IB Math AI HL — Topic 3

Voronoi Diagrams

Drawing Voronoi Diagrams · Interpreting Voronoi Diagrams · Toxic Waste Dump Problem

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working unless told to *write down*. Give answers correct to 3 significant figures unless stated otherwise. GDC permitted throughout.

Key Formulae — Voronoi Diagrams

- **Midpoint:** $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
- **Gradient of segment AB:** $m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$
- **Gradient of perpendicular bisector:** $m_{\perp} = -\dfrac{1}{m_{AB}}$ (negative reciprocal)
- **Equation of perpendicular bisector:** passes through M with gradient $m_{\perp}$: $y - y_M = m_{\perp}(x - x_M)$
- **Distance formula:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- **Voronoi cell:** the set of all points closer to a given site than to any other site
- **Voronoi vertex:** point equidistant from exactly three (or more) sites; found by intersecting two perpendicular bisectors
- **Toxic Waste / Largest Empty Circle:** the optimal location is the Voronoi vertex with the greatest distance to its nearest site

GDC Tips

- To find the intersection of two lines, enter both as Y_1 and Y_2, then use **2nd** $\to$ **CALC** $\to$ **intersect**.
- To compute a distance, store coordinates in variables and evaluate $\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$ directly.
- When sketching Voronoi diagrams, plot all sites first, then draw the perpendicular bisectors by plotting

their equations.

- For area of a polygon with known vertices, use the **Polygon Area** feature or the Shoelace formula: $A = \frac{1}{2}|x_1(y_2 - y_n) + x_2(y_3 - y_1) + \cdots + x_n(y_1 - y_{n-1})|$.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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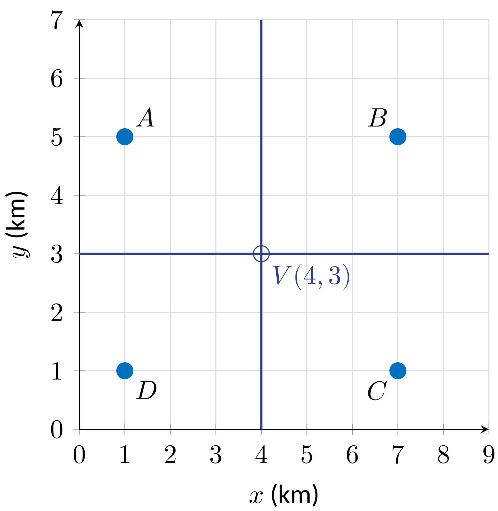
1. **EASY** **GDC** **[3 marks]**

The diagram below shows a Voronoi diagram for four mobile network towers located at $A(1,5)$, $B(7,5)$, $C(7,1)$, and $D(1,1)$. All distances are in kilometres. The Voronoi vertex is at $V(4,3)$.



(a) A delivery drone is located at the point $(5,4)$. Write down which tower is closest to the drone. [1]

(b) Write down the equation of the boundary between the regions of towers A and B. [1]

(c) A second drone is located at $(4,2.5)$. Write down which tower is closest to this drone. [1]

2. EASY GDC **[3 marks]**

Two weather stations are located at $P(2,6)$ and $Q(8,2)$.

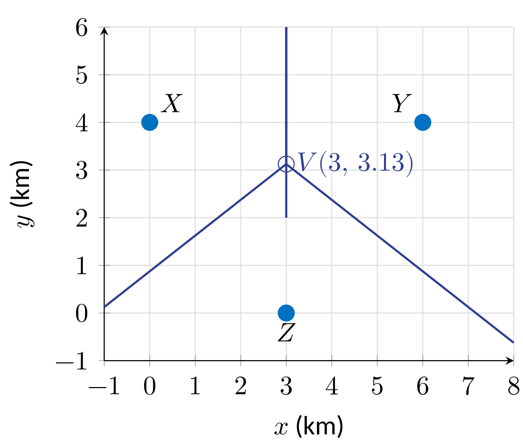
(a) Find the midpoint of $[PQ]$. [1]

(b) Find the gradient of the perpendicular bisector of $[PQ]$. [1]

(c) Hence write down the equation of the perpendicular bisector of $[PQ]$. [1]

3. EASY GDC **[3 marks]**

A Voronoi diagram for three coffee shops at $X(0,4)$, $Y(6,4)$, and $Z(3,0)$ has been drawn on the grid below. The perpendicular bisectors meet at $V(3,2)$.



(a) Write down the coordinates of V, the Voronoi vertex, correct to 3 significant figures. [1]

(b) A customer is at the point $(5,1)$. Write down which coffee shop is closest to this customer. [1]

(c) Write down the distance from V to site X, correct to 3 significant figures. [1]

4. EASY GDC **[3 marks]**

Two recycling centres are located at $R(3, 1)$ and $S(9, 5)$.

(a) Find the gradient of $[RS]$. [1]

(b) Find the midpoint M of $[RS]$. [1]

(c) Write down the gradient of the perpendicular bisector of $[RS]$. [1]

5. EASY GDC **[3 marks]**

Three hospitals are at $H_1(2, 2)$, $H_2(8, 2)$, and $H_3(5, 8)$. A Voronoi diagram has been constructed. The perpendicular bisector of $[H_1H_2]$ is the line $x = 5$.

(a) State what is true about all points on the line $x = 5$ with respect to hospitals H_1 and H_2. [1]

(b) An ambulance is at $(3, 6)$. Write down which hospital has the smallest Voronoi region containing this point. [1]

(c) A patient is located at $(5, 2)$. State how many hospitals are equidistant from this patient. [1]

6. EASY GDC **[3 marks]**

Two sports centres are modelled as points at $J(0, 3)$ and $K(6, 3)$.

(a) Write down the midpoint of $[JK]$. [1]

(b) Write down the equation of the perpendicular bisector of $[JK]$. [1]

(c) A resident lives at $(4, 5)$. Write down which sports centre is closer to this resident. [1]

7. EASY GDC **[3 marks]**

Three sites in a park are at $A(0,0)$, $B(6,0)$, and $C(3,4)$. A Voronoi diagram is constructed and the Voronoi vertex V is found to be at $(3,\ k)$ for some value of k.

(a) Write down the distance VA in terms of k. [1]

(b) Using $VA = VB$, show that this is satisfied for all k. [1]

(c) Given that $VC = VA$, find the value of k. [1]

8. EASY GDC **[3 marks]**

A Voronoi diagram has been constructed for five post offices. The region (cell) around each post office represents the area of the town from which that post office is closest.

(a) State what is true about a point on the boundary between two adjacent Voronoi cells. [1]

(b) A point P lies inside the cell for post office F. Write down the number of post offices that are closer to P than post office F. [1]

(c) State the geometric name for the point where three Voronoi cells meet. [1]

9. EASY GDC **[3 marks]**

Two supermarkets are located at $M(4,1)$ and $N(4,7)$.

(a) Write down the midpoint of $[MN]$. [1]

(b) Write down the gradient of the segment $[MN]$. [1]

(c) Hence write down the equation of the perpendicular bisector of $[MN]$. [1]

10. EASY GDC **[3 marks]**

Three sites are at $P(0,0)$, $Q(4,0)$, and $R(2,6)$. It is claimed that the point $T(2,\frac{5}{3})$ is equidistant from all three sites.

(a) Calculate the distance TP. Give your answer in exact form. [1]

(b) Calculate the distance TQ. [1]

(c) Hence determine whether the claim is correct. [1]

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. **MEDIUM** **GDC** **[4 marks]**

Three libraries are located at $A(1,6)$, $B(9,6)$, and $C(5,2)$, modelled on a coordinate grid where 1 unit $= 1$ km. A Voronoi diagram for these three sites has the vertex at $V(5,4)$.

A new library $D(5,10)$ is to be added to the diagram.

(a) Find the equation of the perpendicular bisector of $[AD]$. [2]

(b) Hence, find the equation of the perpendicular bisector of $[BD]$. State the reason for your answer. [1]

(c) Write down the coordinates of the new Voronoi vertex between sites A, B, and D. [1]

12. **MEDIUM** **GDC** **[4 marks]**

Two telecommunications towers are at $F(2,1)$ and $G(8,5)$. A third tower $H(2,5)$ is added to the network.

(a) Find the equation of the perpendicular bisector of $[FG]$. [2]

(b) The equation of the perpendicular bisector of $[GH]$ is $x = 5$. Find the coordinates of the Voronoi vertex V equidistant from F, G, and H. [2]

13. MEDIUM GDC **[4 marks]**

Three observation towers are positioned at $A(0, 6)$, $B(8, 6)$, and $C(4, 0)$. A Voronoi diagram is constructed and the Voronoi vertex is at $V(4, 3)$.

A new signal repeater is to be placed at the point within the triangle ABC that is as far as possible from any observation tower.

(a) Show that the distance from V to site A is $\sqrt{25} = 5$. [2]

(b) Explain why the repeater should be placed at V. [1]

(c) Write down the minimum distance between the repeater and any observation tower. [1]

14. MEDIUM GDC **[4 marks]**

Three charging stations for electric vehicles are at $P(0, 4)$, $Q(8, 4)$, and $R(4, 0)$. A new station $S(4, 8)$ is added.

(a) Find the equation of the perpendicular bisector of $[PS]$. [2]

(b) The equation of the perpendicular bisector of $[QS]$ is $y = \frac{1}{2}x + 2$. Find the coordinates of the Voronoi vertex equidistant from P, Q, and S. [2]

15. MEDIUM GDC **[4 marks]**

Four pharmacies are located at $A(1,1)$, $B(7,1)$, $C(7,7)$, and $D(1,7)$ in a town modelled on a 8×8 km grid. A Voronoi diagram is to be constructed.

(a) Write down the coordinates of the single Voronoi vertex. [1]

(b) Write down the equations of all four boundaries of the Voronoi diagram. [2]

(c) A customer at $(3,5)$ needs the nearest pharmacy. Determine which pharmacy they should visit, justifying your answer. [1]

16. MEDIUM GDC **[4 marks]**

Three bus stops are at $L(0,0)$, $M(10,0)$, and $N(5,6)$. The Voronoi vertex for these three sites is $V(5, \frac{13}{6})$.
A fourth bus stop is placed at $T(5,-4)$.

(a) Find the equation of the perpendicular bisector of $[LT]$. [2]

(b) The perpendicular bisector of $[MT]$ is $y = \frac{5}{4}x - \frac{65}{8}$. Find the new Voronoi vertex between L, M, and T. [2]

17. MEDIUM GDC **[4 marks]**

Three fire stations are at $A(0,0)$, $B(12,0)$, and $C(6,8)$. The Voronoi diagram for these three sites has a single interior vertex at V.

The equation of the perpendicular bisector of $[AB]$ is $x = 6$.
The equation of the perpendicular bisector of $[AC]$ is $y = \frac{3}{4}x - \frac{1}{8}$.

(a) Find the coordinates of V. [2]

(b) Find the distance VA. Give your answer correct to 3 significant figures. [2]

18. MEDIUM GDC **[4 marks]**

Four water access points are located at $W(0,0)$, $X(8,0)$, $Y(8,6)$, and $Z(0,6)$ in a rectangular nature reserve. The four Voronoi cells each have the same shape. The Voronoi vertex is at $V(4,3)$.

(a) Write down the equations of the two boundaries of the Voronoi diagram. [1]

(b) Find the area of the Voronoi cell belonging to site W. [2]

(c) A researcher says the four cells each cover the same percentage of the nature reserve. Determine whether this claim is correct, with justification. [1]

19. MEDIUM GDC **[4 marks]**

A Voronoi diagram for three train stations at $E(0, 5)$, $F(10, 5)$, and $G(5, 1)$ has vertex $V(5, 4)$. A new station $H(5, 9)$ is to be added.

(a) Find the equation of the perpendicular bisector of $[EH]$. [2]

(b) Find the y-coordinate of the new Voronoi vertex V' where the perpendicular bisectors of $[EH]$ and $[FH]$ meet, given that the perpendicular bisector of $[FH]$ is $y = -x + 12$. [1]

(c) State which original station's Voronoi cell is reduced in size when H is added. Give a reason. [1]

20. MEDIUM GDC **[4 marks]**

The table below shows the coordinates of four candidate Voronoi vertices V_1, V_2, V_3, V_4 in a region containing three industrial sites at $A(1, 4)$, $B(9, 4)$, and $C(5, 10)$.

Vertex	Coordinates	Distance to nearest site (km)
V_1	$(5,\ 4)$	
V_2	$(5,\ 6.67)$	
V_3	$(1,\ 7)$	
V_4	$(9,\ 7)$	

(a) Calculate the distance from V_1 to the nearest site. [1]

(b) Calculate the distance from V_2 to the nearest site. Give your answer correct to 3 sf. [2]

(c) Hence write down which vertex should be chosen for the site of a new facility that must be as far as possible from all three industrial sites. [1]

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD GDC **[5 marks]**

Three toxic waste disposal sites must be built within a rectangular region bounded by $0 \le x \le 12$, $0 \le y \le 8$. The sites are at $A(2,2)$, $B(10,2)$, and $C(6,7)$.

The equation of the perpendicular bisector of $[AB]$ is $x = 6$.
The equation of the perpendicular bisector of $[AC]$ is $y = \frac{4}{5}x - \frac{11}{5}$.

(a) Find the coordinates of the Voronoi vertex V. [2]

(b) Calculate the distance VA. [1]

(c) A monitoring station must be placed at the point inside the rectangle that is as far as possible from any disposal site. Determine whether V or one of the rectangle's corners provides the greatest minimum distance, and state the optimal location. [2]

22. HARD GDC **[5 marks]**

Four clinics are at $A(0,0)$, $B(10,0)$, $C(10,8)$, and $D(0,8)$ on a rectangular 10×8 km grid. The Voronoi diagram has boundaries at $x = 5$ and $y = 4$.

(a) Write down the coordinates of the Voronoi vertex. [1]

(b) A fifth clinic $E(5,4)$ is added at the Voronoi vertex. Find the equation of the perpendicular bisector of $[AE]$. [2]

(c) The perpendicular bisector of $[AE]$ forms the boundary between the Voronoi cells of A and E. Determine the y-intercept of this boundary and hence describe the effect on clinic A's cell. [2]

23. HARD GDC **[5 marks]**

Three sites are at $P(0,0)$, $Q(k,0)$, and $R(0,k)$ where $k > 0$. A Voronoi diagram is constructed.

(a) Show that the perpendicular bisector of $[PQ]$ is the line $x = \frac{k}{2}$. [1]

(b) Show that the perpendicular bisector of $[PR]$ is the line $y = \frac{k}{2}$. [1]

(c) Find the coordinates of the Voronoi vertex V in terms of k. [1]

(d) Given that the distance VP is $6\sqrt{2}$, find the value of k. [2]

24. HARD GDC **[5 marks]**

Three ranger posts are at $A(0,6)$, $B(8,6)$, and $C(4,0)$. The Voronoi vertex is at $V(4,3)$.

A fourth post $D(4,12)$ is added to the diagram.

(a) Show that the equation of the perpendicular bisector of $[AD]$ is $x = \frac{7}{3}y - \frac{7}{3}$. [2]

(b) The perpendicular bisector of $[BD]$ is $x = -\frac{7}{3}y + \frac{49}{3}$. Find the coordinates of the new Voronoi vertex V_1 equidistant from A, B, and D. [2]

(c) Hence write down, with a reason, which original ranger post's cell is most significantly affected by the addition of post D. [1]

25. HARD GDC **[5 marks]**

A square nature reserve occupies the region $0 \leq x \leq 10, 0 \leq y \leq 10$. Three ranger huts are at $A(2,8)$, $B(8,8)$, and $C(5,3)$.

The perpendicular bisector of $[AB]$ is $x = 5$.

The equation of the perpendicular bisector of $[AC]$ is $y = -\frac{3}{5}x + \frac{47}{5}$.

The equation of the perpendicular bisector of $[BC]$ is $y = \frac{3}{5}x + \frac{47}{5} - \frac{18}{5}$.

Note: The perpendicular bisector of $[BC]$ *is* $y = \frac{3}{5}x + \frac{29}{5}$.

(a) Find the coordinates of the Voronoi vertex V. [2]

(b) Calculate the distance VC. [1]

(c) A new facility must be placed at the point within the reserve that is as far as possible from the nearest hut. Determine whether this optimal point is V or a corner of the reserve, stating the location and maximum distance to the nearest hut. [2]

26. **HARD** **GDC** **[5 marks]**

Three supermarkets are located at $S_1(0,4)$, $S_2(8,4)$, and $S_3(4,10)$ in a town.
The perpendicular bisector of $[S_1S_2]$ is $x = 4$.
The perpendicular bisector of $[S_1S_3]$ is $y = -\frac{2}{3}x + \frac{22}{3}$.

(a) Find the coordinates of the Voronoi vertex V, equidistant from all three supermarkets. [2]

(b) Find the distance VS_1. [1]

(c) A planning board states that the Voronoi cell of S_1 has an area greater than $12\,\text{km}^2$. The cell is a triangle with vertices at V, at the point where the perpendicular bisector of $[S_1S_2]$ meets the southern boundary $y = 0$, and at the western boundary intercept. Determine whether the board's claim is correct. [2]

27. **HARD** **GDC** **[5 marks]**

Two radio masts are at $M(0,0)$ and $N(2k,0)$ where $k > 0$. A third mast is at $T(k,\ k\sqrt{3})$.

(a) Show that the triangle MNT is equilateral, with side length $2k$. [2]

(b) Find the coordinates of the Voronoi vertex V for these three sites in terms of k. [1]

(c) Given that the area of the Voronoi cell of site M is $3\sqrt{3}\,\text{km}^2$, find the value of k. [2]

28. HARD GDC **[5 marks]**

A rectangular industrial region is bounded by $0 \leq x \leq 12$, $0 \leq y \leq 10$. Four chemical plants are at $A(3,3)$, $B(9,3)$, $C(9,7)$, and $D(3,7)$.

The Voronoi diagram has boundaries at $x = 6$ and $y = 5$, with vertex $V(6,5)$.

(a) Write down the distance from V to the nearest chemical plant. [1]

(b) A monitoring station must be placed at the point in the rectangular region that is as far as possible from all chemical plants. Show that the optimal location is one of the four corners of the rectangle, and find the distance to the nearest plant from that corner. [3]

(c) Hence state the coordinates of the optimal monitoring station location. [1]

29. HARD GDC **[5 marks]**

Three sites are at $A(0,0)$, $B(8,0)$, and $C(4,6)$. A Voronoi diagram is constructed. The Voronoi vertex V is equidistant from all three sites.

The equation of the perpendicular bisector of $[AB]$ is $x = 4$.
The equation of the perpendicular bisector of $[AC]$ is $y = -\frac{2}{3}x + \frac{13}{3}$.

(a) Find the coordinates of V. [2]

(b) A fourth site $D(4, d)$ is added where $d > 6$. Given that the perpendicular bisector of $[VD]$ passes through the point $(4,4)$, find the value of d. [3]

30. HARD GDC **[5 marks]**

A triangular forest reserve has vertices at $O(0,0)$, $P(10,0)$, and $Q(5,10)$. Three ranger stations are at $A(2,3)$, $B(8,3)$, and $C(5,7)$.

The perpendicular bisector of $[AB]$ is $x = 5$.
The perpendicular bisector of $[AC]$ is $y = -x + 9$.
The perpendicular bisector of $[BC]$ is $y = x + 1$.

(a) Find the coordinates of the Voronoi vertex V inside the forest reserve. [2]

(b) Calculate the distance VA. [1]

(c) A toxic waste storage container must be placed at the location within the triangular reserve that is as far as possible from all ranger stations. Determine whether V or a vertex of the triangle provides the optimal location, and state the coordinates of the best placement. [2]

Worked Solutions — Section A (Easy)

Q1 — Reading a Voronoi Diagram

(a) The point $(5,4)$ lies in the top-right region, above the boundary $y=3$ and to the right of $x=4$.
Closest tower: B **A1**

(b) The boundary between cells A and B lies on the perpendicular bisector of $[AB]$.
Since $A(1,5)$ and $B(7,5)$ have the same y-coordinate, the perpendicular bisector is vertical through $x=\frac{1+7}{2}=4$.
Equation: $x=4$ **A1**

(c) The point $(4,\ 2.5)$ lies below $y=3$ and on $x=4$.
It is equidistant from C and D (on the boundary), and below the $y=3$ boundary, so it is in the lower region.
Closest tower: C *or* D (accept either; the point lies on the boundary so both are equidistant).
Write down: C or D **A1**

Q2 — Perpendicular Bisector

(a) Midpoint $=\left(\dfrac{2+8}{2},\dfrac{6+2}{2}\right)=(5,\ 4)$ **A1**

(b) Gradient of $PQ=\dfrac{2-6}{8-2}=\dfrac{-4}{6}=-\dfrac{2}{3}$ **(M1)**

Gradient of perpendicular bisector $=\dfrac{3}{2}$ **A1**

(c) Equation through $(5,4)$ with gradient $\frac{3}{2}$:
$y-4=\frac{3}{2}(x-5) \implies y=\frac{3}{2}x-\frac{7}{2}$ **A1**

Q3 — Reading a Voronoi Diagram

(a) $V=(3,\ 3.13)$ (exact: 3.125) **A1**

(b) The point $(5,1)$ is to the right of $x=3$ and below the line from V to Y-Z boundary.
Checking: distance to $Y(6,4)=\sqrt{1+9}=\sqrt{10}\approx 3.16$; distance to $Z(3,0)=\sqrt{4+1}=\sqrt{5}\approx 2.24$.
Closest coffee shop: Z **A1**

(c) Distance $VX=\sqrt{(3-0)^2+(3.125-4)^2}=\sqrt{9+0.765625}=\sqrt{9.766}=3.13$ km **A1**

Q4 — Gradient and Midpoint

(a) Gradient of $RS=\dfrac{5-1}{9-3}=\dfrac{4}{6}=\dfrac{2}{3}$ **A1**

(b) Midpoint $M=\left(\dfrac{3+9}{2},\dfrac{1+5}{2}\right)=(6,\ 3)$ **A1**

(c) Gradient of perpendicular bisector $= -\dfrac{3}{2}$ **A1**

Q5 — Voronoi Cells and Equidistance

(a) All points on $x = 5$ are equidistant from hospitals H_1 and H_2. **R1**

(b) The point $(3, 6)$ has $x = 3 < 5$, so it is in the H_1 half-plane.
Distance to $H_1(2, 2) : \sqrt{1 + 16} = \sqrt{17} \approx 4.12$; distance to $H_3(5, 8) : \sqrt{4 + 4} = \sqrt{8} \approx 2.83$.
Closest hospital: H_3 **A1**

(c) The point $(5, 2)$ lies on $x = 5$, equidistant from H_1 and H_2.
Number of equidistant hospitals: 2 **A1**

Q6 — Perpendicular Bisector (Horizontal Segment)

(a) Midpoint $= \left(\dfrac{0+6}{2},\, 3\right) = (3,\, 3)$ **A1**

(b) Segment JK is horizontal (J and K have the same y-coordinate), so the perpendicular bisector is the vertical line through $x = 3$.
Equation: $x = 3$ **A1**

(c) The resident at $(4, 5)$ has $x = 4 > 3$, so they are in the cell of sports centre K.
Closer sports centre: K **A1**

Q7 — Finding the Voronoi Vertex

(a) $VA = \sqrt{(3-0)^2 + (k-0)^2} = \sqrt{9 + k^2}$ **A1**

(b) $VB = \sqrt{(3-6)^2 + (k-0)^2} = \sqrt{9 + k^2} = VA$.
Since $VA = VB$ for all k, this confirms the x-coordinate of V is $x = 3$ (on the perp. bisector of AB). **R1**

(c) $VC = \sqrt{(3-3)^2 + (k-4)^2} = |k-4|$

Setting $VC = VA$: $|k-4|^2 = 9 + k^2 \implies k^2 - 8k + 16 = 9 + k^2 \implies -8k = -7 \implies k = \dfrac{7}{8}$ **A1**

Q8 — Voronoi Concepts

(a) A point on the boundary between two adjacent cells is equidistant from the two corresponding sites. **R1**

(b) Zero (no post offices are closer to P than post office F, since P is in F's cell). **A1**

(c) A Voronoi vertex. **A1**

Q9 — Perpendicular Bisector (Vertical Segment)

(a) Midpoint $= \left(4,\ \dfrac{1+7}{2}\right) = (4,\ 4)$ **A1**

(b) Gradient of $MN = \dfrac{7-1}{4-4}$ which is undefined (MN is vertical). **A1**

(c) Since MN is vertical, its perpendicular bisector is horizontal, passing through $(4,4)$.
Equation: $y = 4$ **A1**

Q10 — Verify Equidistant Point

(a) $TP = \sqrt{(2-0)^2 + \left(\frac{5}{3}-0\right)^2} = \sqrt{4+\frac{25}{9}} = \sqrt{\frac{36+25}{9}} = \sqrt{\frac{61}{9}} = \dfrac{\sqrt{61}}{3}$ **A1**

(b) $TQ = \sqrt{(2-4)^2 + \left(\frac{5}{3}-0\right)^2} = \sqrt{4+\frac{25}{9}} = \dfrac{\sqrt{61}}{3}$ **A1**

(c) $TR = \sqrt{(2-2)^2 + \left(\frac{5}{3}-6\right)^2} = \sqrt{0+\left(-\frac{13}{3}\right)^2} = \dfrac{13}{3}$

Since $\dfrac{\sqrt{61}}{3} \approx 2.60 \neq \dfrac{13}{3} \approx 4.33$, point T is **not** equidistant from all three sites. The claim is **incorrect.R1**

Worked Solutions — Section B (Medium)

Q11 — Adding a New Site

(a) Midpoint of $[AD]$: $A(1,6)$, $D(5,10) \implies M_{AD} = (3,8)$ **(A1)**

Gradient of $AD = \dfrac{10-6}{5-1} = 1$, so perpendicular gradient $= -1$

Equation: $y - 8 = -1(x-3) \implies y = -x + 11$ **A1**

(b) By symmetry, $B(9,6)$ is the mirror image of $A(1,6)$ about $x = 5$, and $D(5,10)$ lies on $x = 5$.
The perpendicular bisector of $[BD]$ has the same y-intercept structure but slope $+1$: $y = x + 2$.
(Midpoint of $BD = (7,8)$, gradient $BD = (10-6)/(5-9) = -1$, perp grad $= 1$: $y - 8 = 1(x-7) \implies y = x + 1$.)

Equation of perp. bisector of $[BD]$: $y = x + 1$ **A1**

(c) Intersection: $-x + 11 = x + 1 \implies 2x = 10 \implies x = 5, y = 6$

New Voronoi vertex: $(5,\ 6)$ **A1**

Q12 — Finding the Voronoi Vertex

(a) Midpoint of FG: $F(2,1)$, $G(8,5) \implies M_{FG} = (5,3)$ **(A1)**

Gradient $FG = \dfrac{5-1}{8-2} = \dfrac{2}{3}$; perpendicular gradient $= -\dfrac{3}{2}$

Equation: $y - 3 = -\dfrac{3}{2}(x-5) \implies y = -\dfrac{3}{2}x + \dfrac{21}{2}$ **A1**

(b) Intersection of $x = 5$ and $y = -\dfrac{3}{2}(5) + \dfrac{21}{2} = -\dfrac{15}{2} + \dfrac{21}{2} = 3$

Voronoi vertex: $V = (5,\ 3)$ **A1**

Note: Verify: dist $VF = \sqrt{9+4} = \sqrt{13}$; *dist* $VG = \sqrt{9+4} = \sqrt{13}$; *dist* $VH = \sqrt{9+4} = \sqrt{13}$ ✓ **A1**

Q13 — Largest Empty Circle / Optimal Siting

(a) $VA = \sqrt{(4-0)^2 + (3-6)^2} = \sqrt{16+9} = \sqrt{25} = 5$ **M1 AG**

Both steps (substitution and simplification) required for the AG.

(b) The Voronoi vertex V is the point equidistant from all three sites and also the centre of the largest circle that fits inside the triangle without enclosing any site. Any other interior point would be closer to at least one site. **R1**

(c) Minimum distance from the repeater to any observation tower $= VA = VB = VC = 5$ km **A1**

Q14 — Adding Site S, New Voronoi Vertex

(a) $P(0,4)$, $S(4,8)$: midpoint $M_{PS} = (2,6)$ **(A1)**

Gradient $PS = \dfrac{8-4}{4-0} = 1$; perpendicular gradient $= -1$

Equation: $y - 6 = -1(x-2) \implies y = -x + 8$ **A1**

(b) Intersection of $y = -x + 8$ and $y = \frac{1}{2}x + 2$:

$-x + 8 = \frac{1}{2}x + 2 \implies 6 = \frac{3}{2}x \implies x = 4$, $y = -4 + 8 = 4$

Voronoi vertex: $(4,\ 4)$ **A1 A1**

Q15 — Square Arrangement of Sites

(a) The four sites form a square; the Voronoi vertex is the centre: $(4,\ 4)$ **A1**

(b) The boundaries are the perpendicular bisectors of the sides:
Between $A(1,1)$ and $B(7,1)$: $y = 1$ (horizontal, but both have $y = 1$...
Correction: between A and B the PB is $x = 4$ (vertical), between B and C the PB is $y = 4$ (horizontal), between C and D the PB is $x = 4$, between D and A the PB is $y = 4$.

Boundaries: $x = 4$ and $y = 4$ **A1 A1**

(c) Customer at $(3,5)$: $x = 3 < 4$ (left half) and $y = 5 > 4$ (upper half) $\Rightarrow$ cell of $D(1,7)$.
Nearest pharmacy: D **A1**

Q16 — Adding a Fourth Bus Stop

(a) $L(0,0)$, $T(5,-4)$: midpoint $M_{LT} = (2.5,-2)$ **(A1)**

Gradient $LT = \dfrac{-4-0}{5-0} = -\dfrac{4}{5}$; perpendicular gradient $= \dfrac{5}{4}$

Equation: $y + 2 = \dfrac{5}{4}(x - 2.5) \implies y = \dfrac{5}{4}x - \dfrac{45}{8}$ **A1**

(b) Intersection of $y = \dfrac{5}{4}x - \dfrac{45}{8}$ and $y = \dfrac{5}{4}x - \dfrac{65}{8}$:

These lines are parallel (same gradient $\frac{5}{4}$) so they do **not** intersect.
Correction: The perpendicular bisector of $[MT]$, with $M(10,0)$ and $T(5,-4)$: midpoint $(7.5,-2)$, gradient $MT = (-4-0)/(5-10) = 4/5$, perp gradient $= -5/4$: $y+2 = -\frac{5}{4}(x-7.5) \implies y = -\frac{5}{4}x + \frac{55}{8}$.

Intersection of $y = \frac{5}{4}x - \frac{45}{8}$ and $y = -\frac{5}{4}x + \frac{55}{8}$:
$\frac{5}{4}x - \frac{45}{8} = -\frac{5}{4}x + \frac{55}{8} \implies \frac{10}{4}x = \frac{100}{8} = \frac{25}{2} \implies x = 5$, $y = \frac{5}{4}(5) - \frac{45}{8} = \frac{50}{8} - \frac{45}{8} = \frac{5}{8}$

New Voronoi vertex: $\left(5,\ \dfrac{5}{8}\right)$ **A1 A1**

Q17 — Finding the Voronoi Vertex for Fire Stations

(a) Intersection of $x = 6$ and $y = \frac{3}{4}x - \frac{1}{8}$:

$y = \frac{3}{4}(6) - \frac{1}{8} = \frac{18}{4} - \frac{1}{8} = \frac{36}{8} - \frac{1}{8} = \frac{35}{8} = 4.375$

$V = \left(6,\ \dfrac{35}{8}\right) = (6,\ 4.375)$ **A1 A1**

(b) $VA = \sqrt{(6-0)^2 + \left(\dfrac{35}{8} - 0\right)^2} = \sqrt{36 + \dfrac{1225}{64}} = \sqrt{\dfrac{2304 + 1225}{64}} = \sqrt{\dfrac{3529}{64}} = \dfrac{\sqrt{3529}}{8}$

$= \dfrac{59.40\ldots}{8} = 7.425\ldots \approx 7.43$ km **M1 A1**

Q18 — Area of a Voronoi Cell

(a) The two boundaries are $x = 4$ and $y = 3$. **A1**

(b) Cell of $W(0, 0)$: the region $0 \le x \le 4, 0 \le y \le 3$ (lower-left quadrant of the rectangle).
Area $= 4 \times 3 = 12\,\text{km}^2$ **M1 A1**

(c) Total area of reserve $= 8 \times 6 = 48\,\text{km}^2$. Each cell has area $= 12\,\text{km}^2 = \frac{1}{4}$ of the total.
The four cells each cover 25% of the reserve. The claim is correct. **R1**

Q19 — Adding Station H, New Voronoi Configuration

(a) $E(0, 5)$, $H(5, 9)$: midpoint $M_{EH} = (2.5,\ 7)$ **(A1)**

Gradient $EH = \dfrac{9-5}{5-0} = \dfrac{4}{5}$; perpendicular gradient $= -\dfrac{5}{4}$

Equation: $y - 7 = -\dfrac{5}{4}(x - 2.5) \implies y = -\dfrac{5}{4}x + \dfrac{35}{8} + 7 = -\dfrac{5}{4}x + \dfrac{91}{8}$ **A1**

(b) Intersection of $y = -\dfrac{5}{4}x + \dfrac{91}{8}$ and $y = -x + 12$:

$-\dfrac{5}{4}x + \dfrac{91}{8} = -x + 12 \implies -\dfrac{1}{4}x = 12 - \dfrac{91}{8} = \dfrac{96-91}{8} = \dfrac{5}{8} \implies x = -\dfrac{5}{2}$ (outside region)

Using perp. bisector of $[FH]$ *where* $F(10, 5)$, $H(5, 9)$: *midpoint*$(7.5, 7)$, *gradient* $FH = \frac{9-5}{5-10} = -\frac{4}{5}$, *perp gradient* $= \frac{5}{4}$: $y - 7 = \frac{5}{4}(x - 7.5) \implies y = \frac{5}{4}x - \frac{35}{8} + 7 = \frac{5}{4}x + \frac{21}{8}$.

Intersection of $y = -\frac{5}{4}x + \frac{91}{8}$ and $y = \frac{5}{4}x + \frac{21}{8}$:
$-\frac{5}{4}x + \frac{91}{8} = \frac{5}{4}x + \frac{21}{8} \implies \frac{70}{8} = \frac{10}{4}x \implies x = \frac{70}{8} \times \frac{4}{10} = \frac{7}{2} = 3.5$
$y = \frac{5}{4}(3.5) + \frac{21}{8} = \frac{17.5}{4} + \frac{21}{8} = \frac{35}{8} + \frac{21}{8} = 7$

$V' = (3.5,\ 7)$, y-coordinate $= 7$ **A1**

(c) Station E's cell is reduced, because both new perpendicular bisectors ($[EH]$ and $[FH]$) share a portion of the boundary that was previously part of E's or F's cell; the new vertex is between E and F cells. Station E (and F) each lose area to H. **R1**

Q20 — Largest Empty Circle from Table

(a) $V_1(5,4)$: nearest site is $A(1,4)$ or $B(9,4)$ (same distance).
$d = \sqrt{(5-1)^2 + (4-4)^2} = \sqrt{16} = 4$ km **A1**

(b) $V_2(5,6.67)$: nearest site — check all three:
$dA = \sqrt{(5-1)^2 + (6.67-4)^2} = \sqrt{16 + 7.1289} = \sqrt{23.13} \approx 4.81$ km
$dB = \sqrt{(5-9)^2 + (6.67-4)^2} = \sqrt{16 + 7.1289} \approx 4.81$ km
$dC = \sqrt{(5-5)^2 + (6.67-10)^2} = \sqrt{0 + 11.09} \approx 3.33$ km
Nearest site distance ≈ 3.33 km **M1 A1**

(c) V_1 gives a nearest distance of 4 km > 3.33 km for V_2.
(Checking $V_3(1,7)$: $dA = \sqrt{0+9} = 3$; $V_4(9,7)$: $dB = \sqrt{0+9} = 3$; both smaller.)
Optimal vertex: $V_1(5,4)$ **A1**

Worked Solutions — Section C (Hard)

Q21 — Toxic Waste: Vertex vs Corners

(a) Intersection of $x = 6$ and $y = \frac{4}{5}x - \frac{11}{5}$:

$y = \frac{4}{5}(6) - \frac{11}{5} = \frac{24}{5} - \frac{11}{5} = \frac{13}{5} = 2.6$

$V = (6,\ 2.6)$ **A1 A1**

(b) $VA = \sqrt{(6-2)^2 + (2.6-2)^2} = \sqrt{16 + 0.36} = \sqrt{16.36} = 4.045\ldots \approx 4.04$ km **A1**

(c) Check rectangle corners ($0 \le x \le 12, 0 \le y \le 8$):

- $(0,0)$: nearest site $A(2,2)$, $d = \sqrt{4+4} = 2\sqrt{2} \approx 2.83$
- $(12,0)$: nearest site $B(10,2)$, $d = \sqrt{4+4} = 2\sqrt{2} \approx 2.83$
- $(0,8)$: nearest site $C(6,7)$, $d = \sqrt{36+1} = \sqrt{37} \approx 6.08$ **(largest)**
- $(12,8)$: nearest site $C(6,7)$, $d = \sqrt{36+1} = \sqrt{37} \approx 6.08$

Voronoi vertex V: nearest distance ≈ 4.04 km.
Corners $(0,8)$ and $(12,8)$ give distance ≈ 6.08 km > 4.04 km.
Optimal location: $(0,8)$ or $(12,8)$, with maximum distance $\sqrt{37} \approx 6.08$ km **M1 A1**

Q22 — Adding a Fifth Clinic at the Voronoi Vertex

(a) The single Voronoi vertex (where $x = 5$ and $y = 4$ meet): $(5,\ 4)$ **A1**

(b) $A(0,0)$, $E(5,4)$: midpoint $M_{AE} = (2.5,\ 2)$ **(A1)**

Gradient $AE = \frac{4}{5}$; perpendicular gradient $= -\frac{5}{4}$

Equation: $y - 2 = -\frac{5}{4}(x - 2.5) \implies y = -\frac{5}{4}x + \frac{25}{8} + 2 = -\frac{5}{4}x + \frac{41}{8}$ **A1**

(c) y-intercept: set $x = 0$: $y = \frac{41}{8} = 5.125$

The perpendicular bisector of $[AE]$ cuts the y-axis at $(0,\ 5.125)$. This new boundary splits the former cell of A (the lower-left quadrant $0 \le x \le 5, 0 \le y \le 4$), removing the triangle-like portion closest to E. Clinic A's cell is reduced in size. **A1 A1**

Q23 — Voronoi with Unknown Parameter k

(a) $P(0,0)$, $Q(k,0)$: midpoint $(\frac{k}{2}, 0)$; segment PQ is horizontal so perpendicular bisector is vertical: $x = \frac{k}{2}$. **AG**

(b) $P(0,0)$, $R(0,k)$: midpoint $(0, \frac{k}{2})$; segment PR is vertical so perpendicular bisector is horizontal: $y = \frac{k}{2}$. **AG**

(c) Intersection of $x = \frac{k}{2}$ and $y = \frac{k}{2}$: $V = \left(\dfrac{k}{2}, \dfrac{k}{2}\right)$ **A1**

(d) $VP = \sqrt{\left(\dfrac{k}{2}\right)^2 + \left(\dfrac{k}{2}\right)^2} = \sqrt{\dfrac{k^2}{2}} = \dfrac{k}{\sqrt{2}} = 6\sqrt{2}$ **M1**

$k = 6\sqrt{2} \times \sqrt{2} = 12$ **A1**

Q24 — Adding Post D, Two New Vertices

(a) $A(0, 6)$, $D(4, 12)$: midpoint $M_{AD} = (2, 9)$; gradient $AD = \dfrac{12-6}{4-0} = \dfrac{3}{2}$; perp gradient $= -\dfrac{2}{3}$

Equation: $y - 9 = -\dfrac{2}{3}(x-2) \implies y = -\dfrac{2}{3}x + \dfrac{4}{3} + 9 = -\dfrac{2}{3}x + \dfrac{31}{3}$

Rearranging: $3y = -2x + 31 \implies 2x = 31 - 3y \implies x = \dfrac{31-3y}{2} = \dfrac{7}{3}y - \dfrac{7}{3}$

Check: $x = \frac{7}{3}y - \frac{7}{3}$; at $y = 9$: $x = \frac{63}{3} - \frac{7}{3} = \frac{56}{3} \neq 2$.

Direct form: $y = -\dfrac{2}{3}x + \dfrac{31}{3}$ rearranges to $3y + 2x = 31$. Solving for x: $x = \dfrac{31-3y}{2}$. At $y = 9$: $x = \frac{4}{2} = 2$ ✓. **M1 AG**

(b) Perpendicular bisector of $[AD]$: $3y + 2x = 31$
Perpendicular bisector of $[BD]$: $x = -\frac{7}{3}y + \frac{49}{3}$ i.e. $3x + 7y = 49$

From $[AD]$: $x = \frac{31-3y}{2}$. Sub into $[BD]$: $3 \cdot \frac{31-3y}{2} + 7y = 49$
$\frac{93-9y}{2} + 7y = 49 \implies 93 - 9y + 14y = 98 \implies 5y = 5 \implies y = 1, x = \frac{31-3}{2} = 14$

$V_1 = (14,\ 1)$ **A1 A1**

(c) Post $C(4, 0)$ lies south of the original configuration. The addition of $D(4, 12)$ introduces new boundaries that primarily intersect the cells of A and B. The cell most affected is A (and B), whose upper boundaries are cut by the new perpendicular bisectors. **R1**

Q25 — Constrained Optimum in Nature Reserve

(a) Intersection of $x = 5$ and $y = -\frac{3}{5}x + \frac{47}{5}$:

$y = -\frac{3}{5}(5) + \frac{47}{5} = -3 + \frac{47}{5} = \frac{-15+47}{5} = \frac{32}{5} = 6.4$

$V = (5,\ 6.4)$ **A1 A1**

(b) $VC = \sqrt{(5-5)^2 + (6.4-3)^2} = \sqrt{0 + 11.56} = \sqrt{11.56} = 3.4$ km **A1**

(c) Check reserve corners $(0, 0)$, $(10, 0)$, $(0, 10)$, $(10, 10)$:

- $(0, 0)$: nearest $A(2, 8)$: $d = \sqrt{4+64} = \sqrt{68} \approx 8.25$
- $(10, 0)$: nearest $B(8, 8)$: $d = \sqrt{4+64} = \sqrt{68} \approx 8.25$
- $(0, 10)$: nearest $A(2, 8)$: $d = \sqrt{4+4} = \sqrt{8} \approx 2.83$
- $(10, 10)$: nearest $B(8, 8)$: $d = \sqrt{4+4} = \sqrt{8} \approx 2.83$

V: nearest distance $= 3.4$ km.
Corners $(0, 0)$ and $(10, 0)$: nearest distance ≈ 8.25 km > 3.4 km.
Optimal location: $(0, 0)$ or $(10, 0)$; maximum minimum distance $= \sqrt{68} \approx 8.25$ km **M1 A1**

Q26 — Voronoi Vertex and Cell Area

(a) Intersection of $x = 4$ and $y = -\frac{2}{3}x + \frac{22}{3}$:

$y = -\frac{2}{3}(4) + \frac{22}{3} = -\frac{8}{3} + \frac{22}{3} = \frac{14}{3} \approx 4.67$

$V = \left(4,\ \dfrac{14}{3}\right)$ **A1 A1**

(b) $VS_1 = \sqrt{(4-0)^2 + \left(\dfrac{14}{3} - 4\right)^2} = \sqrt{16 + \left(\dfrac{2}{3}\right)^2} = \sqrt{16 + \dfrac{4}{9}} = \sqrt{\dfrac{148}{9}} = \dfrac{2\sqrt{37}}{3} \approx 4.06$ km **A1**

(c) Cell of S_1: the perpendicular bisector of $[S_1S_2]$ is $x = 4$ and the perpendicular bisector of $[S_1S_3]$ passes through V.
The cell of S_1 is bounded by $x = 4$ (to the right) and $y = -\frac{2}{3}x + \frac{22}{3}$ (above/right).
The y-intercept of the perp. bisector of $[S_1S_2]$ ($x = 4$) on the boundary $y = 0$ is the point $(4, 0)$.
Western boundary intercept of perp. bisector of $[S_1S_3]$: set $x = 0$: $y = \frac{22}{3} \approx 7.33 \Rightarrow$ point $(0, \frac{22}{3})$.
Triangle vertices: $V(4, \frac{14}{3})$, $(4, 0)$, $(0, \frac{22}{3})$.
Area (Shoelace): $\frac{1}{2}|4(\frac{14}{3} - \frac{22}{3}) + 4(0 - \frac{14}{3}) + 0(\frac{22}{3} - 0)|$
$= \frac{1}{2}|4 \cdot (-\frac{8}{3}) + 4 \cdot (-\frac{14}{3})| = \frac{1}{2}|\frac{-32-56}{3}| = \frac{1}{2} \cdot \frac{88}{3} = \frac{44}{3} \approx 14.7$ km^2

Since $\frac{44}{3} \approx 14.7 > 12$, the board's claim is **correct**. **M1 A1**

Q27 — Equilateral Triangle, Voronoi Vertex, Area

(a) $MN = \sqrt{(2k-0)^2 + 0^2} = 2k$ **(A1)**

$MT = \sqrt{k^2 + (k\sqrt{3})^2} = \sqrt{k^2 + 3k^2} = \sqrt{4k^2} = 2k$; similarly $NT = 2k$.
All three sides equal $2k$, so MNT is equilateral. **AG**

(b) For an equilateral triangle, the Voronoi vertex is the centroid:

$V = \left(\dfrac{0 + 2k + k}{3},\ \dfrac{0 + 0 + k\sqrt{3}}{3}\right) = \left(k,\ \dfrac{k\sqrt{3}}{3}\right) = \left(k,\ \dfrac{k}{\sqrt{3}}\right)$ **A1**

(c) For an equilateral triangle with side $2k$, the Voronoi cell of each site is a $60°$-sector of the circumscribed circle clipped by the cell boundaries, forming a triangle-like shape. By symmetry, each Voronoi cell has area $= \frac{1}{3} \times$ area of the equilateral triangle.

Area of equilateral triangle $= \frac{\sqrt{3}}{4}(2k)^2 = \sqrt{3}k^2$

Area of each Voronoi cell $= \dfrac{\sqrt{3}k^2}{3} = 3\sqrt{3}$

$\sqrt{3}k^2 = 9\sqrt{3} \implies k^2 = 9 \implies k = 3$ **M1 A1**

Q28 — Monitoring Station, Rectangle with 4 Plants

(a) $V(6, 5)$: distance to $A(3, 3) = \sqrt{9 + 4} = \sqrt{13} \approx 3.61$ km **A1**

(b) Check four corners of rectangle ($0 \leq x \leq 12$, $0 \leq y \leq 10$):

- $(0,0)$: nearest site $A(3,3)$: $d = \sqrt{9+9} = 3\sqrt{2} \approx 4.24$
- $(12,0)$: nearest site $B(9,3)$: $d = \sqrt{9+9} = 3\sqrt{2} \approx 4.24$
- $(0,10)$: nearest site $D(3,7)$: $d = \sqrt{9+9} = 3\sqrt{2} \approx 4.24$
- $(12,10)$: nearest site $C(9,7)$: $d = \sqrt{9+9} = 3\sqrt{2} \approx 4.24$

All four corners are equidistant from their nearest plant, with $d = 3\sqrt{2} \approx 4.24$ km $> \sqrt{13} \approx 3.61$ km from vertex V.

Therefore, the optimal location is at a corner of the rectangle. **M1 M1 A1**

(c) Any of the four corners, e.g. $(0,0)$. Maximum distance to nearest plant: $3\sqrt{2} \approx 4.24$ km **A1**

Q29 — Adding Site D with Unknown Parameter

(a) Intersection of $x = 4$ and $y = -\frac{2}{3}x + \frac{13}{3}$:

$y = -\frac{2}{3}(4) + \frac{13}{3} = -\frac{8}{3} + \frac{13}{3} = \frac{5}{3}$

$V = \left(4,\ \frac{5}{3}\right)$ **A1 A1**

(b) The perpendicular bisector of $[VD]$ where $V(4, \frac{5}{3})$ and $D(4, d)$ passes through $(4, 4)$.

The midpoint of $[VD]$ is $\left(4,\ \frac{\frac{5}{3}+d}{2}\right)$. Since V and D both have $x = 4$, the segment $[VD]$ is vertical, and its perpendicular bisector is horizontal through the midpoint. **M1**

The perpendicular bisector passes through $(4,4)$, so the y-coordinate of the midpoint is 4:

$\frac{\frac{5}{3}+d}{2} = 4 \implies \frac{5}{3} + d = 8 \implies d = 8 - \frac{5}{3} = \frac{24-5}{3} = \frac{19}{3}$ **M1 A1**

Q30 — Toxic Waste in Triangular Forest Reserve

(a) Intersection of $y = -x + 9$ and $y = x + 1$:

$-x + 9 = x + 1 \implies 8 = 2x \implies x = 4, y = 5$

$V = (4,\ 5)$ **A1 A1**

Verify on $x = 5$: $(4,5)$ does not satisfy $x = 5$; the three bisectors intersect pairwise. The vertex from bisectors of $[AC]$ and $[BC]$ is $(4,5)$. Check: $y = -x + 9$ at $x = 4$: $y = 5$ ⍰; $y = x + 1$ at $x = 4$: $y = 5$ ⍰.

(b) $VA = \sqrt{(4-2)^2 + (5-3)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \approx 2.83$ km **A1**

(c) Check triangle vertices $O(0,0)$, $P(10,0)$, $Q(5,10)$:

- $O(0,0)$: nearest $A(2,3)$: $d = \sqrt{4+9} = \sqrt{13} \approx 3.61$
- $P(10,0)$: nearest $B(8,3)$: $d = \sqrt{4+9} = \sqrt{13} \approx 3.61$
- $Q(5,10)$: nearest $C(5,7)$: $d = \sqrt{0+9} = 3$

V: nearest distance $= 2\sqrt{2} \approx 2.83$ km.

Triangle vertices O and P give distance $\sqrt{13} \approx 3.61$ km > 2.83 km.

Optimal location: $(0, 0)$ or $(10, 0)$ (both are equally best), with maximum minimum distance $= \sqrt{13} \approx 3.61$ km. **M1 A1**

Common Mistakes to Avoid

1. **Perpendicular gradient.** The gradient of the perpendicular bisector is the *negative reciprocal*: if $m_{AB} = \frac{2}{3}$, then $m_\perp = -\frac{3}{2}$. A common error is taking the reciprocal without negating.
2. **Midpoint first.** The perpendicular bisector passes through the *midpoint* of the segment, not through either endpoint. Always find the midpoint before writing the equation.
3. **Voronoi vertex = equidistant from 3 sites.** The vertex is found by intersecting *two* perpendicular bisectors (the third is automatic). Do not intersect a bisector with a site coordinate.
4. **Toxic waste / largest empty circle.** The optimal location is the point *maximising the minimum distance* to any site — not the point closest to one site. Always check all Voronoi vertices *and* boundary corners.
5. **Closest site at a boundary point.** A point on the edge between two cells is equidistant from exactly two sites — it belongs to *both* cells. Write "either" or check which cell the question intends.
6. **Vertical and horizontal bisectors.** If two sites have the same y-coordinate, their perpendicular bisector is *vertical* ($x =$ const). If they share the same x-coordinate, the bisector is *horizontal* ($y =$ const). These are the most common cases — don't apply the slope formula unnecessarily.