IB Math AI HL — Topic 5 [Set 1]

Differential Equations

Separation of Variables · Modelling · Slope Fields · Euler's Method

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Grand total: **120 marks**

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Name:______________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. A GDC is permitted throughout.

Key Formulae

Separation of Variables

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx + C$$

Exponential Growth / Decay

$$\frac{dN}{dt} = kN \;\Rightarrow\; N = N_0\, e^{kt}$$

Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_{\text{env}}) \;\Rightarrow\; T = T_{\text{env}} + Ae^{-kt}$$

Logistic Growth

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) \;\Rightarrow\; P = \frac{L}{1 + Ae^{-kt}}$$

Euler's Method (step size h)

$$y_{n+1} = y_n + h \cdot f(x_n,\, y_n)$$

Isocline of slope c

Set $\dfrac{dy}{dx} = c$ and sketch the resulting curve

GDC Tips

- **Euler's Method table:** Use the GDC spreadsheet. Column A: x_n; Column B: y_n; Column C: $f(x_n, y_n)$; Column D: $y_n + h \cdot C$. Drag down to complete all steps.
- **Finding constants after separating:** After integrating, substitute the initial condition and use the GDC's solve function or algebra to find C.
- **Logistic check:** Once you have k, L, A, plot $P(t)$ on the GDC and use the graph to verify long-term behaviour and read off values at specific times.
- **Slope fields on GDC:** Use the ODE plotter (e.g. TI-84: `MODE` → Diff Eq, or Casio: `GRAPH` → Diff Eq) to visualise families — but always sketch by hand in the exam.

Common Mistakes to Avoid

1. **Forgetting the constant of integration.** Always write $+C$ after integrating, then find C from the initial condition.
2. **Dividing by zero.** Check $g(y) \neq 0$ before separating variables; singular points break Euler's method.
3. **Euler gradient evaluated at wrong point.** Use $f(x_n, y_n)$ — the gradient at the *current* step, not the next one.
4. **Logistic long-term value.** As $t \to \infty$, $P \to L$ (the carrying capacity) — not zero.
5. **Sign in Newton's cooling.** $k > 0$ already appears with a minus sign: $dT/dt = -k(T - T_{\text{env}})$. Do not add another minus when separating.
6. **Slope field curves cannot cross.** By the uniqueness theorem, two distinct solution curves never intersect.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. **EASY** **GDC** **[3 marks]**

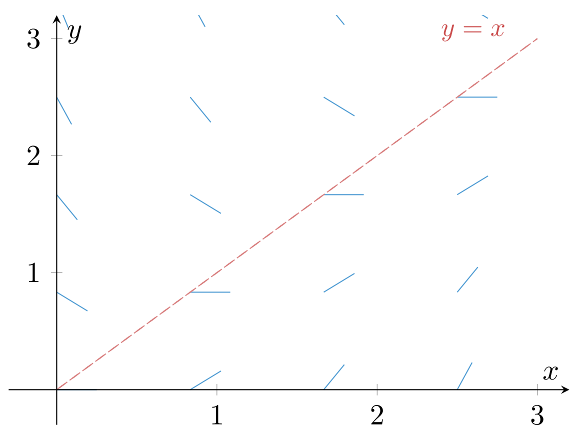
The number of bacteria, N, in a laboratory culture increases at a rate proportional to the number present. Initially there are 500 bacteria.

(a) Write down a differential equation for N in terms of time t (hours) and a positive constant k. [1]

(b) Show that $N = 500\,e^{kt}$ is the solution satisfying this initial condition. [2]

2. **EASY** **GDC** **[3 marks]**

The slope field below is for the differential equation $\dfrac{dy}{dx} = x - y$.



(a) Write down the value of $\dfrac{dy}{dx}$ at the point $(2,\,1)$. [1]

(b) On the slope field above, sketch the solution curve passing through the point $(0,\,2)$. [2]

3. EASY GDC **[3 marks]**

Consider the differential equation $\frac{dy}{dx} = 2x + 1$, with $y = 3$ when $x = 0$. Use Euler's method with step size $h = 0.5$ to find an approximation for y when $x = 1.5$.

n	x_n	y_n	$f(x_n, y_n) = 2x_n + 1$
0	0	3	1
1	0.5		
2	1.0		
3	1.5		

4. EASY GDC **[3 marks]**

A radioactive substance decays so that its mass m (g) satisfies $\frac{dm}{dt} = -0.04m$, where t is time in years. Initially $m = 80$ g.

(a) Solve the differential equation to find m in terms of t. [2]

(b) Find the mass after 10 years, correct to 3 significant figures. [1]

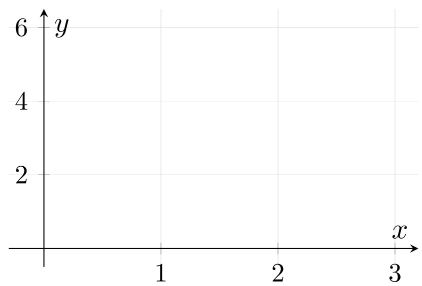
5. **EASY** **GDC** **[3 marks]**

A differential equation is given by $\frac{dy}{dx} = y - 2x$.

(a) Write down the equation of the isocline on which $\frac{dy}{dx} = 0$. [1]

(b) On the axes below, sketch this isocline. [1]

(c) State what the slope field looks like along this isocline. [1]



6. **EASY** **GDC** **[3 marks]**

A cup of tea cools in a room at $20\,°\text{C}$. Let $T\ (°\text{C})$ be the temperature of the tea at time t (minutes) after it is made. The cooling follows Newton's Law.

(a) Write down a differential equation for T in terms of t and a positive constant k. [1]

(b) Write down the initial condition if the tea is initially at $90\,°\text{C}$. [1]

(c) Show that $T = 20 + 70\,e^{-kt}$ is the solution satisfying these conditions. [1]

7. EASY GDC **[3 marks]**

Consider the differential equation $\frac{dy}{dx} = y$, with $y = 1$ when $x = 0$. Use Euler's method with step size $h = 0.25$ to complete the table and find an approximation for y when $x = 1$.

n	x_n	y_n	$f(x_n, y_n) = y_n$
0	0.00	1.0000	1.0000
1	0.25		
2	0.50		
3	0.75		
4	1.00		

8. EASY GDC **[3 marks]**

The rate of change of the height h (cm) of water in a tank is given by $\frac{dh}{dt} = 3 - h$, where t is time in minutes. When $t = 0$, $h = 1$ cm.

(a) Separate variables and solve to find h in terms of t. [2]

(b) Find the value of h as $t \to \infty$. [1]

9. EASY GDC **[3 marks]**

Three differential equations are listed below.

$$\textbf{P: } \frac{dy}{dx} = x^2 \qquad \textbf{Q: } \frac{dy}{dx} = -y \qquad \textbf{R: } \frac{dy}{dx} = x + y$$

Match each description to the correct equation (P, Q, or R).

(a) The slope depends only on x and is always non-negative. [1]

(b) The slope is zero along the line $y = -x$. [1]

(c) The slope is zero along the x-axis, with the slope field symmetric about it. [1]

10. EASY GDC **[3 marks]**

A population of fish P in a lake satisfies $\frac{dP}{dt} = kP$ (t in years). When $t = 0$, $P = 2000$, and when $t = 3$, $P = 5000$.

(a) Write down the solution $P = P_0 e^{kt}$, stating the value of P_0. [1]

(b) Find the value of k, correct to 3 significant figures. [2]

Section B — Medium MEDIUM (10 questions $\times$ 4 marks = 40 marks)

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11. MEDIUM GDC **[4 marks]**

A vine grows so that its length L (m) at time t (weeks) satisfies $\dfrac{dL}{dt} = Lt$, with $L = 2$ when $t = 0$.

(a) Show that $L = 2\,e^{t^2/2}$. [3]

(b) Find the length of the vine after 4 weeks, correct to 3 significant figures. [1]

12. MEDIUM GDC **[4 marks]**

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$, with $y = 2$ when $x = 0$.

(a) Use Euler's method with step size $h = 0.4$ to complete the table below. [3]

n	x_n	y_n	$f(x_n, y_n) = x_n/y_n$
0	0.0	2.000	0.000
1	0.4		
2	0.8		
3	1.2		

(b) State one reason why Euler's method would fail if y became close to zero. [1]

13. MEDIUM GDC **[4 marks]**

A metal rod is removed from a furnace. Its temperature T ($^\circ$C) at time t (minutes) after removal satisfies $\frac{dT}{dt} = -k(T - 25)$, where $25\,^\circ$C is ambient temperature and $k > 0$. When $t = 0$, $T = 600\,^\circ$C, and when $t = 10$, $T = 360\,^\circ$C.

(a) Show that $T = 25 + 575\,e^{-kt}$. [1]

(b) Find the value of k, correct to 3 significant figures. [2]

(c) Find the time at which the rod reaches $100\,^\circ$C, correct to 3 significant figures. [1]

14. MEDIUM GDC **[4 marks]**

A population of deer P in a wildlife reserve satisfies $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{800}\right)$, where t is in years. Initially $P = 100$.

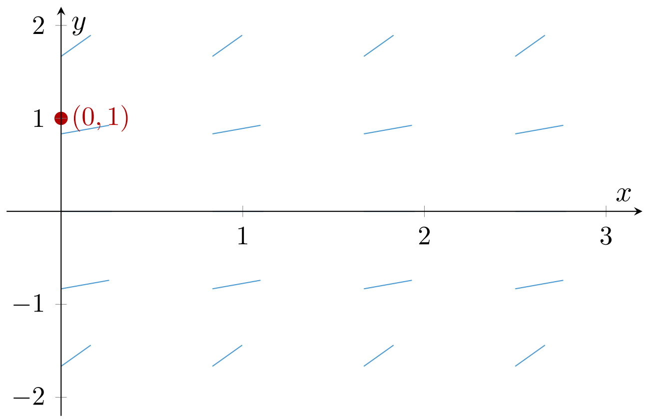
(a) Write down the carrying capacity of the reserve. [1]

(b) Show that $P = \dfrac{800}{1 + 7\,e^{-0.3t}}$. [2]

(c) Find the population after 5 years, correct to the nearest whole number. [1]

15. MEDIUM GDC **[4 marks]**

The slope field below represents the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{2}$.



(a) On the slope field above, sketch the solution curve passing through $(0, 1)$. [2]

(b) Find the exact solution $y(x)$ passing through $(0, 1)$, and state the x-value at which the solution has a vertical asymptote. [2]

16. MEDIUM GDC **[4 marks]**

The velocity v (m/s) of a particle at time t (s) satisfies $\dfrac{dv}{dt} = \cos t$, with $v = 0$ when $t = 0$.

(a) Solve the differential equation to find v in terms of t. [2]

(b) Find the displacement of the particle from its starting position at $t = \pi$, given $s = 0$ when $t = 0$. [2]

17. MEDIUM GDC **[4 marks]**

The spread of a rumour through a group of 200 students is modelled by $\frac{dR}{dt} = 0.05R(200 - R)$, where R is the number of students who have heard the rumour at time t (hours). Initially $R = 4$. Use Euler's method with step size $h = 0.1$ to complete the table and find an approximate value for R when $t = 0.3$.

n	t_n	R_n	$f(t_n, R_n) = 0.05R_n(200 - R_n)$
0	0.0	4.000	39.20
1	0.1		
2	0.2		
3	0.3		

18. MEDIUM GDC **[4 marks]**

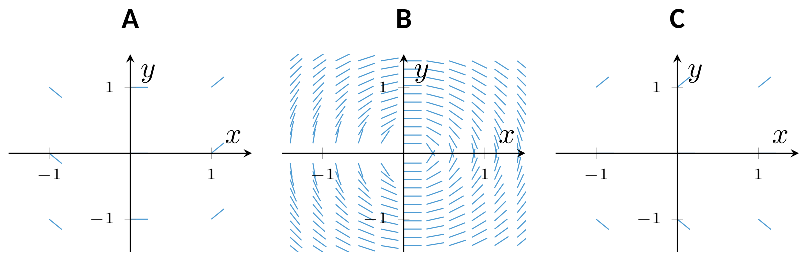
The concentration C (mol/L) of a chemical satisfies $\frac{dC}{dt} = -\frac{C}{1+t}$, with $C = 6$ when $t = 0$ (minutes).

(a) Solve the differential equation to find C in terms of t. [3]

(b) Find the concentration when $t = 5$, correct to 3 significant figures. [1]

19. MEDIUM GDC **[4 marks]**

Three slope fields A, B, and C are shown below.



(a) Match each slope field to the correct differential equation, giving a reason for each. [3]

$$\text{(i) } \frac{dy}{dx} = y \qquad \text{(ii) } \frac{dy}{dx} = x \qquad \text{(iii) } \frac{dy}{dx} = -\frac{x}{y}$$

(b) On field **C**, sketch the solution curve through $(0, 1)$. [1]

20. MEDIUM GDC **[4 marks]**

The rate of change of the depth d (m) of snow on a roof satisfies $\dfrac{dd}{dt} = \dfrac{1}{\sqrt{d}}$, with $d = 4$ when $t = 0$ (hours).

(a) Show that $d = \left(\dfrac{t}{2} + 2\right)^2$. [3]

(b) Find the depth of snow after 6 hours. [1]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **GDC** **[5 marks]**

A biologist models cell culture growth using $\dfrac{dN}{dt} = kN\left(1 - \dfrac{N}{L}\right)$, where N is the number of cells (thousands) at time t (days). The table gives recorded data.

Time t (days)	**Cells N (thousands)**
0	2
5	8
∞	40

(a) Write down the value of L. [1]

(b) Show that the particular solution is $N = \dfrac{40}{1 + 19\,e^{-kt}}$. [2]

(c) Find the value of k, correct to 3 significant figures. [1]

(d) Find the time at which the culture reaches $30\,000$ cells, correct to 3 significant figures. [1]

22. HARD GDC **[5 marks]**

Consider the differential equation $\dfrac{dy}{dx} = y^2$, with $y = 0.5$ when $x = 0$.

(a) Use Euler's method with step size $h = 0.5$ to find an approximation for y when $x = 1$. [2]

(b) By solving the differential equation exactly, find the true value of y when $x = 1$. [2]

(c) Find the percentage error in the Euler approximation, correct to 3 significant figures. [1]

23. HARD GDC **[5 marks]**

The population P (millions) of a city satisfies $\dfrac{dP}{dt} = \dfrac{t}{P}$, with $P = 3$ when $t = 2$ (years since a census).

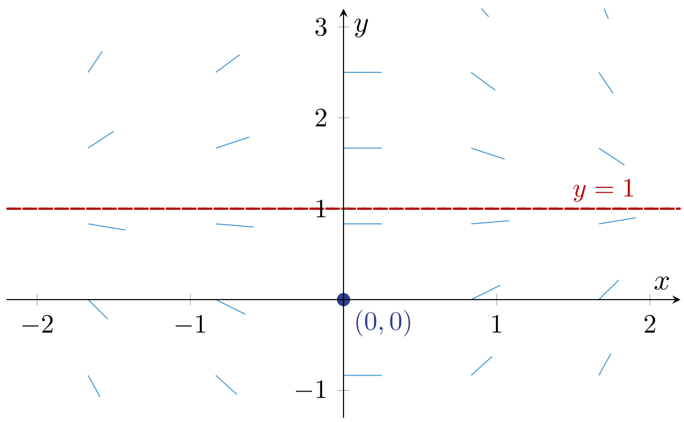
(a) Show that $P^2 = t^2 + K$ for some constant K. [2]

(b) Apply the condition $P(2) = 3$ to find the particular solution $P(t)$. [2]

(c) Find the population when $t = 6$, correct to 3 significant figures. [1]

24. HARD GDC **[5 marks]**

The slope field below represents the differential equation $\dfrac{dy}{dx} = x(1-y)$. The dashed line shows $y = 1$.



(a) On the slope field above, sketch the solution curve through $(0,\,0)$. [2]

(b) By separating variables, show that $y = 1 - e^{-x^2/2}$ is the solution with $y(0) = 0$. [2]

(c) Verify that this solution approaches $y = 1$ as $x \to \infty$, and state the significance of the dashed line. [1]

25. HARD GDC **[5 marks]**

A chemical reaction produces a compound whose mass m (g) satisfies $\dfrac{dm}{dt} = 4 - m^2$, with $m = 0$ when $t = 0$.

(a) Use Euler's method with $h = 0.25$ to estimate m when $t = 0.5$. [2]

(b) By solving the differential equation using partial fractions, show that

$$m = 2 \cdot \frac{e^{4t} - 1}{e^{4t} + 1}.$$

[3]

26. HARD GDC **[5 marks]**

A fish population P (hundreds) in a lake satisfies

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{120}\right) - 3,$$

where t is time in months. When $P = 60$, the population is neither growing nor declining.

(a) Show that $k = \frac{1}{10}$. [2]

(b) Find the second value of P at which $\frac{dP}{dt} = 0$. [2]

(c) Explain what happens to the fish population if P falls below the lower equilibrium value. [1]

27. HARD GDC **[5 marks]**

The temperature T (°C) of a chemical sample satisfies $\frac{dT}{dt} = -k(T-15)^2$, where t is time in minutes and $k > 0$.

(a) Solve the differential equation to show that $\frac{1}{T-15} = kt + C$ for some constant C. [2]

(b) Given $T = 65$ when $t = 0$ and $T = 40$ when $t = 5$, find the values of k and C. [2]

(c) Find the temperature when $t = 20$, correct to 3 significant figures. [1]

28. HARD GDC **[5 marks]**

Consider the differential equation $\dfrac{dy}{dx} = x + y$, with $y(0) = 1$.

(a) Use Euler's method with $h = 1$ to find an approximation for $y(2)$. [2]

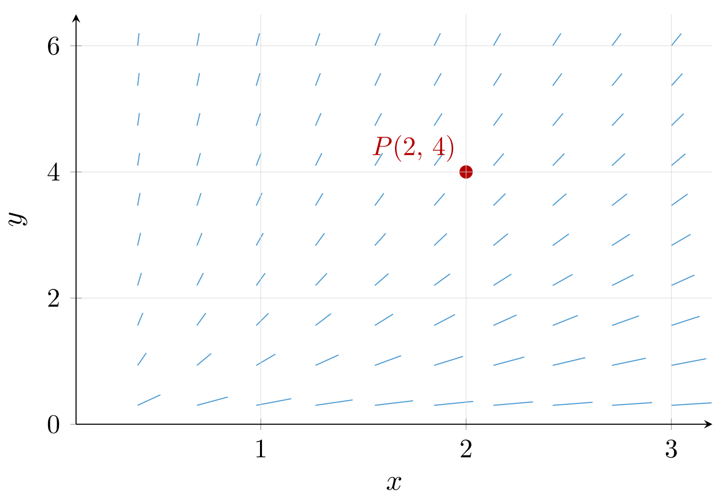
(b) The exact solution is $y = 2e^x - x - 1$. Find the exact value of $y(2)$. [1]

(c) Find the percentage error in the Euler approximation, correct to 3 significant figures. [1]

(d) Explain why a smaller step size would reduce this percentage error. [1]

29. HARD GDC **[5 marks]**

The slope field below represents $\dfrac{dy}{dx} = \dfrac{2y}{x}$ for $x > 0, y > 0$. The point $P(2,\ 4)$ is marked.



(a) Solve the differential equation to find the general solution $y = Ax^2$ (where $A > 0$). [2]

(b) Find the particular solution passing through $P(2,\ 4)$, and sketch it on the slope field above. [2]

(c) Write down the gradient of the solution curve at the point where $x = 3$. [1]

30. **HARD** **GDC** **[5 marks]**

A tank initially contains 100 litres of pure water. Brine containing 0.5 kg of salt per litre flows in at 4 litres per minute, and the well-mixed solution drains out at the same rate. Let S kg be the amount of salt in the tank at time t (minutes).

(a) Show that $\dfrac{dS}{dt} = 2 - \dfrac{S}{25}$. [2]

(b) Solve this differential equation with $S(0) = 0$ to find $S(t)$. [2]

(c) Find the amount of salt in the tank when $t = 50$ minutes, and state the long-term value of S as $t \to \infty$. [1]

Worked Solutions

IB Math AI HL — Differential Equations

Q1 (3 marks) EASY — Exponential Growth

(a) $\dfrac{dN}{dt} = kN$ **A1**

(b) Separate and integrate: $\displaystyle\int \frac{dN}{N} = \int k\,dt$ **M1**

$\ln N = kt + c \Rightarrow N = Ae^{kt}$. Apply $N(0) = 500$: $A = 500$, so $N = 500e^{kt}$ **A1 AG**

Q2 (3 marks) EASY — Slope Field: Reading and Sketching

(a) $\left.\dfrac{dy}{dx}\right|_{(2,1)} = 2 - 1 = \mathbf{1}$ **A1**

(b) Curve starts at $(0, 2)$ with initial slope -2 (downward); follows field elements and approaches the equilibrium line $y = x$ from above. **A1A1**

A1 for smooth curve through $(0, 2)$; A1 for correct direction and long-run approach to $y = x$.

Q3 (3 marks) EASY — Euler Method – dy/dx = 2x+1

$y_{n+1} = y_n + 0.5(2x_n + 1)$

$n = 0$: $y_1 = 3 + 0.5(1) = 3.500$ **(M1)**

$n = 1$: $y_2 = 3.5 + 0.5(3) = 5.000$ **A1**

$n = 2$: $y_3 = 5.0 + 0.5(5) = \mathbf{7.500}$ **A1**

Exact: $y = x^2 + x + 3$, so $y(1.5) = 6.75$. Euler overestimates (function is convex). FT from y_1.

Q4 (3 marks) EASY — Radioactive Decay

(a) $\displaystyle\int \frac{dm}{m} = \int -0.04\,dt \Rightarrow \ln m = -0.04t + c \Rightarrow m = Ae^{-0.04t}$ **M1**

$m(0) = 80$: $m = 80e^{-0.04t}$ **A1**

(b) $m(10) = 80e^{-0.4} = 53.6\ldots \approx$ **53.6 g** **A1**

Q5 (3 marks) EASY — Isocline

(a) $y - 2x = 0 \Rightarrow y = 2x$ **A1**

(b) Straight line through origin, gradient 2. **A1**

(c) Along $y = 2x$, $dy/dx = 0$, so all field segments are **horizontal**. **A1**

Q6 (3 marks) EASY — Newton's Law of Cooling

(a) $\dfrac{dT}{dt} = -k(T - 20)$ **A1**

(b) $T(0) = 90$ **A1**

(c) $\displaystyle\int \frac{dT}{T - 20} = \int -k\,dt \Rightarrow \ln|T - 20| = -kt + c \Rightarrow T - 20 = Ae^{-kt}$

$T(0) = 90$: $A = 70$; therefore $T = 20 + 70e^{-kt}$ **A1 AG**

Q7 (3 marks) EASY — Euler Method – dy/dx = y

$y_{n+1} = 1.25\,y_n$ **(M1)**

$y_1 = 1.2500,\ y_2 = 1.5625,\ y_3 = 1.9531,\ y_4 = 2.4414$ **A1A1**

Approximation: $y(1) \approx$ **2.44** (exact: $e^1 \approx 2.718$; Euler underestimates on convex e^x)

A1 for y_2 correct; A1 for y_4 correct. FT from y_1.

Q8 (3 marks) EASY — Separable ODE – dh/dt = 3 minus h

(a) $\displaystyle\int \frac{dh}{3 - h} = \int dt \Rightarrow -\ln|3 - h| = t + c \Rightarrow 3 - h = Be^{-t}$ **M1**

$h(0) = 1$: $B = 2$, so $h = 3 - 2e^{-t}$ **A1**

(b) As $t \to \infty$: $h \to$ **3** cm **A1**

Q9 (3 marks) EASY — Matching ODEs to Descriptions

(a) P: $dy/dx = x^2 \geq 0$; depends only on x. **A1**

(b) R: $dy/dx = x + y = 0 \Leftrightarrow y = -x$. **A1**

(c) Q: $dy/dx = -y$; zero on x-axis; symmetric since $-(-y) = y$. **A1**

Q10 (3 marks) EASY — Fish Population: Find k

(a) $P_0 =$ **2000**; $P = 2000e^{kt}$ **A1**

(b) $5000 = 2000e^{3k} \Rightarrow e^{3k} = 2.5$ **M1**

$k = \dfrac{\ln 2.5}{3} = 0.30543\ldots \approx$ **0.305** **A1**

Q11 (4 marks) MEDIUM — Vine Growth – dL/dt = Lt

(a) $\displaystyle\int \frac{dL}{L} = \int t\,dt \Rightarrow \ln L = \frac{t^2}{2} + c \Rightarrow L = Ae^{t^2/2}$ **M1A1**

$L(0) = 2$: $A = 2$, so $L = 2e^{t^2/2}$ **A1 AG**

(b) $L(4) = 2e^8 = 5961.9\ldots \approx$ **5960 m** **A1**

Q12 (4 marks) MEDIUM — Euler Method (dy/dx = x/y) and Limitation

(a) $y_{n+1} = y_n + 0.4(x_n/y_n)$

$n = 0$: $f_0 = 0$; $y_1 = 2.000$ **(M1)**

$n = 1$: $f_1 = 0.4/2.000 = 0.2000$; $y_2 = 2.000 + 0.4(0.2000) = 2.080$ **A1**

$n = 2$: $f_2 = 0.8/2.080 = 0.38462$; $y_3 = 2.080 + 0.4(0.38462) = 2.234$ **A1**

(b) As $y \to 0$, $x/y \to \infty$; a tiny error in y produces a huge error in the gradient, causing the method to break down (the singularity at $y = 0$ violates the assumption of bounded gradients). **R1**

Q13 (4 marks) MEDIUM — Newton's Cooling: Find k and Time

(a) Separate: $T - 25 = Ae^{-kt}$; $T(0) = 600 \Rightarrow A = 575$; so $T = 25 + 575e^{-kt}$ **A1 AG**

(b) $360 = 25 + 575e^{-10k} \Rightarrow e^{-10k} = 335/575$ **M1**

$k = -\frac{1}{10}\ln(335/575) = 0.053747\ldots \approx$ **0.0537** **A1**

(c) $100 = 25 + 575e^{-kt} \Rightarrow e^{-kt} = 75/575$ **(M1)**

$t = \dfrac{\ln(575/75)}{0.053747} = 37.9\ldots \approx$ **37.9** min **A1**

Q14 (4 marks) MEDIUM — Logistic Deer Population

(a) Carrying capacity: **800** deer **A1**

(b) $P = \dfrac{800}{1 + Ae^{-0.3t}}$; $P(0) = 100$: $1 + A = 8 \Rightarrow A = 7$ **M1A1 AG**

(c) $P(5) = \dfrac{800}{1+7e^{-1.5}} = \dfrac{800}{2.5619\ldots} = 312.3\ldots \approx$ **312** deer **A1**

Q15 (4 marks) MEDIUM — Slope Field (dy/dx=y squared/2) Vertical Asymptote

(a) Curve starts at $(0,1)$ with slope $\frac{1}{2}$; steepens increasingly; curves upward toward a vertical asymptote. **A1A1**

A1 for curve through $(0,1)$; A1 for correct concave-up shape that steepens markedly.

(b) $\displaystyle\int y^{-2}\,dy = \int \tfrac{1}{2}\,dx \Rightarrow -\tfrac{1}{y} = \tfrac{x}{2} + C$ **M1**

$(0,1)$: $C = -1$; $y = \dfrac{2}{2-x}$. Asymptote at **x = 2** **A1**

Q16 (4 marks) MEDIUM — Velocity and Displacement

(a) $v = \displaystyle\int \cos t\,dt = \sin t + C$; $v(0) = 0 \Rightarrow C = 0$; $v = \sin t$ **M1A1**

(b) $s = \displaystyle\int_0^{\pi} \sin t\,dt = \big[-\cos t\big]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 =$ **2 m** **M1A1**

Q17 (4 marks) MEDIUM — Euler for Rumour Spread

$R_{n+1} = R_n + 0.1 \times 0.05R_n(200 - R_n)$

$n = 0$: $f_0 = 0.05(4)(196) = 39.20$; $R_1 = 4 + 3.920 = 7.920$ **M1**

$n = 1$: $f_1 = 0.05(7.920)(192.08) = 76.10$; $R_2 = 7.920 + 7.610 = 15.530$ **A1**

$n = 2$: $f_2 = 0.05(15.530)(184.47) = 143.2$; $R_3 = 15.530 + 14.320 =$ **29.9** **A1**

Carry 4 s.f. in intermediate values. Accept $R_3 \in [29.8,\ 30.0]$.

Q18 (4 marks) MEDIUM — Separable ODE – dC/dt = -C/(1+t)

(a) $\displaystyle\int \frac{dC}{C} = \int \frac{-1}{1+t}\,dt \Rightarrow \ln C = -\ln(1+t) + c \Rightarrow C = \frac{A}{1+t}$ **M1A1**

$C(0) = 6$: $A = 6$; $C = \dfrac{6}{1+t}$ **A1**

(b) $C(5) = 6/6 =$ **1.00** mol/L **A1**

Q19 (4 marks) MEDIUM — Slope Field Matching

(a)

(i) $dy/dx = y$ **matches field C**: slopes depend only on y (constant along horizontal lines); zero on x-axis. **A1**

(ii) $dy/dx = x$ **matches field A**: slopes depend only on x (constant along vertical lines); zero on y-axis. **A1**

(iii) $dy/dx = -x/y$ **matches field B**: produces circular field (slope perpendicular to position vector). **A1**

(b) On field C: solution through $(0, 1)$ is $y = e^x$ (exponentially increasing). **A1**

Q20 (4 marks) MEDIUM — Snow Depth - dd/dt = 1 over root d

(a) $\int d^{1/2}\,dd = \int dt \Rightarrow \frac{2}{3}d^{3/2} = t + c$ **M1A1**

$d(0) = 4$: $c = \frac{2}{3}(8) = \frac{16}{3}$, so $\frac{2}{3}d^{3/2} = t + \frac{16}{3}$.

Verify: if $d = \left(\frac{t}{2} + 2\right)^2$, then $d^{1/2} = \frac{t}{2} + 2$ and $\frac{dd}{dt} = 2\left(\frac{t}{2} + 2\right) \cdot \frac{1}{2} = \frac{t}{2} + 2 = d^{1/2}$ ✓ **A1 AG**

(b) $d(6) = \left(\frac{6}{2} + 2\right)^2 = 5^2 =$ **25 m** **A1**

Q21 (5 marks) HARD — Logistic with Unknown Parameters

(a) $N(\infty) = L =$ **40** (thousands) **A1**

(b) $N = \dfrac{40}{1 + Ae^{-kt}}$; $N(0) = 2$: $A = 19$; so $N = \dfrac{40}{1 + 19e^{-kt}}$ **M1A1 AG**

(c) $N(5) = 8$: $1 + 19e^{-5k} = 5 \Rightarrow e^{-5k} = 4/19$ **(M1)**

$k = \dfrac{\ln(19/4)}{5} = 0.31487\ldots \approx$ **0.315** **A1**

(d) $30 = \dfrac{40}{1 + 19e^{-kt}} \Rightarrow 19e^{-kt} = 1/3 \Rightarrow t = \dfrac{\ln 57}{0.315} = 12.84\ldots \approx$ **12.8** days **A1**

Q22 (5 marks) HARD — Euler vs Exact for dy/dx = y-squared

(a) $y_0 = 0.5$, $h = 0.5$.

$f_0 = (0.5)^2 = 0.25$; $y_1 = 0.5 + 0.5(0.25) = 0.625$ **M1**

$f_1 = (0.625)^2 = 0.39063$; $y_2 = 0.625 + 0.5(0.39063) =$ **0.820** **A1**

(b) $\int y^{-2}\,dy = \int dx \Rightarrow -\dfrac{1}{y} = x + C$; $C = -2$; $y = \dfrac{1}{2 - x}$ **M1**

$y(1) = \dfrac{1}{1} =$ **1.00** **A1**

(c) % error $= \dfrac{|0.820 - 1.00|}{1.00} \times 100 =$ **18.0%** **A1**

Q23 (5 marks) HARD — Separable ODE – External Boundary Condition

(a) $\displaystyle\int P\,dP = \int t\,dt \Rightarrow \frac{P^2}{2} = \frac{t^2}{2} + C' \Rightarrow P^2 = t^2 + K$ **M1A1 AG**

(b) $P(2) = 3$: $9 = 4 + K \Rightarrow K = 5$ **M1**

$P = \sqrt{t^2+5}$ $(P > 0)$ **A1**

(c) $P(6) = \sqrt{36+5} = \sqrt{41} = 6.4031\ldots \approx$ **6.40** million **A1**

Q24 (5 marks) HARD — Slope Field, Exact Solution, Verification

(a) At $(0,0)$: slope $= 0$; curve rises toward $y = 1$, S-shaped, approaching it asymptotically from below. **A1A1**

A1 for curve through $(0,0)$; A1 for correct concave-up shape approaching $y = 1$ from below.

(b) $\displaystyle\int \frac{dy}{1-y} = \int x\,dx \Rightarrow -\ln|1-y| = \frac{x^2}{2} + C$ **M1**

$y(0) = 0$: $C = 0$; $1 - y = e^{-x^2/2}$; $y = 1 - e^{-x^2/2}$ **A1 AG**

(c) As $x \to \infty$: $e^{-x^2/2} \to 0$, so $y \to 1$. The dashed line $y = 1$ is a **stable equilibrium** — all solution curves with $0 < y < 1$ approach it from below. **R1**

Q25 (5 marks) HARD — Euler then Exact for dm/dt = 4 minus m-squared

(a) $m_0 = 0$, $h = 0.25$.

$f_0 = 4 - 0 = 4$; $m_1 = 0 + 0.25(4) = 1.000$ **M1**

$f_1 = 4 - 1 = 3$; $m_2 = 1.000 + 0.25(3) =$ **1.750** **A1**

(b) Partial fractions: $\dfrac{1}{4-m^2} = \dfrac{1}{4}\left(\dfrac{1}{2-m} + \dfrac{1}{2+m}\right)$ **M1**

$\dfrac{1}{4}\ln\left(\dfrac{2+m}{2-m}\right) = t$; $(m(0) = 0 \Rightarrow C = 0)$ **A1**

$\dfrac{2+m}{2-m} = e^{4t} \Rightarrow 2 + m = (2-m)e^{4t} \Rightarrow m(1+e^{4t}) = 2(e^{4t}-1) \Rightarrow m = 2 \cdot \dfrac{e^{4t}-1}{e^{4t}+1}$ **A1 AG**

Q26 (5 marks) HARD — Logistic with Harvesting: Equilibria

(a) At $P = 60$, $dP/dt = 0$: $\frac{60k}{1}\left(1 - \frac{60}{120}\right) = 3 \Rightarrow 30k = 3 \Rightarrow k = \frac{1}{10}$ **M1A1 AG**

(b) Set $\frac{P}{10}\left(1 - \frac{P}{120}\right) = 3$: $P^2 - 120P + 3600 = 0 \Rightarrow (P-60)^2 = 0$ **M1**

Double root: both equilibria coincide at $P =$ **60** (hundred) **A1**

(c) For any $P < 60$, $dP/dt < 0$ throughout (the saddle-node means no lower stable equilibrium); the population will decline continuously to **extinction**. **R1**

Q27 (5 marks) HARD — Separable ODE with Quadratic Factor

(a) $\int (T-15)^{-2}\,dT = \int -k\,dt \Rightarrow -\frac{1}{T-15} = -kt + c' \Rightarrow \frac{1}{T-15} = kt + C$ **M1A1 AG**

(b) $T(0) = 65$: $C = \frac{1}{50} = 0.02$ **A1**

$T(5) = 40$: $\frac{1}{25} = 5k + 0.02 \Rightarrow 5k = 0.02 \Rightarrow k =$ **0.004** **A1**

(c) $\frac{1}{T-15} = 0.004(20) + 0.02 = 0.10 \Rightarrow T - 15 = 10 \Rightarrow T =$ **25.0 °C** **A1**

Q28 (5 marks) HARD — Euler Error Analysis – dy/dx = x+y

(a) $y_0 = 1$, $h = 1$.

$f_0 = 0 + 1 = 1$; $y_1 = 1 + 1 = 2$; $f_1 = 1 + 2 = 3$; $y_2 = 2 + 3 =$ **5** **M1A1**

(b) $y(2) = 2e^2 - 2 - 1 = 2(7.38906) - 3 = 11.778\ldots \approx$ **11.8** **A1**

(c) % error $= \frac{|5 - 11.778|}{11.778} \times 100 = 57.5\ldots\% \approx$ **57.5%** **A1**

(d) A smaller h means each linear approximation covers a shorter interval; the curvature error per step is smaller and the accumulated error is less, so the approximation stays closer to the true (exponentially curving) solution. **R1**

Q29 (5 marks) HARD — Slope Field and Exact Solution (dy/dx = 2y/x)

(a) $\int \frac{dy}{y} = \int \frac{2}{x}\,dx \Rightarrow \ln y = 2\ln x + c \Rightarrow y = Ax^2$ **M1A1**

(b) $P(2,4)$: $4 = A(4) \Rightarrow A = 1$; particular solution $y = x^2$ **M1**

Sketch: parabola through $(0,0)$ and $(2,4)$ following the field. **A1**

(c) $\left.\dfrac{dy}{dx}\right|_{x=3} = \left.\dfrac{2y}{x}\right|_{y=9,\,x=3} = \dfrac{18}{3} = \mathbf{6}$ **A1**

Q30 (5 marks) HARD — Salt-Tank Mixing Model

(a) Rate in $= 4 \times 0.5 = 2$ kg/min; rate out $= 4 \times (S/100) = S/25$ kg/min **A1**

$\dfrac{dS}{dt} = 2 - \dfrac{S}{25}$ **A1 AG**

(b) $\displaystyle\int \frac{dS}{2 - S/25} = \int dt \Rightarrow -25\ln\left|2 - \frac{S}{25}\right| = t + c$ **M1**

$S(0) = 0$: $c = -25\ln 2$; $2 - S/25 = 2e^{-t/25} \Rightarrow S = 50(1 - e^{-t/25})$ **A1**

(c) $S(50) = 50(1 - e^{-2}) = 50(0.86466) = 43.23\ldots \approx$ **43.2** kg **A1**

As $t \to \infty$: $S \to$ **50** kg

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