0.2$ or $p=0.8$ **A1**

Q24 – Parameterised Distribution: Show EX, Find t, Find VarX HARD

(a) Verify sum: $t + 2t + (1 - 4t) + t = 4t + 1 - 4t = 1$ **M1**

$E(X) = 0(t) + 1(2t) + 2(1 - 4t) + 3(t) = 2t + 2 - 8t + 3t = 2 - 3t$ **AG A1**

(b) $2 - 3t = 1.4 \implies 3t = 0.6 \implies t = 0.2$ **A1**

Verify: $P(X = 0) = 0.2,\ P(X = 1) = 0.4,\ P(X = 2) = 0.2,\ P(X = 3) = 0.2$; *sum* $= 1$ $\surd$

(c) $E(X^2) = 0^2(0.2) + 1^2(0.4) + 2^2(0.2) + 3^2(0.2) = 0 + 0.4 + 0.8 + 1.8 = 3.0$ **M1**

Var$(X) = 3.0 - (1.4)^2 = 3.0 - 1.96 = 1.04$ **A1**

Q25 – Prime-Number Game: Distribution, EX, Fair Value HARD

Primes on a die: 2, 3, 5 $\Rightarrow$ 3 out of 6. Non-primes: 1, 4, 6 $\Rightarrow$ 3 out of 6.

(a) Distribution of net gain X (winnings $-d$):

- $X = 10 - d$: roll a prime; $P = \frac{1}{2}$
- $X = -d$: roll a non-prime; $P = \frac{1}{2}$

A1 A1

(b) $E(X) = (10 - d)\cdot\frac{1}{2} + (-d)\cdot\frac{1}{2} = \frac{10-d-d}{2} = 5 - d$ **A1**

(c) Game favours house when $E(X) < 0$: $5 - d < 0 \implies d > 5$ **A1**

(d) Fair game: $E(X) = 0 \implies 5 - d = 0 \implies d = \$5$ **A1**

Q26 – Derived Variable Y=2X-5: Var and EY and VarY HARD

(a) Var$(X) = E(X^2) - [E(X)]^2 = 11 - 3^2 = 11 - 9 = 2$ **A1**

(b) $E(Y) = E(2X - 5) = 2E(X) - 5 = 2(3) - 5 = 1$ **A1**

(c) Var$(Y) =$ Var$(2X - 5) = 4\,$Var$(X) = 4 \times 2 = 8$ **A1**

(d) $E(Y^2) =$ Var$(Y) + [E(Y)]^2 = 8 + 1^2 = 9$ **M1 A1**

Q27 – Quality Control: EX and VarX and Expected Cost HARD

(a) $E(X) = 0(0.168) + 1(0.336) + 2(0.294) + 3(0.147) + 4(0.043) + 5(0.009) + 6(0.002) + 7(0.001)$
$= 0 + 0.336 + 0.588 + 0.441 + 0.172 + 0.045 + 0.012 + 0.007 = 1.601\ldots \approx 1.60$ **M1 A1**

(b) $E(X^2) = 0 + 0.336 + 1.176 + 1.323 + 0.688 + 0.225 + 0.072 + 0.049 = 3.869$

Var$(X) = 3.869 - (1.601)^2 = 3.869 - 2.563 = 1.306\ldots \approx 1.31$ **M1 A1**

(c) Expected repair cost $= 35 \times E(X) = 35 \times 1.601 = \$56.04\ldots \approx \$56.0$ **A1**

Q28 – Independent Variables: EW and VarW HARD

$W = 3X - 2Y + 1$

(a) $E(W) = 3E(X) - 2E(Y) + 1 = 3(4) - 2(2) + 1 = 12 - 4 + 1 = 9$ **M1 A1**

(b) Since X and Y are independent:

$\text{Var}(W) = 3^2\,\text{Var}(X) + (-2)^2\,\text{Var}(Y) = 9(3) + 4(5) = 27 + 20 = 47$ **M1 A1**

(c) $\text{SD}(W) = \sqrt{47} = 6.855\ldots \approx 6.86$ **A1**

Q29 – Find Unknown Integer n from VarX Condition HARD

(a) $E(X) = 1(0.3) + 2(0.4) + 3(0.2) + n(0.1) = 0.3 + 0.8 + 0.6 + 0.1n = 1.7 + 0.1n$ **A1**

(b) $E(X^2) = 1(0.3) + 4(0.4) + 9(0.2) + n^2(0.1) = 0.3 + 1.6 + 1.8 + 0.1n^2 = 3.7 + 0.1n^2$

$\text{Var}(X) = E(X^2) - [E(X)]^2 = (3.7 + 0.1n^2) - (1.7 + 0.1n)^2$ **M1**

$= 3.7 + 0.1n^2 - (2.89 + 0.34n + 0.01n^2)$

$= 3.7 + 0.1n^2 - 2.89 - 0.34n - 0.01n^2$

$= 0.81 + 0.09n^2 - 0.34n$ **A1**

Set equal to 2.84:

$$0.09n^2 - 0.34n + 0.81 = 2.84 \implies 0.09n^2 - 0.34n - 2.03 = 0$$

Multiply by $\frac{100}{9}$: $9n^2 - 34n - 203 = 0$ **M1**

$n = \dfrac{34 \pm \sqrt{34^2 + 4(9)(203)}}{18} = \dfrac{34 \pm \sqrt{1156 + 7308}}{18} = \dfrac{34 \pm \sqrt{8464}}{18} = \dfrac{34 \pm 92}{18}$

$n = \dfrac{34 + 92}{18} = \dfrac{126}{18} = 7$ (reject negative root)

$\therefore n = 7$ **A1**

Verify: $E(X) = 1.7 + 0.7 = 2.4$; $E(X^2) = 3.7 + 0.1(49) = 8.6$; *Var*$(X) = 8.6 - 5.76 = 2.84$ $\checkmark$

Q30 – Raffle: Distribution, EX, Fair-Game Determination HARD

Cost per ticket: \$3. Prizes and net gains (= prize − \$3):

- Win \$200: net gain = \$197; $P = \frac{1}{500}$
- Win \$50: net gain = \$47; $P = \frac{2}{500}$
- Win \$20: net gain = \$17; $P = \frac{5}{500}$
- Win nothing: net gain = −\$3; $P = \frac{492}{500}$

(a) Probability distribution of X:

x	197	47	17	-3
$P(X=x)$	$\frac{1}{500}$	$\frac{2}{500}$	$\frac{5}{500}$	$\frac{492}{500}$

A1 A1

(b) $E(X) = 197 \cdot \frac{1}{500} + 47 \cdot \frac{2}{500} + 17 \cdot \frac{5}{500} + (-3) \cdot \frac{492}{500}$ **M1**

$= \dfrac{197 + 94 + 85 - 1476}{500} = \dfrac{-1100}{500} = -\$2.20$ **A1**

(c) $E(X) = -2.20 < 0$; the raffle is **not** fair for Farida — she expects to lose \$2.20 per ticket on average. **R1**

Common Mistakes to Avoid

1. **Forgetting $\sum P = 1$.** Always verify the probabilities sum to exactly 1 before computing $E(X)$. A missing or wrong k value cascades through the whole question.
2. **$E(X^2) \neq [E(X)]^2$.** These are different quantities. Var$(X) = E(X^2) - [E(X)]^2 \geq 0$, never negative.
3. **Variance of a linear transform.** Var$(aX + b) = a^2$ Var(X). The constant b does *not* affect the variance.
4. **Combining variances of independent variables.** Var$(aX \pm bY) = a^2$ Var$(X) + b^2$ Var(Y) — note the $+$ sign even when subtracting Y.
5. **Fair-game condition.** A game is fair when $E(\text{net gain}) = 0$. Net gain $=$ winnings $-$ cost to play.
6. **Probability vs probability distribution.** Writing a single number instead of a complete table loses the method marks for constructing the distribution.