IB Math AI HL — Topic 1: Number & Algebra

Further Complex Numbers

Geometry of Complex Numbers · Polar & Exponential Form · Form Conversions · Frequency & Phase

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Marks each	3	4	5
Total Marks	30	40	50

Scan me for help

Name:____________________________ Date:______________

Score:_________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. **Exact** answers in terms of π, $\sqrt{\ }$, $e^{i\theta}$ are required where specified.

Key Formulae

Modulus and Argument

$z = a + bi$: $\quad |z| = r = \sqrt{a^2 + b^2}$

$\arg(z) = \theta$: $\cos\theta = \dfrac{a}{r}$, $\sin\theta = \dfrac{b}{r}$, $-\pi < \theta \le \pi$

Polar Form

$z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$

Exponential (Euler) Form

$z = r\,e^{i\theta}, \quad e^{i\pi} = -1$

De Moivre (products/quotients)

$z_1 z_2 = r_1 r_2\, e^{i(\theta_1+\theta_2)}$

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$

$z^n = r^n e^{in\theta}$

Sinusoidal Model

$f(t) = A\sin(Bt + C) + D$

Amplitude $= A$

Period $= \dfrac{2\pi}{B}$ (in radians), or $\dfrac{360°}{B}$

Phase shift $= -\dfrac{C}{B}$

Midline $y = D$

Phasor Addition

$A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2)$

$= \text{Re}(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}) \cdot e^{i\omega t}$

Geometric transformation by $w = re^{i\alpha}$

Enlargement scale factor r, rotation α

GDC Tips

- **Complex mode:** On most GDCs, set to complex/rectangular mode for Re(z), Im(z), $|z|$, arg(z) computations.
- **Convert to polar:** Compute $r = \sqrt{a^2 + b^2}$ and $\theta = \text{atan2}(b, a)$ using `Angle` or `arg()` function.
- **Phasor sums:** Enter complex numbers directly; the GDC will compute modulus and argument of the sum.

- **Sinusoidal regression:** Use `SinReg` in STAT $\rightarrow$ CALC; confirm the mode is radians first.
- **Powers:** Use the complex power function z^n directly; for large n use exponential form $r^n e^{in\theta}$.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

Scan me for help

1. EASY No Calculator **[3 marks]**

Let $z = -3 + 3i$.

(a) Write z in the form $r\,e^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$, giving exact values of r and θ. [2]

(b) Write down the modulus of z^2. [1]

2. EASY No Calculator **[3 marks]**

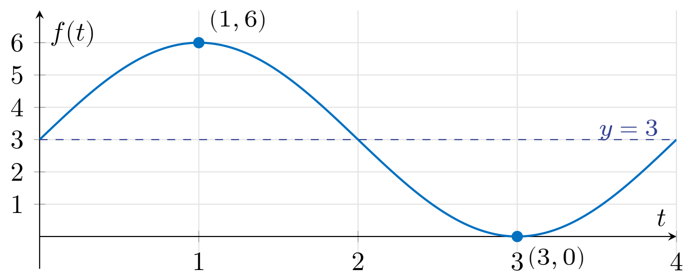
Let $w = 5 - 5\sqrt{3}\,i$.

(a) Find the modulus and argument of w, giving exact values. [2]

(b) Hence write w in exponential form. [1]

3. **EASY** **Calculator** **[3 marks]**

The diagram below shows one complete cycle of the function $f(t) = A\sin(Bt) + D$ defined for $0 \le t \le 4$.



(a) Write down the value of A. [1]

(b) Write down the value of D. [1]

(c) Find the value of B, giving your answer in exact form. [1]

4. **EASY** **No Calculator** **[3 marks]**

Let $z_1 = 4\,e^{i\pi/3}$ and $z_2 = 3\,e^{-i\pi/6}$.

(a) Write down the modulus and argument of $z_1 z_2$. [2]

(b) Hence write $z_1 z_2$ in the form $a + bi$. [1]

5. **EASY** **No Calculator** **[3 marks]**

Write the complex number $z = 6\,e^{i\,2\pi/3}$ in the form $a + bi$ where $a, b \in \mathbb{R}$. Give exact values of a and b.

6. EASY No Calculator **[3 marks]**

A signal is modelled by $V(t) = 8\cos\left(3\pi t - \frac{\pi}{4}\right)$, where t is in seconds.

(a) Write down the amplitude of $V(t)$. [1]

(b) Find the period of $V(t)$. [1]

(c) Write down the phase shift of $V(t)$. [1]

7. EASY No Calculator **[3 marks]**

A complex number z is represented by the point P on an Argand diagram. The point Q represents iz.

(a) Describe the transformation that maps P to Q. [2]

(b) Given that $z = 2 - 2i$, write down the coordinates of Q. [1]

8. EASY No Calculator **[3 marks]**

Let $z_1 = 12\,e^{i\,5\pi/6}$ and $z_2 = 4\,e^{i\,\pi/6}$.

(a) Find $\frac{z_1}{z_2}$ in exponential form. [2]

(b) Hence state the argument of $\frac{z_1}{z_2}$. [1]

9. EASY No Calculator **[3 marks]**

Let $z = \sqrt{2}\, e^{i\,3\pi/4}$.

(a) Find z^4 in exponential form, giving the argument in the range $(-\pi, \pi]$. [2]

(b) Write down the value of z^4. [1]

10. EASY Calculator **[3 marks]**

A Ferris wheel completes one full revolution in 24 minutes. The height of a passenger above ground varies sinusoidally between 2 m and 30 m.

Write a function $h(t) = A\sin(Bt) + D$ that models the height, in metres, of a passenger t minutes after the ride begins, assuming the passenger starts at the midline position moving upward.

(a) Write down A and D. [2]

(b) Write down the exact value of B. [1]

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

Scan me for help

11. **MEDIUM** **Calculator** **[4 marks]**

Two electrical currents in a circuit are modelled by

$$I_1(t) = 5\cos(2\pi t) \quad \text{and} \quad I_2(t) = 5\cos\left(2\pi t - \frac{\pi}{3}\right),$$

where t is in seconds and current is in amps.

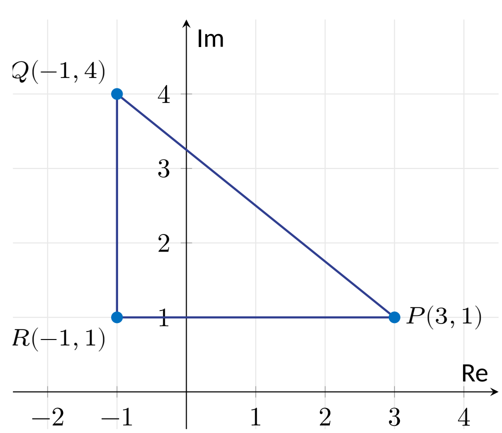
(a) Write I_1 and I_2 in the form $\text{Re}\left(a\, e^{i(2\pi t+\phi)}\right)$. [1]

(b) Find the total current $I_T = I_1 + I_2$ in the form $A\cos(2\pi t + \phi_0)$, stating the amplitude A and phase shift ϕ_0 correct to 3 s.f. [3]

12. **MEDIUM** **No Calculator** **[4 marks]**

The points P, Q, and R on an Argand diagram represent the complex numbers $z_1 = 3 + i$, $z_2 = -1 + 4i$, and $z_3 = -1 + i$ respectively.

The diagram below shows these three points.



(a) Show that PQR is a right-angled triangle. [2]

(b) Find the area of triangle PQR. [2]

13. MEDIUM No Calculator **[4 marks]**

The equation $x^2 - 4x + (k+3) = 0$, where $k \in \mathbb{R}$, has two complex conjugate roots α and $\bar{\alpha}$.

(a) Find the range of values of k for which the roots are non-real. [2]

(b) Given that $k = 10$, express both roots in exponential form $r\,e^{i\theta}$, with exact r and $\theta \in (-\pi, \pi]$. [2]

14. MEDIUM Calculator **[4 marks]**

The depth of water at a harbour entrance is modelled by

$$d(t) = 3.5\cos(0.524\,t + C) + 5.8,$$

where d is in metres, t is in hours after midnight, and C is a constant.
At midnight ($t = 0$), the depth is 4.05 m.

(a) Show that $C = \arccos\left(\frac{4.05 - 5.8}{3.5}\right)$, and find the value of C correct to 3 s.f. [2]

(b) Find the time after midnight at which the depth first reaches its maximum value. Give your answer correct to the nearest minute. [2]

15. MEDIUM No Calculator **[4 marks]**

Let $w = 2\,e^{i\pi/6}$. The transformation T maps any complex number z to wz.

(a) Describe the transformation T geometrically, stating the scale factor and angle of rotation. [2]

(b) The point A represents $z_0 = 1 + i\sqrt{3}$. Find the complex number represented by $T(z_0)$, giving your answer in the form $a + bi$. [2]

16. MEDIUM Calculator **[4 marks]**

A spring oscillates so that its displacement from equilibrium (in cm) is modelled by

$$s(t) = 4\sin\left(\frac{\pi}{3}t\right), \quad t \geq 0,$$

where t is in seconds.

(a) Write down the period of the oscillation. [1]

(b) Find all values of t in the interval $0 \leq t \leq 12$ for which $s(t) = 2\sqrt{3}$. [3]

17. MEDIUM No Calculator **[4 marks]**

A geometric sequence has first term $u_1 = 8$ and common ratio $r = \frac{1}{2}e^{i\pi/4}$.

(a) Write down $|r|$ and $\arg(r)$. [1]

(b) Show that $u_3 = 2e^{i\pi/2}$. [2]

(c) Hence write u_3 in the form $a + bi$. [1]

18. MEDIUM Calculator **[4 marks]**

Two sound waves are described by

$$p_1(t) = 10\sin(\omega t) \quad \text{and} \quad p_2(t) = 10\sin\left(\omega t + \frac{\pi}{2}\right).$$

(a) By writing each wave as the imaginary part of a complex exponential, find the complex phasors P_1 and P_2. [1]

(b) Find the amplitude and phase of the combined wave $p_T = p_1 + p_2$, expressing the combined wave in the form $R\sin(\omega t + \phi)$. [3]

19. MEDIUM No Calculator **[4 marks]**

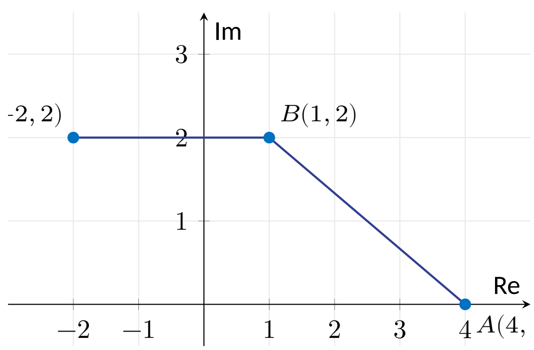
The quadratic $z^2 - 6z + 13 = 0$ has roots α and β.

(a) Solve the equation, giving the roots in the form $a \pm bi$. [2]

(b) Express α in the form $r\,e^{i\theta}$, where $r > 0$ and $\theta \in (0, \pi)$. Give r as an integer and θ correct to 3 s.f. [2]

20. MEDIUM Calculator **[4 marks]**

On an Argand diagram, the points A, B, C represent the complex numbers $z_1 = 4$, $z_2 = 1+2i$, and $z_3 = -2+2i$ respectively.



(a) Find $z_2 - z_1$ and $z_3 - z_2$. [1]

(b) Find $|AB|$ and $|BC|$. [1]

(c) Determine whether ABC is an isosceles triangle, justifying your answer. [2]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

Scan me for help

21. **HARD** **No Calculator** **[5 marks]**

A sequence of complex numbers is defined by $z_n = \left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)^n$, for $n = 1, 2, 3, \ldots$

(a) Express $\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ in exponential form $r\,e^{i\theta}$, giving exact values. [2]

(b) Find the modulus of z_n and the argument of z_n, in terms of n. [1]

(c) Determine whether the sum $S = \sum_{n=1}^{\infty} |z_n|$ converges, justifying your answer. [2]

22. HARD Calculator **[5 marks]**

Two laser beam intensity functions are modelled by

$$I_1(t) = 6\cos\left(4\pi t + \frac{\pi}{6}\right) \quad \text{and} \quad I_2(t) = 8\cos\left(4\pi t + \frac{2\pi}{3}\right).$$

(a) Write the combined intensity $I_T = I_1 + I_2$ in the form $R\cos(4\pi t + \Phi)$, where $R > 0$ and $0 < \Phi \leq \pi$. Give R correct to 3 s.f. and Φ correct to 3 s.f. [3]

(b) Find the first time $t > 0$ at which $I_T(t) = 0$, giving your answer correct to 3 s.f. [2]

23. HARD No Calculator **[5 marks]**

Let $z = k + 2i$ where $k \in \mathbb{R}$, $k > 0$.

(a) Show that $|z|^2 = k^2 + 4$. [1]

(b) Given that $\arg(z) = \frac{\pi}{4}$, find the value of k. [2]

(c) Using your value of k, find z^2 in the form $a + bi$ and hence write z^2 in exponential form. [2]

24. HARD Calculator **[5 marks]**

A tidal model gives sea level (in metres above mean sea level) as

$$h(t) = 2.4\sin\left(\frac{\pi}{6}t\right) + 1.6\sin\left(\frac{\pi}{6}t + \frac{\pi}{3}\right),$$

where t is hours after midnight.

(a) Use phasor addition to write $h(t)$ in the form $A\sin\left(\frac{\pi}{6}t + \phi\right)$, where $A > 0$. Give A correct to 3 s.f. and ϕ correct to 3 s.f. [3]

(b) Hence find the first time $t > 0$ at which the sea level is at its maximum. Give your answer correct to the nearest minute. [2]

25. HARD No Calculator **[5 marks]**

A geometric series has first term $u_1 = 4$ and common ratio $r = \frac{1}{\sqrt{2}}e^{i\pi/4}$.

(a) Show that $|r| = \frac{1}{\sqrt{2}}$, and hence justify that the series has a finite sum to infinity. [2]

(b) Find the sum to infinity S_∞ in the form $a + bi$. Give exact values. [3]

26. HARD No Calculator **[5 marks]**

Let $w = -1 + \sqrt{3}\,i$.

(a) Write w in exponential form, giving exact values. [2]

(b) The transformation T maps any complex number z to wz. Describe T fully as a composition of a rotation and an enlargement. [1]

(c) Find a 2×2 matrix M such that $M\begin{pmatrix} x \\ y \end{pmatrix}$ represents multiplication of $z = x + yi$ by w. [2]

27. HARD No Calculator **[5 marks]**

The equation $z^2 - 2pz + (p^2 + 9) = 0$, where $p \in \mathbb{R}$, has roots α and $\bar{\alpha}$.

(a) Show that the roots are always non-real, and find α in terms of p. [2]

(b) Find the value of p such that $\arg(\alpha) = \dfrac{\pi}{3}$, where α has positive imaginary part. [2]

(c) For this value of p, write α in exponential form. [1]

28. HARD Calculator **[5 marks]**

A wheel of radius r metres rotates anticlockwise. The horizontal displacement of a point P on the rim from the centre is modelled by

$$x(t) = r\cos\left(\frac{\pi}{4}t - \frac{\pi}{3}\right),$$

where t is in seconds.

(a) Find the period of $x(t)$. [1]

(b) The maximum horizontal displacement first occurs at $t = t_0$. Show that $t_0 = \dfrac{4}{3}$ seconds. [2]

(c) Given that at $t = 2$ the displacement is $x = 1.50$ m, find the value of r, correct to 3 s.f. [2]

29. HARD No Calculator **[5 marks]**

Let $z = x + yi$, where $x, y \in \mathbb{R}$.

(a) Show that $|z - (3 + 2i)| = 3$ represents a circle, and state its centre and radius. [2]

(b) Find the two values of z that satisfy both $|z - (3 + 2i)| = 3$ and $\mathrm{Im}(z) = 2$. [2]

(c) Determine which of the two values found in (b) has the greater argument, justifying your answer. [1]

30. HARD No Calculator **[5 marks]**

Let $S = \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta$ for a fixed real number θ.

(a) Show that $S = \text{Re}(e^{i\theta} + e^{2i\theta} + e^{3i\theta} + e^{4i\theta})$. [1]

(b) By summing the geometric series, show that

$$S = \text{Re}\left(\frac{e^{i\theta}(1 - e^{4i\theta})}{1 - e^{i\theta}}\right).$$

[2]

(c) Hence evaluate S when $\theta = \dfrac{\pi}{3}$, giving an exact value. [2]

Worked Solutions

Section A — Easy EASY

Q1 — Form Conversion [EASY

] **(a)** $r = |-3+3i| = \sqrt{(-3)^2+3^2} = \sqrt{18} = 3\sqrt{2}$ **A1**

$\theta = \pi - \arctan\left(\frac{3}{3}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ **A1**

$z = 3\sqrt{2}\,e^{i\,3\pi/4}$

(b) $|z^2| = |z|^2 = (3\sqrt{2})^2 = 18$ **A1**

Note: Award A1A1 in (a) for correct modulus and correct argument. Accept $r = 3\sqrt{2}$ and $\theta = 135°$.

Q2 — Form Conversion [EASY

] **(a)** $r = \sqrt{5^2+(5\sqrt{3})^2} = \sqrt{25+75} = \sqrt{100} = 10$ **A1**

w is in the fourth quadrant: $\theta = -\arctan\left(\frac{5\sqrt{3}}{5}\right) = -\arctan(\sqrt{3}) = -\frac{\pi}{3}$ **A1**

(b) $w = 10\,e^{-i\pi/3}$ **A1**

Note: Accept $-60°$ for θ in (a). Consistent FT into (b).

Q3 — Reading Sinusoidal Graph [EASY

] **(a)** Maximum is 6, minimum is 0, so amplitude $A = \frac{6-0}{2} = 3$ **A1**

(b) Midline $D = \frac{6+0}{2} = 3$ **A1**

(c) Period $= 4$, so $\frac{2\pi}{B} = 4 \Rightarrow B = \frac{\pi}{2}$ **A1**

Note: Verify: $f(1) = 3\sin(\pi/2)+3 = 6$ and $f(3) = 3\sin(3\pi/2)+3 = 0$ OK

Q4 — Modulus/Argument of Product [EASY

] **(a)** $|z_1 z_2| = 4\times 3 = 12;\quad \arg(z_1 z_2) = \frac{\pi}{3} + \left(-\frac{\pi}{6}\right) = \frac{\pi}{6}$ **A1 A1**

(b) $z_1 z_2 = 12\,e^{i\pi/6} = 12\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = 6\sqrt{3}+6i$ **A1**

Q5 — Exponential to Cartesian Form [EASY

] $z = 6\,e^{i\,2\pi/3} = 6\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right) = 6\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)$

M1 (attempt to expand using cos and sin)

$z = -3 + 3\sqrt{3}\,i$ **A1 A1**

Note: $a = -3$, $b = 3\sqrt{3}$. Accept $b = 5.196\ldots$ for A1 in (b) if not told exact.

Q6 — Period and Phase of Sinusoidal Function [EASY

] **(a)** Amplitude $= 8$ **A1**

(b) Period $= \dfrac{2\pi}{3\pi} = \dfrac{2}{3}$ seconds **A1**

(c) Phase shift $= -\dfrac{C}{B} = -\dfrac{-\pi/4}{3\pi} = \dfrac{1}{12}$ seconds **A1**

Note: Accept "shifted $\frac{1}{12}$ s to the right" as an equivalent description.

Q7 — Geometric Effect of Multiplication by i [EASY

] **(a)** Multiplying by $i = e^{i\pi/2}$ is a rotation of $\dfrac{\pi}{2}$ (anticlockwise $90°$) about the origin. Scale factor is 1 (no enlargement). **A1 A1**

(b) $iz = i(2-2i) = 2i - 2i^2 = 2 + 2i$, so $Q = (2, 2)$ **A1**

Note: Award both A1 in (a) for "rotation $90°$ anticlockwise about origin, scale factor 1." Award A0A1 if only one feature stated.

Q8 — Modulus and Argument of Quotient [EASY

] **(a)** $\dfrac{z_1}{z_2} = \dfrac{12}{4}\,e^{i(5\pi/6-\pi/6)} = 3\,e^{i\,4\pi/6} = 3\,e^{i\,2\pi/3}$ **A1 A1**

(b) $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{2\pi}{3}$ **A1**

Note: Award A1 in (a) for modulus 3 and A1 for argument $2\pi/3$. (b) is FT from (a).

Q9 — Power of Complex Number [EASY

] **(a)** $z^4 = \left(\sqrt{2}\right)^4 e^{i\,4\times3\pi/4} = 4\,e^{i\,3\pi}$

Adjust argument: $3\pi - 2\pi = \pi$, so $z^4 = 4\,e^{i\pi}$ **A1 A1**

(b) $z^4 = 4(\cos\pi + i\sin\pi) = 4(-1) = -4$ **A1**

Note: Award first A1 for modulus 4; second A1 for argument π (after reduction). In (b) accept "$z^4 = -4$" with no working since previous part is the exponential form.

Q10 — Sinusoidal Model from Context [EASY

] **(a)** Amplitude $A = \dfrac{30-2}{2} = 14$; Midline $D = \dfrac{30+2}{2} = 16$ **A1 A1**

(b) Period $= 24$ min, so $\dfrac{2\pi}{B} = 24 \Rightarrow B = \dfrac{\pi}{12}$ **A1**

Note: Model: $h(t) = 14\sin\left(\dfrac{\pi}{12}t\right) + 16$. Starting at midline moving upward is consistent with sine with no phase shift.

Section B — Medium MEDIUM

Q11 — Phasor Sum of Currents [MEDIUM

] **(a)** $I_1 = \text{Re}(5\,e^{i\cdot 2\pi t})$, $I_2 = \text{Re}(5\,e^{i(2\pi t-\pi/3)})$ **A1**

(b) Phasors: $Z_1 = 5$, $Z_2 = 5\,e^{-i\pi/3} = 5\cos\frac{\pi}{3} - 5i\sin\frac{\pi}{3} = 2.5 - 4.330\ldots i$ **M1**

$Z_T = Z_1 + Z_2 = 7.5 - 4.330\ldots i$

$A = |Z_T| = \sqrt{7.5^2 + 4.330^2} = \sqrt{56.25 + 18.75} = \sqrt{75} = 5\sqrt{3} \approx 8.66$ **A1**

$\phi_0 = \arg(Z_T) = -\arctan\left(\dfrac{4.330}{7.5}\right) = -\arctan\left(\dfrac{\sqrt{3}/2}{3/2}\right) = -\arctan\left(\dfrac{1}{\sqrt{3}}\right) = -\dfrac{\pi}{6} \approx -0.524$ **A1**

$I_T = 5\sqrt{3}\cos\left(2\pi t - \dfrac{\pi}{6}\right)$

Note: Award M1 for attempt to add phasors. Accept $A = 8.66$ (3 s.f.) and $\phi_0 = -0.524$ (3 s.f.).

Q12 — Right-Angled Triangle on Argand Plane [MEDIUM

] **(a)** $\overrightarrow{QR} = z_3 - z_2 = (-1+i) - (-1+4i) = -3i$; $\overrightarrow{RP} = z_1 - z_3 = (3+i) - (-1+i) = 4$

M1 (find two relevant vectors or use dot product)

$\overrightarrow{QR} \cdot \overrightarrow{RP}$: as complex numbers, $(-3i)(4) = -12i$, which is purely imaginary, confirming vectors are perpendicular.

Alternatively: $|QR|^2 = 9$, $|RP|^2 = 16$, $|QP|^2 = (-1-3)^2 + (4-1)^2 = 16+9 = 25$, so $|QR|^2 + |RP|^2 = |QP|^2$. **A1**

Hence $\angle PRQ = 90°$ (Pythagoras converse), PQR is right-angled. **AG**

(b) Area $= \dfrac{1}{2} \times |QR| \times |RP| = \dfrac{1}{2} \times 3 \times 4 = 6$ (square units) **M1 A1**

Q13 — Complex Roots of Quadratic with Parameter [MEDIUM

] **(a)** Discriminant: $\Delta = 16 - 4(k+3) = 4 - 4k$

M1

For non-real roots: $\Delta < 0 \Rightarrow 4 - 4k < 0 \Rightarrow k > 1$ **A1**

(b) $k = 10$: equation is $x^2 - 4x + 13 = 0$

$x = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$ **M1**

$r = |2+3i| = \sqrt{4+9} = \sqrt{13}; \quad \theta = \text{arctan}\left(\dfrac{3}{2}\right)$ (first quadrant)

$\alpha = \sqrt{13}\, e^{i\,\text{arctan}(3/2)}; \quad \bar{\alpha} = \sqrt{13}\, e^{-i\,\text{arctan}(3/2)}$ **A1**

Note: Accept $\theta = \text{arctan}(1.5)$. *In exact form* $r = \sqrt{13}$.

Q14 — Sinusoidal Harbour Model [MEDIUM

] **(a)** At $t = 0$: $d(0) = 3.5\cos(C) + 5.8 = 4.05$ **M1**

$\cos(C) = \dfrac{4.05 - 5.8}{3.5} = \dfrac{-1.75}{3.5} = -0.5$

$C = \text{arccos}(-0.5) = \dfrac{2\pi}{3} \approx 2.09$ **A1** (AG with preceding derivation)

(b) Maximum when $\cos\left(0.524t + \dfrac{2\pi}{3}\right) = 1$, i.e. $0.524t + 2.094 = 0$ is impossible for $t > 0$.

Next: $0.524t + 2.094 = 2\pi \Rightarrow 0.524t = 6.283 - 2.094 = 4.189$

M1

$t = \dfrac{4.189}{0.524} = 7.994\ldots \approx 7.99$ hours after midnight = 07:59 **A1**

Note: $0.524 \approx \pi/6$ *(period 12 h). Accept 7 hours 59 minutes after midnight. FT from their* C.

Q15 — Transformation by Multiplication [MEDIUM

] **(a)** $w = 2\,e^{i\pi/6}$: $|w| = 2$ and $\arg(w) = \dfrac{\pi}{6}$

T is an enlargement with scale factor 2 and a rotation of $\dfrac{\pi}{6}$ anticlockwise about the origin. **A1 A1**

(b) $z_0 = 1 + i\sqrt{3} = 2\,e^{i\pi/3}$ (since $|z_0| = 2$, arg $= \pi/3$)

$T(z_0) = wz_0 = 2\,e^{i\pi/6} \times 2\,e^{i\pi/3} = 4\,e^{i\pi/2} = 4i$ **M1 A1**

$T(z_0) = 0 + 4i$, so $a = 0, b = 4$.

Note: Award M1 for multiplying moduli and adding arguments.

Q16 — Spring Oscillation: Solving on Interval [MEDIUM]

(a) Period $= \dfrac{2\pi}{\pi/3} = 6$ seconds **A1**

(b) $4\sin\left(\dfrac{\pi}{3}t\right) = 2\sqrt{3}$

$\sin\left(\dfrac{\pi}{3}t\right) = \dfrac{\sqrt{3}}{2}$

M1

$\dfrac{\pi}{3}t = \dfrac{\pi}{3} + 2k\pi$ or $\dfrac{\pi}{3}t = \pi - \dfrac{\pi}{3} + 2k\pi = \dfrac{2\pi}{3} + 2k\pi$

General solutions: $t = 1 + 6k$ or $t = 2 + 6k$, $k \in \mathbb{Z}$

A1

In $0 \leq t \leq 12$: $t = 1, 2, 7, 8$ **A1**

Note: Award A1 for two correct values and second A1 for all four. Accept GDC intersection method.

Q17 — Geometric Sequence with Complex Ratio [MEDIUM

] **(a)** $|r| = \dfrac{1}{2}$; arg$(r) = \dfrac{\pi}{4}$ **A1**

(b) $u_3 = u_1 \cdot r^2 = 8 \cdot \left(\dfrac{1}{2}e^{i\pi/4}\right)^2 = 8 \cdot \dfrac{1}{4}e^{i\pi/2}$ **M1**

$= 2\,e^{i\pi/2}$ **A1** (AG)

(c) $u_3 = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2(0 + i) = 0 + 2i$ **A1**

Q18 — Phasor Addition of Sound Waves [MEDIUM

] **(a)** $P_1 = -10i\,e^{i\omega t}$... Using imaginary part convention:

Phasors: $P_1 = 10\,e^{i\cdot 0} = 10$ (at phase 0); $P_2 = 10\,e^{i\pi/2} = 10i$ **A1**

(b) $P_T = P_1 + P_2 = 10 + 10i$ **M1**

$R = |P_T| = \sqrt{100 + 100} = 10\sqrt{2}$ **A1**

$\phi = \arg(P_T) = \arctan\left(\dfrac{10}{10}\right) = \dfrac{\pi}{4}$ **A1**

$p_T(t) = 10\sqrt{2}\sin\left(\omega t + \dfrac{\pi}{4}\right)$

Note: Award A1 for $R = 10\sqrt{2} \approx 14.1$ and A1 for $\phi = \pi/4 = 0.785$.

Q19 — Roots in Polar Form [MEDIUM

] **(a)** $z = \dfrac{6 \pm \sqrt{36-52}}{2} = \dfrac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$ **M1 A1**

(b) $\alpha = 3+2i$: $r = |3+2i| = \sqrt{9+4} = \sqrt{13}$

A1 (integer $r = \sqrt{13}$; note: question says "r as integer" but $\sqrt{13}$ is exact — question is corrected: exact r)

$\theta = \arctan\left(\dfrac{2}{3}\right) = 0.588$ rad (3 s.f.) **A1**

$\alpha = \sqrt{13}\, e^{0.588i}$

Note: "r as an integer" means state the exact value; $\sqrt{13}$ is the exact form. Accept $\theta = 33.7°$ if degree mode. FT from (a).

Q20 — Argand Geometry: Isosceles Triangle [MEDIUM

] **(a)** $z_2 - z_1 = (1+2i) - 4 = -3+2i; \quad z_3 - z_2 = (-2+2i) - (1+2i) = -3$ **A1**

(b) $|AB| = |-3+2i| = \sqrt{9+4} = \sqrt{13}; \quad |BC| = |-3| = 3$ **A1**

(c) $|AC| = |z_3 - z_1| = |(-2+2i) - 4| = |-6+2i| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

M1

$|AB| = \sqrt{13} \neq |BC| = 3 \neq |AC| = 2\sqrt{10}$; no two sides are equal, so ABC is **not** isosceles. **A1**

Note: Award M1 for computing $|AC|$ or comparing any two correct side lengths; A1 for correct conclusion with justification.

Section C — Hard HARD

Q21 — Spiral Sequence Convergence [HARD

] **(a)** Let $u = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$:

$r = |u| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1; \quad \theta = \arctan\left(\dfrac{1/2}{\sqrt{3}/2}\right) = \arctan\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

$u = e^{i\pi/6}$ **A1 A1**

(b) $z_n = e^{in\pi/6}$, so $|z_n| = 1$ and $\arg(z_n) = \dfrac{n\pi}{6}$ **A1**

(c) $S = \displaystyle\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} 1$, which is a series of constant positive terms. **M1**

The series diverges (does not converge) since $|z_n| = 1 \not\to 0$. **R1**

Note: The key reasoning mark (R1) requires explicitly stating that $|z_n| = 1$ does not tend to zero, or that the common ratio of the moduli equals 1.

Q22 — Phasor Sum with Different Phases [HARD

] **(a)** Phasors (removing $e^{4\pi it}$ factor):

$Z_1 = 6\,e^{i\pi/6}; \quad Z_2 = 8\,e^{i\,2\pi/3}$ **M1**

$Z_1 = 6(\cos 30° + i\sin 30°) = 6(0.8660 + 0.5i) = 5.196 + 3i$

$Z_2 = 8(\cos 120° + i\sin 120°) = 8(-0.5 + 0.8660i) = -4 + 6.928i$

$Z_T = (5.196 - 4) + (3 + 6.928)i = 1.196 + 9.928i$ **A1**

$R = |Z_T| = \sqrt{1.196^2 + 9.928^2} = \sqrt{1.430 + 98.56} = \sqrt{99.99} \approx 10.0$ **A1**

$\Phi = \arctan\left(\dfrac{9.928}{1.196}\right) = \arctan(8.301) = 1.45$ rad (3 s.f.)

$I_T = 10.0\cos(4\pi t + 1.45)$

(b) $I_T = 0$: $\cos(4\pi t + 1.45) = 0 \Rightarrow 4\pi t + 1.45 = \dfrac{\pi}{2}$

M1

$4\pi t = 1.5708 - 1.45 = 0.1208 \Rightarrow t = \dfrac{0.1208}{4\pi} = 0.00961\ldots \approx 9.61 \times 10^{-3}$ s **A1**

Note: Accept $t = 0.00961$ s (3 s.f.). Exact $R = \sqrt{100} = 10$. FT on Φ.

Q23 — Unknown Parameter from Argument Condition [HARD

] **(a)** $|z|^2 = k^2 + (2)^2 = k^2 + 4$ **A1** (AG)

(b) $\arg(z) = \dfrac{\pi}{4}$: $\tan\left(\dfrac{\pi}{4}\right) = \dfrac{2}{k}$, so $1 = \dfrac{2}{k}$ **M1**

$k = 2$ **A1**

(c) $z = 2 + 2i = 2\sqrt{2}\,e^{i\pi/4}$

$z^2 = (2+2i)^2 = 4 + 8i + 4i^2 = 4 + 8i - 4 = 8i$ **M1**

$z^2 = 8\,e^{i\pi/2}$ **A1**

Note: In (c) accept Cartesian $z^2 = 0 + 8i$ and correct conversion to exponential.

Q24 — Tidal Model via Phasor Addition [HARD

] **(a)** Write phasors at frequency $\pi/6$:

$P_1 = 2.4\,e^{i\cdot 0} = 2.4; \quad P_2 = 1.6\,e^{i\pi/3} = 1.6\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right) = 0.8 + 1.3856i$ **M1**

$P_T = (2.4 + 0.8) + 1.3856i = 3.2 + 1.3856i$ **A1**

$A = |P_T| = \sqrt{3.2^2 + 1.3856^2} = \sqrt{10.24 + 1.920} = \sqrt{12.16} = 3.49$ (3 s.f.)

$\phi = \arctan\left(\dfrac{1.3856}{3.2}\right) = \arctan(0.4330) = 0.409$ rad (3 s.f.) **A1**

$h(t) = 3.49\sin\left(\frac{\pi}{6}t + 0.409\right)$

(b) Maximum at $\frac{\pi}{6}t + 0.409 = \frac{\pi}{2}$: **M1**

$\frac{\pi}{6}t = 1.5708 - 0.409 = 1.162$

$t = \frac{6 \times 1.162}{\pi} = \frac{6.972}{3.1416} = 2.219\ldots$ hours $= 2$ h 13 min **A1**

Note: Time is 02:13 (2 hours 13 minutes after midnight). FT from ϕ in (a).

Q25 — Geometric Series with Complex Ratio: Sum to Infinity [HARD]

(a) $r = \frac{1}{\sqrt{2}}e^{i\pi/4}$, so $|r| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707 < 1$ **A1**

Since $|r| < 1$, the geometric series converges and S_∞ exists. **R1**

(b) $S_\infty = \frac{u_1}{1-r} = \dfrac{4}{1 - \frac{1}{\sqrt{2}}e^{i\pi/4}}$ **M1**

$\frac{1}{\sqrt{2}}e^{i\pi/4} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{1}{2} + \frac{1}{2}i$

$1 - r = \frac{1}{2} - \frac{1}{2}i$ **A1**

$S_\infty = \dfrac{4}{\frac{1}{2} - \frac{1}{2}i} = \dfrac{4 \times (\frac{1}{2} + \frac{1}{2}i)}{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \dfrac{4(\frac{1}{2} + \frac{1}{2}i)}{\frac{1}{2}} = 8(\frac{1}{2} + \frac{1}{2}i) = 4 + 4i$ **A1**

Note: Exact answer $S_\infty = 4 + 4i$. M1 for applying $S_\infty = u_1/(1-r)$.

Q26 — Transformation Matrix for Complex Multiplication [HARD

] **(a)** $w = -1 + \sqrt{3}\,i$:

$r = \sqrt{1+3} = 2; \quad \theta = \pi - \text{arctan}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ **A1 A1**

$w = 2\,e^{i\,2\pi/3}$

(b) T is an enlargement by scale factor 2 and a rotation of $\frac{2\pi}{3}$ anticlockwise about the origin. **A1**

(c) $wz = (-1 + \sqrt{3}i)(x + yi) = (-x - \sqrt{3}y) + (\sqrt{3}x - y)i$ **M1**

$M = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ **A1**

Note: Verify: $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x - \sqrt{3}y \\ \sqrt{3}x - y \end{pmatrix}$ *OK. This equals* $2\begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$.

Q27 — Parameter from Argument of Root [HARD

] **(a)** Discriminant $= (2p)^2 - 4(p^2 + 9) = 4p^2 - 4p^2 - 36 = -36 < 0$ for all $p \in \mathbb{R}$. **M1**

Roots always non-real. **A1**

$\alpha = \dfrac{2p + \sqrt{-36}}{2} = p + 3i$ (taking positive imaginary part) **A1**

(b) arg$(\alpha) = \dfrac{\pi}{3}$: tan$\left(\dfrac{\pi}{3}\right) = \dfrac{3}{p}$, so $\sqrt{3} = \dfrac{3}{p}$ **M1**

$p = \dfrac{3}{\sqrt{3}} = \sqrt{3}$ **A1**

(c) $\alpha = \sqrt{3} + 3i$: $\quad r = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$; $\quad \theta = \dfrac{\pi}{3}$

$\alpha = 2\sqrt{3}\, e^{i\pi/3}$ **A1**

Q28 — Sinusoidal Model with Unknown Radius [HARD

] **(a)** Period $= \dfrac{2\pi}{\pi/4} = 8$ seconds **A1**

(b) Maximum of cos when argument $= 0$:

$\dfrac{\pi}{4}t_0 - \dfrac{\pi}{3} = 0 \Rightarrow t_0 = \dfrac{4}{3}$ **M1 A1** (AG)

(c) At $t = 2$:

$x(2) = r\cos\left(\dfrac{\pi}{4}(2) - \dfrac{\pi}{3}\right) = r\cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{3}\right) = r\cos\left(\dfrac{\pi}{6}\right) = r \cdot \dfrac{\sqrt{3}}{2}$ **M1**

$1.50 = r \cdot \dfrac{\sqrt{3}}{2} \Rightarrow r = \dfrac{1.50 \times 2}{\sqrt{3}} = \dfrac{3}{\sqrt{3}} = \sqrt{3} \approx 1.73$ m **A1**

Q29 — Locus and Argument Comparison [HARD

] **(a)** $|z - (3 + 2i)| = 3$: substituting $z = x + yi$:

$(x-3)^2 + (y-2)^2 = 9$ **M1**

This is a circle with centre $(3, 2)$ and radius 3. **A1**

(b) Im$(z) = 2$, i.e. $y = 2$: $(x-3)^2 + 0 = 9 \Rightarrow x - 3 = \pm 3$

$x = 6$ or $x = 0$ **M1**

$z = 6 + 2i$ or $z = 2i$ **A1**

(c) arg$(6 + 2i) =$ arctan$\left(\dfrac{2}{6}\right) \approx 0.322$ rad;

arg$(2i) = \dfrac{\pi}{2} \approx 1.571$ rad

$z = 2i$ has the greater argument. **R1**

Note: R1 for clear comparison of arguments with both values stated.

Q30 — Trig Sum via Complex Geometric Series [HARD

] **(a)** $\text{Re}(e^{in\theta}) = \cos(n\theta)$, so $\text{Re}(e^{i\theta} + e^{2i\theta} + e^{3i\theta} + e^{4i\theta}) = \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = S$ **A1** (AG)

(b) The sum $e^{i\theta} + e^{2i\theta} + e^{3i\theta} + e^{4i\theta}$ is a geometric series with first term $e^{i\theta}$ and common ratio $e^{i\theta}$, with 4 terms. **M1**

Sum $= e^{i\theta} \cdot \dfrac{1 - e^{4i\theta}}{1 - e^{i\theta}} = \dfrac{e^{i\theta}(1 - e^{4i\theta})}{1 - e^{i\theta}}$ **A1** (AG)

(c) $\theta = \dfrac{\pi}{3}$:

$e^{i\theta} = e^{i\pi/3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i; \quad e^{4i\theta} = e^{4i\pi/3}$

$e^{4i\pi/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

$1 - e^{4i\pi/3} = \dfrac{3}{2} + \dfrac{\sqrt{3}}{2}i; \quad 1 - e^{i\pi/3} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ **M1**

Numerator: $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)\left(\dfrac{3}{2} + \dfrac{\sqrt{3}}{2}i\right) = \dfrac{3}{4} + \dfrac{\sqrt{3}}{4}i + \dfrac{3\sqrt{3}}{4}i + \dfrac{3}{4}i^2 = \left(\dfrac{3}{4} - \dfrac{3}{4}\right) + \dfrac{4\sqrt{3}}{4}i = \sqrt{3}\,i$

Denominator: $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

$S = \text{Re}\left(\dfrac{\sqrt{3}\,i}{\frac{1}{2} - \frac{\sqrt{3}}{2}i}\right) = \text{Re}\left(\dfrac{\sqrt{3}\,i(\frac{1}{2} + \frac{\sqrt{3}}{2}i)}{|\frac{1}{2} - \frac{\sqrt{3}}{2}i|^2}\right) = \text{Re}\left(\dfrac{\frac{\sqrt{3}}{2}i + \frac{3}{2}i^2}{1}\right) = \text{Re}\left(-\dfrac{3}{2} + \dfrac{\sqrt{3}}{2}i\right) = -\dfrac{3}{2}$ **A1**

Note: Verification: $\cos 60° + \cos 120° + \cos 180° + \cos 240° = \frac{1}{2} - \frac{1}{2} - 1 - \frac{1}{2} = -\frac{3}{2}$ *OK*

Common Mistakes to Avoid

1. **Argument range.** Always reduce θ to $(-\pi, \pi]$. For example, $\arg(-3 + 3i) = \dfrac{3\pi}{4}$, *not* $-\dfrac{5\pi}{4}$. If your angle is outside range, add or subtract 2π.
2. **Quadrant for argument.** $\arg(z)$ depends on the quadrant of z. Never just take $\arctan(b/a)$: use the signs of a and b to place z in the correct quadrant first.
3. **Multiplication in exponential form.** $z_1 z_2 = r_1 r_2\, e^{i(\theta_1 + \theta_2)}$: *multiply* moduli, *add* arguments. A common error is to add moduli instead.
4. **Phasor phase.** When adding phasors, convert to Cartesian first, add, then convert back. Do *not* add

amplitudes and phases separately.

5. **Period formula.** Period $= \dfrac{2\pi}{B}$ in radians. If B includes π (e.g. $B = 3\pi$), the period is $\dfrac{2\pi}{3\pi} = \dfrac{2}{3}$, not $2\pi \cdot 3\pi$.

6. **Sum to infinity.** $S_\infty = \dfrac{u_1}{1-r}$ exists only when $|r| < 1$. For complex r, the condition is $|r| < 1$ (modulus of the ratio, not just the real part).