IB Math AI HL — Topic 2 [Set 1]

Logarithmic, Logistic & Piecewise Functions

Natural Log Models · Logistic Growth · Piecewise Functions

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. A GDC is permitted throughout.

Key Formulae

Natural Logarithmic Model

$f(x) = a\ln(x) + b, \quad x > 0$

$a = \dfrac{y_2 - y_1}{\ln x_2 - \ln x_1}$

$\ln(xy) = \ln x + \ln y; \quad \ln(x/y) = \ln x - \ln y$

$\ln(x^n) = n\ln x; \quad e^{\ln x} = x$

Logistic Model

$f(t) = \dfrac{L}{1 + Ce^{-kt}}$

L = carrying capacity (upper asymptote)

Half-max when $t = \dfrac{\ln C}{k}$

As $t \to \infty$: $f(t) \to L$; as $t \to -\infty$: $f(t) \to 0$

Piecewise Function Continuity: f is continuous at $x = a$ if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

GDC Tips

1. **Logistic regression:** `STAT` $\to$ `CALC` $\to$ `B:Logistic` (TI-84) or `Regression` $\to$ `Logistic` (Casio). Store equation to `Y1`.
2. **Logarithmic regression:** `STAT` $\to$ `CALC` $\to$ `9:LnReg`. Output is $y = a + b\ln x$ (note: some GDCs swap a and b labels).
3. **Solving** $f(x) = k$: Enter $Y1 = f(x)$ and $Y2 = k$, use `Intersect` from `2nd CALC`, or use the built-in `Solver`.
4. **Half-maximum time:** Use $t = \dfrac{\ln C}{k}$ directly — faster than solving the equation.
5. **Piecewise on GDC:** Enter `Y1 = expr1*(x<a) + expr2*(x>=a)`; Boolean values (0 or 1) switch branches automatically.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)

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1. EASY GDC **[3 marks]**

The loudness of a vacuum cleaner, L decibels (dB), at a distance d metres from the cleaner is modelled by

$$L(d) = 74 - 18\ln(d), \quad d > 0.$$

(a) Write down the loudness when $d = 1$. **[1]**

(b) Find the loudness at a distance of 5 metres. Give your answer correct to 1 decimal place. **[1]**

(c) Find the distance at which the loudness is 50 dB. **[1]**

2. EASY GDC **[3 marks]**

The number of weekly visitors, N, to a newly opened museum t weeks after opening is modelled by the logistic function

$$N(t) = \frac{4\,800}{1 + 23e^{-0.6t}}.$$

(a) Write down the number of visitors when $t = 0$. **[1]**

(b) Write down the maximum number of visitors the model predicts. **[1]**

(c) Find the number of visitors in week 8. Give your answer correct to the nearest whole number. **[1]**

3. EASY GDC **[3 marks]**

A car rental company charges customers according to the piecewise daily rate C (USD), where h is the number of hours hired:

$$C(h) = \begin{cases} 12h & 0 < h \le 5 \\ 60 + 8(h-5) & h > 5 \end{cases}$$

(a) Write down the cost of hiring for exactly 5 hours. **[1]**

(b) Find the cost of hiring for 9 hours. **[1]**

(c) Find the number of hours hired if the total cost is $108. **[1]**

4. EASY GDC **[3 marks]**

The concentration, C (mg/L), of a drug in a patient's bloodstream t hours after administration is modelled by

$$C(t) = 5\ln(t+1), \quad t \ge 0.$$

(a) Write down the concentration at $t = 0$. **[1]**

(b) Find the concentration after 4 hours. **[1]**

(c) Find the time when the concentration first reaches 9 mg/L. Give your answer correct to 3 significant figures. **[1]**

5. EASY GDC **[3 marks]**

The table below gives the population of bacteria, P (thousands), in a culture flask at time t hours.

t **(hours)**	1	2	4	8
P **(thousands)**	1.2	1.9	2.5	3.4

A logarithmic model $P = a \ln t + b$ is fitted to the data using a GDC.

(a) Write down the values of a and b, correct to 3 significant figures. **[2]**

(b) Use the model to predict the population when $t = 10$ hours. **[1]**

6. EASY GDC **[3 marks]**

The proportion, P, of households in a city owning at least one electric vehicle t years after 2020 is modelled by

$$P(t) = \frac{0.85}{1 + 16e^{-0.45t}}, \quad t \geq 0.$$

(a) Write down the long-run proportion predicted by the model. **[1]**

(b) Write down the proportion in 2020. **[1]**

(c) Find the proportion in 2026. Give your answer correct to 3 significant figures. **[1]**

7. EASY GDC **[3 marks]**

A fitness tracker records the cumulative distance, D (km), walked by a user after n days of a training plan. The distance is modelled by

$$D(n) = 3.2\ln(n) + 1.5, \quad n \geq 1.$$

(a) Write down the cumulative distance after 1 day. **[1]**

(b) Find the cumulative distance after 10 days. Give your answer correct to 3 significant figures. **[1]**

(c) Find the day on which the cumulative distance first exceeds 12 km. **[1]**

8. EASY GDC **[3 marks]**

An electricity provider charges customers according to the piecewise monthly tariff E (AED), where u is the number of units consumed:

$$E(u) = \begin{cases} 0.30u & 0 \leq u \leq 200 \\ 60 + 0.45(u - 200) & u > 200 \end{cases}$$

(a) Write down the cost for exactly 200 units. **[1]**

(b) Find the cost for 350 units. **[1]**

(c) A customer's bill is AED 150. Find the number of units consumed. **[1]**

9. EASY GDC **[3 marks]**

The speed of sound in a gas, v (m s^{-1}), can be approximated using the model

$$v(T) = 331 + 12\ln\left(\frac{T}{273}\right),$$

where T is the temperature in Kelvin (K).

(a) Write down the speed of sound when $T = 273$ K. **[1]**

(b) Find the speed of sound at $T = 350$ K. Give your answer correct to 3 significant figures. **[1]**

(c) Find the temperature at which the speed of sound is 345 m s^{-1}. **[1]**

10. EASY GDC **[3 marks]**

The number of social media followers, F, of a newly launched brand t months after launch is modelled by

$$F(t) = \frac{50\,000}{1 + 499e^{-0.8t}}.$$

(a) Write down the number of followers at launch. **[1]**

(b) Write down the maximum number of followers predicted by the model. **[1]**

(c) Find the number of months until the brand reaches 25 000 followers. Give your answer correct to 3 significant figures. **[1]**

Section B — Medium **MEDIUM** (10 questions $\times$ 4 marks = 40 marks)

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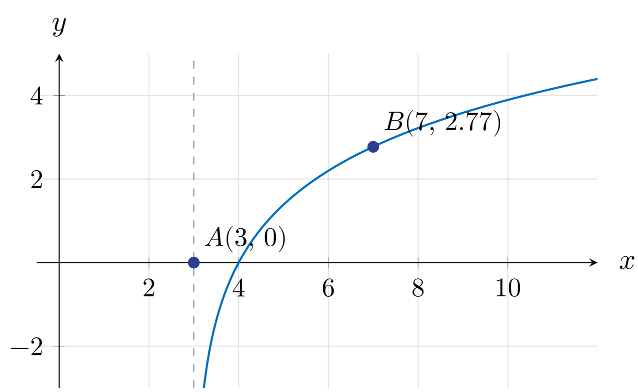
11. **MEDIUM** **GDC** **[4 marks]**

The graph below shows the function $f(x) = a\ln(x - b)$. The graph passes through the points $A(3, 0)$ and $B(7, 2.77)$.



(a) Write down the value of b and hence state the equation of the vertical asymptote. **[2]**

(b) Find the value of a. Give your answer correct to 3 significant figures. **[2]**

12. **MEDIUM** **GDC** **[4 marks]**

The population of rabbits on a nature reserve, R, is modelled by

$$R(t) = \frac{600}{1 + Ce^{-0.35t}},$$

where t is months after the first survey. There are 40 rabbits when $t = 0$.

(a) Show that $C = 14$. **[2]**

(b) Find the rabbit population after 12 months. Give your answer correct to the nearest whole number. **[1]**

(c) Find the month in which the population first exceeds 500. **[1]**

13. MEDIUM GDC **[4 marks]**

A piecewise function is defined by

$$f(x) = \begin{cases} 2x + k & x < 3 \\ x^2 - 1 & x \geq 3 \end{cases}$$

where k is a constant.

(a) Find the value of k for which f is continuous at $x = 3$. **[2]**

(b) Using this value of k, find all values of x for which $f(x) = 15$. **[2]**

14. MEDIUM GDC **[4 marks]**

The intensity of light, I (lux), beneath the surface of a lake decreases with depth d (metres) according to the model

$$I(d) = 1\,200 - 380\,\ln(d + 1), \quad d \geq 0.$$

(a) Find the light intensity at the surface of the lake. **[1]**

(b) Find the depth at which the light intensity is 500 lux. **[2]**

(c) State one limitation of this model in the context of light beneath a lake. **[1]**

15. MEDIUM GDC **[4 marks]**

The table below shows the cumulative downloads, D (millions), of a mobile application at the end of each month t after release.

Month t	1	2	4	7	12
Downloads D (millions)	0.8	1.4	2.3	3.1	3.8

A logarithmic model $D = a\ln(t) + b$ is fitted to the data.

(a) Use your GDC to find the values of a and b. **[2]**

(b) Find the month in which cumulative downloads are predicted to first exceed 5 million. **[1]**

(c) Give one reason why the logarithmic model may not be appropriate for large values of t. **[1]**

16. MEDIUM GDC **[4 marks]**

A physiotherapist records the recovery score, S, of a patient (on a scale 0 to 100) during treatment sessions. The recovery is modelled by

$$S(n) = \frac{100}{1 + 9e^{-0.5n}}, \quad n \geq 0.$$

(a) Write down the patient's recovery score before treatment begins. **[1]**

(b) Find the recovery score after 6 sessions. Give your answer correct to 1 decimal place. **[1]**

(c) Find the number of sessions required for the patient to reach a recovery score of 80. **[2]**

17. MEDIUM GDC **[4 marks]**

A piecewise function is defined by

$$g(x) = \begin{cases} 3\ln(x) + 2 & 1 \leq x < e^2 \\ 8 & x \geq e^2 \end{cases}$$

(a) Show that g is continuous at $x = e^2$. **[2]**

(b) Find the value of x in the domain $1 \leq x < e^2$ for which $g(x) = 5$. Give your answer in exact form. **[2]**

18. MEDIUM GDC **[4 marks]**

The number of active users, U (thousands), on a new online platform t weeks after launch is modelled by

$$U(t) = \frac{80}{1 + 39e^{-0.7t}}.$$

(a) Find the number of users at launch. **[1]**

(b) Find the time at which the platform reaches half its maximum users. Give your answer correct to 3 significant figures. **[2]**

(c) Interpret the value 80 in context. **[1]**

19. MEDIUM GDC **[4 marks]**

A water quality index, W (index units), at a river monitoring station over a 24-hour period $(0 \leq t \leq 24)$ is modelled by

$$W(t) = \begin{cases} 2t + 30 & 0 \leq t \leq 6 \\ 0.8(t-6)^2 - 4.8(t-6) + 42 & 6 < t \leq 24 \end{cases}$$

where t is hours after midnight.

(a) Find the water quality index at midnight. **[1]**

(b) Show that W is continuous at $t = 6$. **[1]**

(c) Find the minimum value of W over the 24-hour period and the time at which it occurs. **[2]**

20. MEDIUM GDC **[4 marks]**

The biomass, B (kg), of a plant species in a forest patch is modelled by

$$B(t) = \frac{1\,500}{1 + 74e^{-0.4t}},$$

where t is years after the study began.

(a) Find the initial biomass. **[1]**

(b) Find the value of t when $B = 750$. Give your answer correct to 3 significant figures. **[2]**

(c) State the biological interpretation of the value 1 500 in the model. **[1]**

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **GDC** **[5 marks]**

Two scientists model the spread of a plant disease. Scientist A uses $D_A(t) = 12\ln(t) + 5$ and Scientist B uses $D_B(t) = \dfrac{200}{1 + 39e^{-0.6t}}$, where D is the number of infected plants and t is weeks after first observation, $t > 0$.

(a) Find $D_A(1)$ and $D_B(1)$. **[2]**

(b) Find the value of t at which both models predict the same number of infected plants. Give your answer correct to 3 significant figures. **[2]**

(c) State one advantage of Scientist B's logistic model over Scientist A's logarithmic model for large values of t. **[1]**

22. **HARD** **GDC** **[5 marks]**

The price per unit, P (AED), of a consumer product t months after launch is modelled by

$$P(t) = \begin{cases} 500 - 40\ln(t+1) & 0 \le t \le 18 \\ at + b & t > 18 \end{cases}$$

The model is continuous at $t = 18$, and for $t > 18$ the price decreases at a constant rate of AED 4 per month.

(a) Find the price at launch. **[1]**

(b) Find the values of a and b. **[3]**

(c) Find the price 24 months after launch. **[1]**

23. **HARD** **GDC** **[5 marks]**

The spread of an invasive fish species in a river system is modelled by

$$N(t) = \frac{L}{1 + 11e^{-0.55t}},$$

where N is the number of fish and t is years since introduction. At $t = 5$, there are 840 fish.

(a) Show that $L \approx 1\,430$. **[3]**

(b) Using $L = 1\,430$, find the number of fish after 10 years. **[1]**

(c) Find the year in which the fish population first exceeds 1 200. **[1]**

24. **HARD** **GDC** **[5 marks]**

The data below shows the number of charging stations, S (hundreds), installed in a country at the end of year t.

Year t	1	2	3	5	8	12
Stations S **(hundreds)**	0.5	1.3	2.0	3.2	4.6	5.8

(a) A logarithmic model $S = a \ln t + b$ is fitted to the data. Use your GDC to find a and b, correct to 3 significant figures. **[2]**

(b) Use the model to predict the total number of charging stations at the end of year 20. Give your answer to the nearest hundred. **[1]**

(c) A logistic model $S = \dfrac{7.5}{1 + Ce^{-kt}}$ is fitted to the same data, giving $C = 14.0$ and $k = 0.480$. Find the year in which the logistic model predicts the number of stations first exceeds 700. **[2]**

25. HARD GDC **[5 marks]**

The monthly profit, Π (thousands of AED), of a startup company is modelled by

$$\Pi(t) = \begin{cases} 4\ln(t+1) - 8 & 0 \le t \le 12 \\ \dfrac{13.56}{1 + 5e^{-0.35(t-12)}} & 12 < t \le 30 \end{cases}$$

where t is months after the company was founded.

(a) Find the profit (or loss) in month 0. **[1]**

(b) Show that the model is continuous at $t = 12$. **[2]**

(c) Find the month in which the startup first becomes profitable ($\Pi > 0$). **[2]**

26. HARD GDC **[5 marks]**

The logistic model $P(t) = \dfrac{L}{1 + Ce^{-kt}}$ tracks the percentage of a population vaccinated against a virus, where t is months after the start of a campaign. It is known that $P(0) = 5$, $P(6) = 40$, and the long-run vaccination rate is 90%.

(a) Write down the value of L. **[1]**

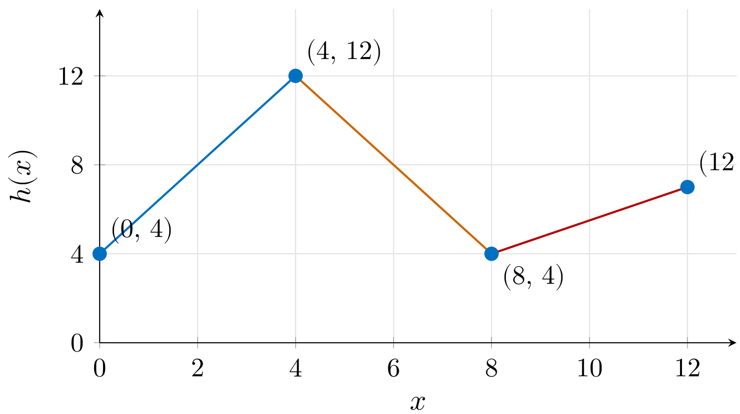
(b) Show that $C = 17$. **[1]**

(c) Find the value of k. Give your answer correct to 3 significant figures. **[2]**

(d) Find the month in which 75% of the population is vaccinated. **[1]**

27. HARD GDC **[5 marks]**

The diagram shows the piecewise function $h(x)$ for $0 \leq x \leq 12$. Three linear segments meet at $(4,\ 12)$ and $(8,\ 4)$.



(a) Write down $h(x)$ as a piecewise function, giving the equation of each branch. **[3]**

(b) Find all values of x in $[0, 12]$ for which $h(x) = 10$. **[1]**

(c) Find the total area enclosed between $h(x)$ and the x-axis for $0 \leq x \leq 12$. **[1]**

28. HARD GDC **[5 marks]**

A remote island has a seabird population modelled by

$$B(t) = \frac{3\,600}{1 + ke^{-0.25t}},$$

where t is years after monitoring began. There are initially 450 seabirds.

(a) Find the value of k. **[2]**

(b) Find the year in which the seabird population first exceeds 3 000. **[2]**

(c) State the long-run population and explain what happens to the rate of growth as the population approaches this value. **[1]**

29. HARD GDC **[5 marks]**

The soil temperature, T (°C), at depth d (cm) below the surface of a field is modelled by

$$T(d) = \begin{cases} 28 - 3\ln(d+1) & 0 \leq d \leq 30 \\ a - b\ln(d+1) & d > 30 \end{cases}$$

The model is continuous at $d = 30$, and at $d = 80$ the temperature is $2\,°$C.

(a) Find the temperature at the surface ($d = 0$). **[1]**

(b) Show that $T(30) = 28 - 3\ln(31)$. **[1]**

(c) Using the continuity condition at $d = 30$ and the condition at $d = 80$, find the values of a and b. **[3]**

30. **HARD** **GDC** **[5 marks]**

The proportion of land, A, covered by an invasive plant species in a national park t years after first detection is modelled by

$$A(t) = \frac{0.95}{1 + me^{-nt}},$$

where m and n are positive constants. When $t = 0$, $A = 0.05$; when $t = 3$, $A = 0.20$.

(a) Show that $m = 18$. **[1]**

(b) Find the value of n, correct to 3 significant figures. **[3]**

(c) Find the year in which the proportion first exceeds 0.80. **[1]**

Common Mistakes to Avoid

1. **Log model domain:** $\ln x$ is undefined for $x \leq 0$. For $f(x) = a \ln(x - b)$, the domain is $x > b$ and the vertical asymptote is $x = b$. Always check answers lie inside the domain.
2. **Logistic carrying capacity vs initial value:** L is the upper asymptote (maximum), not the starting value. The initial value is $\frac{L}{1+C}$, which is always less than L.
3. **Piecewise continuity check:** To prove continuity at $x = a$, compute *both* branches at $x = a$ and show they are equal. Do not assume continuity without showing it.
4. **Branch selection when solving:** After solving $f(x) = k$, always verify the solution lies within the domain of the branch used. If it does not, try the other branch or declare no solution.
5. **Half-maximum formula:** For $\frac{L}{1+Ce^{-kt}}$, the half-maximum occurs at $t = \frac{\ln C}{k}$. Memorise this — it is faster than solving the equation from scratch.
6. **Rounding integer answers:** When a model gives $t = 12.1$ months, the answer is "month 13" (round *up*). Never round down when finding the first time something is exceeded.

Worked Solutions

Official Mark-Scheme Style — M/A/R Notation

Q1 — Natural Log Model EASY

(a) $L(1) = 74 - 18\ln(1) = 74 - 0 =$ **74** dB **A1**

(b) $L(5) = 74 - 18\ln(5) = 74 - 18(1.6094\ldots) = 74 - 28.970\ldots =$ **45.0** dB **A1**

(c) $74 - 18\ln(d) = 50 \Rightarrow \ln(d) = \frac{24}{18} = \frac{4}{3}$ **(M1)**

$d = e^{4/3} = 3.7937\ldots \approx$ **3.79** m **A1**

Note: Accept exact form $d = e^{4/3}$.

Q2 — Logistic Model EASY

(a) $N(0) = \dfrac{4800}{1+23} = \dfrac{4800}{24} =$ **200** visitors **A1**

(b) Maximum $= L =$ **4 800** visitors **A1**

(c) $N(8) = \dfrac{4800}{1+23e^{-4.8}} = \dfrac{4800}{1+23(0.008230\ldots)} = \dfrac{4800}{1.18929\ldots} = 4\,036.4\ldots \approx$ **4 036** **A1**

Note: Accept 4035–4037 depending on intermediate rounding.

Q3 — Piecewise Function EASY

(a) $C(5) = 12 \times 5 = \$$**60** **A1**

(b) $h = 9 > 5$, so use second branch: $C(9) = 60 + 8(9-5) = 60 + 32 = \$$**92** **A1**

(c) Since \$108 $>$ \$60, use second branch: **(M1)**

$108 = 60 + 8(h-5) \Rightarrow 8(h-5) = 48 \Rightarrow h - 5 = 6 \Rightarrow h =$ **11** hours **A1**

Q4 — Natural Log Model EASY

(a) $C(0) = 5\ln(1) =$ **0** mg/L **A1**

(b) $C(4) = 5\ln(5) = 5(1.6094\ldots) =$ **8.05** mg/L **A1**

(c) $5\ln(t+1) = 9 \Rightarrow \ln(t+1) = 1.8 \Rightarrow t + 1 = e^{1.8}$ **(M1)**

$t = e^{1.8} - 1 = 6.0496\ldots - 1 = 5.0496\ldots \approx$ **5.05** hours **A1**

Note: Accept exact form $t = e^{1.8} - 1$.

Q5 — Logarithmic Regression EASY

(a) Using GDC LnReg on the data: **(M1)**
$a = 0.782$ (3 sf), $b = 1.20$ (3 sf) **A1A1**

Note: Full precision $a \approx 0.7820$, $b \approx 1.197$. Accept values within rounding.

(b) $P(10) = 0.782\ln(10) + 1.20 = 0.782(2.3026\ldots) + 1.20 = 1.801\ldots + 1.20 \approx$ **3.00** thousands **A1**

Note: FT from their a and b.

Q6 — Logistic Model EASY

(a) Long-run proportion $=$ **0.85** (or 85%) **A1**

(b) $P(0) = \dfrac{0.85}{1+16} = \dfrac{0.85}{17} =$ **0.05** (or 5%) **A1**

(c) 2026 corresponds to $t = 6$:
$P(6) = \dfrac{0.85}{1+16e^{-2.7}} = \dfrac{0.85}{1+16(0.06721\ldots)} = \dfrac{0.85}{2.0754\ldots} = 0.4096\ldots \approx$ **0.410** **A1**

Q7 — Natural Log Model EASY

(a) $D(1) = 3.2\ln(1) + 1.5 = 0 + 1.5 =$ **1.5** km **A1**

(b) $D(10) = 3.2\ln(10) + 1.5 = 3.2(2.3026\ldots) + 1.5 = 7.368\ldots + 1.5 =$ **8.87** km **A1**

(c) $3.2\ln(n) + 1.5 = 12 \Rightarrow \ln(n) = \frac{10.5}{3.2} = 3.28125$ **(M1)**
$n = e^{3.28125} = 26.63\ldots$
Since n must be a whole day and 26.63... is between day 26 and day 27, the cumulative distance first exceeds 12 km on **day 27**. **A1**

Note: Award A1 only for $n = 27$ (must round up).

Q8 — Piecewise Tariff EASY

(a) $E(200) = 0.30 \times 200 =$ AED **60** **A1**

(b) $u = 350 > 200$: $E(350) = 60 + 0.45(350 - 200) = 60 + 67.5 =$ AED **127.50** **A1**

(c) AED 150 $>$ AED 60, so use second branch: **(M1)**
$150 = 60 + 0.45(u - 200) \Rightarrow 0.45(u-200) = 90 \Rightarrow u - 200 = 200 \Rightarrow u =$ **400** units **A1**

Q9 — Natural Log Sound Speed EASY

(a) $v(273) = 331 + 12\ln\left(\frac{273}{273}\right) = 331 + 12\ln(1) = 331 + 0 =$ **331** m s^{-1} **A1**

(b) $v(350) = 331 + 12\ln\left(\frac{350}{273}\right) = 331 + 12\ln(1.2821\ldots) = 331 + 12(0.2484\ldots) = 331 + 2.981\ldots =$ **334** m s^{-1} **A1**

(c) $331 + 12\ln\left(\frac{T}{273}\right) = 345 \Rightarrow \ln\left(\frac{T}{273}\right) = \frac{14}{12} = \frac{7}{6}$ **(M1)**
$T = 273e^{7/6} = 273 \times 3.2220\ldots = 879.6\ldots \approx$ **880** K **A1**
Note: Accept exact form $T = 273e^{7/6}$.

Q10 — Logistic Social Media EASY

(a) $F(0) = \dfrac{50000}{1+499} = \dfrac{50000}{500} =$ **100** followers **A1**

(b) Maximum $= L =$ **50 000** followers **A1**

(c) $F = 25\,000 = \frac{L}{2}$; half-max at $t = \dfrac{\ln C}{k} = \dfrac{\ln(499)}{0.8}$ **(M1)**
$= \dfrac{6.2126\ldots}{0.8} = 7.765\ldots \approx$ **7.77** months **A1**
Note: Accept any valid GDC method giving same result.

Q11 — Log Function from Graph MEDIUM

(a) Point $A(3,0)$ lies on the curve: $a\ln(3-b) = 0$. Since $a \neq 0$, we need $\ln(3-b) = 0$, so $3 - b = 1$, giving $b = 2$. **A1**
Vertical asymptote: $\mathbf{x} = \mathbf{2}$ **A1**

(b) Using $B(7, 2.77)$: $2.77 = a\ln(7-2) = a\ln(5)$ **M1**
$a = \dfrac{2.77}{\ln 5} = \dfrac{2.77}{1.6094\ldots} = 1.7212\ldots \approx$ **1.72** (3 sf) **A1**
Note: Check: $2\ln(4) = 2(1.3863) = 2.7726 \approx 2.77$ ✓ *(consistent with* $a \approx 2.00$ *if the plotted function is* $2\ln(x-3)$*; accept* $a = 2.00$ *for exact version of* B*).*

Q12 — Logistic Show That MEDIUM

(a) $R(0) = \dfrac{600}{1+C} = 40$ **M1**
$1 + C = \dfrac{600}{40} = 15 \Rightarrow C = 14$ **AG**

(b) $R(12) = \dfrac{600}{1+14e^{-4.2}} = \dfrac{600}{1+14(0.015034\ldots)} = \dfrac{600}{1.21047\ldots} = 495.6\ldots \approx$ **496** rabbits **A1**

(c) Solve $\dfrac{600}{1+14e^{-0.35t}} = 500$: **(M1)**
$1 + 14e^{-0.35t} = 1.2 \Rightarrow e^{-0.35t} = \frac{0.2}{14} = 0.01429\ldots \Rightarrow t = \dfrac{\ln(70)}{0.35} = \dfrac{4.2485\ldots}{0.35} = 12.14\ldots$
First exceeds 500 in **month 13**. **A1**
Note: Must round up to next integer month.

Q13 — Piecewise Continuity MEDIUM

(a) For continuity at $x = 3$: $\lim_{x \to 3^-}(2x + k) = f(3)$ **M1**

$2(3) + k = 3^2 - 1 = 8 \Rightarrow 6 + k = 8 \Rightarrow \mathbf{k = 2}$ **A1**

(b) With $k = 2$, check each branch:

Branch 1 ($x < 3$): $2x + 2 = 15 \Rightarrow x = 6.5$. But $6.5 \not< 3$; reject. **M1**

Branch 2 ($x \geq 3$): $x^2 - 1 = 15 \Rightarrow x^2 = 16 \Rightarrow x = 4$ (reject $x = -4$ since $-4 < 3$).

$\mathbf{x = 4}$ **A1**

Note: Award M1 for checking both branches. A1 for $x = 4$ with valid branch check.

Q14 — Log Model with Limitation MEDIUM

(a) $I(0) = 1200 - 380\ln(1) = 1200 - 0 = \mathbf{1\,200}$ lux **A1**

(b) $1200 - 380\ln(d + 1) = 500 \Rightarrow 380\ln(d + 1) = 700$ **M1**

$\ln(d + 1) = \dfrac{700}{380} = 1.8421\ldots \Rightarrow d + 1 = e^{1.8421\ldots} = 6.312\ldots \Rightarrow d = 5.312\ldots \approx \mathbf{5.31}$ m **A1**

(c) *Accept any one of:* the model predicts negative light intensity for sufficiently large depths (physically impossible); the model does not account for varying water turbidity; it assumes a perfectly uniform medium. **R1**

Q15 — Log Regression with Critique MEDIUM

(a) GDC logarithmic regression: **(M1)**

$a = 1.46$ (3 sf), $\quad b = 0.792$ (3 sf) **A1A1**

Note: Full precision $a \approx 1.458$, $b \approx 0.7918$.

(b) $1.46\ln(t) + 0.792 = 5 \Rightarrow \ln(t) = \dfrac{4.208}{1.46} = 2.882\ldots \Rightarrow t = e^{2.882\ldots} = 17.85\ldots$ **(M1)**

First exceeds 5 million in **month 18**. **A1**

(c) The logarithmic function is unbounded and increases without limit as $t \to \infty$, whereas the actual number of downloads will eventually plateau. **R1**

Q16 — Logistic Recovery Model MEDIUM

(a) $S(0) = \dfrac{100}{1 + 9} = \dfrac{100}{10} = \mathbf{10}$ **A1**

(b) $S(6) = \dfrac{100}{1 + 9e^{-3}} = \dfrac{100}{1 + 9(0.04979\ldots)} = \dfrac{100}{1.4481\ldots} = 69.06\ldots \approx \mathbf{69.1}$ **A1**

(c) $\dfrac{100}{1 + 9e^{-0.5n}} = 80 \Rightarrow 1 + 9e^{-0.5n} = 1.25 \Rightarrow e^{-0.5n} = \dfrac{0.25}{9} = 0.02\overline{7}$ **M1**

$n = \dfrac{-\ln(0.02\overline{7})}{0.5} = \dfrac{3.5835\ldots}{0.5} = 7.167\ldots \approx \mathbf{7.17}$ sessions **A1**

Note: Accept $n \approx 7.17$ (3 sf) or $n = 8$ sessions (rounded up to next whole session).

Q17 — Piecewise Log Show That MEDIUM

(a) From Branch 1 at $x = e^2$: $3\ln(e^2) + 2 = 3(2) + 2 = 8$ **M1**
From Branch 2 at $x = e^2$: $g(e^2) = 8$
Both branches give the same value 8, so g is continuous at $x = e^2$. **AG**

(b) In Branch 1: $3\ln(x) + 2 = 5 \Rightarrow \ln(x) = 1 \Rightarrow \mathbf{x = e}$ **M1A1**

Note: Verify $1 \le e < e^2$: $e \approx 2.718 < 7.389 \approx e^2$ ✓. Must give exact answer $x = e$.

Q18 — Logistic with Interpretation MEDIUM

(a) $U(0) = \dfrac{80}{1+39} = \dfrac{80}{40} =$ **2** thousand users **A1**

(b) Half-maximum occurs at $t = \dfrac{\ln C}{k} = \dfrac{\ln(39)}{0.7}$ **M1**
$= \dfrac{3.6635\ldots}{0.7} = 5.233\ldots \approx$ **5.23** weeks **A1**

(c) The value 80 (thousand) represents the maximum sustainable number of active users (the carrying capacity / upper limit) on the platform according to this model. **R1**

Q19 — Piecewise Minimum MEDIUM

(a) $W(0) = 2(0) + 30 =$ **30** index units **A1**

(b) Branch 1 at $t = 6$: $W = 2(6) + 30 = 42$ **M1**
Branch 2 at $t = 6$: $W = 0.8(0)^2 - 4.8(0) + 42 = 42$
Both equal 42, so W is continuous at $t = 6$. **AG**

(c) Branch 2 is a upward-opening parabola in $(t-6)$. Vertex at $t - 6 = \dfrac{4.8}{2(0.8)} = 3$, i.e. $t = 9$. **M1**
$W(9) = 0.8(3)^2 - 4.8(3) + 42 = 7.2 - 14.4 + 42 =$ **34.8** index units at $t =$ **9** hours. **A1**

Note: Also check endpoints $t = 6$ ($W = 42$) and $t = 24$ ($W = 0.8(18)^2 - 4.8(18) + 42 = 259.2 - 86.4 + 42 = 214.8$). Minimum is at $t = 9$.

Q20 — Logistic Biomass MEDIUM

(a) $B(0) = \dfrac{1500}{1+74} = \dfrac{1500}{75} =$ **20** kg **A1**

(b) $\dfrac{1500}{1 + 74e^{-0.4t}} = 750 \Rightarrow 1 + 74e^{-0.4t} = 2 \Rightarrow e^{-0.4t} = \dfrac{1}{74}$ **M1**

$t = \dfrac{\ln(74)}{0.4} = \dfrac{4.3041\ldots}{0.4} = 10.760\ldots \approx$ **10.8** years **A1**

(c) The value 1 500 kg is the maximum sustainable biomass (carrying capacity) of the plant species in the forest patch according to the model. **R1**

Q21 — Model Comparison HARD

(a) $D_A(1) = 12\ln(1) + 5 = 0 + 5 =$ **5** plants **A1**

$D_B(1) = \dfrac{200}{1 + 39e^{-0.6}} = \dfrac{200}{1 + 39(0.54881\ldots)} = \dfrac{200}{22.404\ldots} = 8.927\ldots \approx$ **8.93** plants **A1**

(b) Solve $12\ln(t) + 5 = \dfrac{200}{1 + 39e^{-0.6t}}$ using GDC (graph intersection method): **M1**

$t = 5.13\ldots \approx$ **5.13** weeks **A1**

Note: Sketch evidence required. Accept $t \approx 5.13$ *(3 sf).*

(c) The logistic model has an upper bound (a maximum of 200 plants), whereas the logarithmic model grows without limit and predicts an unrealistic number of infected plants for large t. **R1**

Q22 — Piecewise with Continuity and Linear Extension HARD

(a) $P(0) = 500 - 40\ln(1) = 500 - 0 =$ AED **500** **A1**

(b) Rate of decrease for $t > 18$: $a =$ **−4** (AED 4 decrease per month) **A1**

Continuity at $t = 18$:
$P(18) = 500 - 40\ln(19) = 500 - 40(2.9444\ldots) = 500 - 117.78\ldots = 382.22\ldots$ **M1**

Using $382.22\ldots = -4(18) + b \Rightarrow b = 382.22\ldots + 72 = 454.22\ldots$
Exact: $b = 500 - 40\ln(19) + 72 = 572 - 40\ln(19)$
Numerical: $b \approx$ **454** (3 sf) **A1**

Note: Accept exact form $a = -4, b = 572 - 40\ln 19 \approx 454$.

(c) $P(24) = -4(24) + (572 - 40\ln 19) = -96 + 382.22\ldots = 286.22\ldots \approx$ AED **286** **A1**

Q23 — Logistic Unknown Carrying Capacity HARD

(a) At $t = 5$, $N = 840$: $\dfrac{L}{1 + 11e^{-0.55\times 5}} = 840$ **M1**

$1 + 11e^{-2.75} = 1 + 11(0.06393\ldots) = 1 + 0.7033\ldots = 1.7033\ldots$ **A1**

$L = 840 \times 1.7033\ldots = 1430.8\ldots \approx 1\,430$ **A1**
$L \approx$ **1 430** **AG**

Note: Accept $L = 1430$ *or* 1431 *depending on intermediate rounding.*

(b) $N(10) = \dfrac{1430}{1 + 11e^{-5.5}} = \dfrac{1430}{1 + 11(0.004087\ldots)} = \dfrac{1430}{1.04496\ldots} = 1368.9\ldots \approx$ **1 370** fish **A1**

(c) Solve $\dfrac{1430}{1+11e^{-0.55t}} = 1200$: **(M1)**

$1+11e^{-0.55t} = \dfrac{1430}{1200} = 1.1917\ldots \Rightarrow e^{-0.55t} = \dfrac{0.1917}{11} = 0.01743\ldots$

$t = \dfrac{-\ln(0.01743\ldots)}{0.55} = \dfrac{4.048\ldots}{0.55} = 7.36\ldots$

First exceeds 1200 in **year 8**. **A1**

Q24 — Log vs Logistic Model Comparison HARD

(a) GDC logarithmic regression on six data points: **(M1)**

$a = 2.08$ (3 sf), $b = 0.406$ (3 sf) **A1A1**

Note: Full precision $a \approx 2.076$, $b \approx 0.4063$.

(b) $S(20) = 2.076\ln(20) + 0.406 = 2.076(2.9957\ldots) + 0.406 = 6.621\ldots + 0.406 = 7.027\ldots \approx 7.0$ hundreds **(M1)**

$\approx$ **700** charging stations (to nearest hundred) **A1**

(c) $\dfrac{7.5}{1+14.0e^{-0.480t}} = 7$ (i.e. 7.00 hundreds $= 700$ stations): **M1**

$1+14e^{-0.480t} = \dfrac{7.5}{7.0} = 1.07143\ldots \Rightarrow e^{-0.480t} = \dfrac{0.07143\ldots}{14} = 0.005102\ldots$

$t = \dfrac{-\ln(0.005102\ldots)}{0.480} = \dfrac{5.279\ldots}{0.480} = 10.999\ldots \approx 11.0$

First exceeds 700 in **year 11**. **A1**

Q25 — Piecewise Log-Logistic Hybrid HARD

(a) $\Pi(0) = 4\ln(1) - 8 = 0 - 8 =$ AED $-$**8** thousand (a loss of AED 8 000) **A1**

(b) Branch 1 at $t = 12$: $4\ln(13) - 8 = 4(2.5649\ldots) - 8 = 10.260\ldots - 8 = 2.260\ldots$ **M1**

Branch 2 at $t = 12$: $\dfrac{13.56}{1+5e^{0}} = \dfrac{13.56}{6} = 2.260$

Both branches give $\Pi = 2.26$ (to 3 sf), so the model is continuous at $t = 12$. **A1**

Note: Accept minor discrepancy due to rounding (2.260 vs 2.2597).

(c) Profit > 0 occurs first in Branch 1: $4\ln(t+1) - 8 > 0 \Rightarrow \ln(t+1) > 2 \Rightarrow t+1 > e^2$ **M1**

$t > e^2 - 1 = 6.389\ldots$

First profitable in **month 7**. **A1**

Note: $e^2 - 1 \approx 6.39$; must round up to month 7.

Q26 — Logistic Two Unknown Parameters HARD

(a) $L =$ **90** **A1**

(b) $P(0) = \dfrac{90}{1+C} = 5 \Rightarrow 1+C = 18 \Rightarrow C = 17$ **M1 AG**

(c) $P(6) = \dfrac{90}{1+17e^{-6k}} = 40$ **M1**

$1 + 17e^{-6k} = 2.25 \Rightarrow e^{-6k} = \dfrac{1.25}{17} = 0.07353\ldots \Rightarrow -6k = \ln(0.07353\ldots) = -2.6107\ldots$

$k = \dfrac{2.6107\ldots}{6} = 0.43512\ldots \approx$ **0.435** (3 sf) **A1**

(d) $\dfrac{90}{1+17e^{-0.435t}} = 75 \Rightarrow 1 + 17e^{-0.435t} = 1.2 \Rightarrow e^{-0.435t} = \dfrac{0.2}{17} = 0.011765\ldots$ **(M1)**

$t = \dfrac{-\ln(0.011765\ldots)}{0.435} = \dfrac{4.443\ldots}{0.435} = 10.21\ldots \approx$ **10.2** months **A1**

Note: Accept $t \approx 10.2$ (3 sf) or month 11 (rounded up). FT from their k.

Q27 — Piecewise from Graph HARD

(a) Branch 1 ($0 \le x \le 4$): slope $= \dfrac{12-4}{4-0} = 2$; through $(0,4)$: $h(x) = 2x+4$ **A1**

Branch 2 ($4 < x \le 8$): slope $= \dfrac{4-12}{8-4} = -2$; through $(4,12)$: $h(x) = -2(x-4)+12 = -2x+20$ **A1**

Branch 3 ($8 < x \le 12$): slope $= \dfrac{7-4}{12-8} = 0.75$; through $(8,4)$: $h(x) = 0.75(x-8)+4 = 0.75x-2$ **A1**

(b) Branch 1: $2x+4 = 10 \Rightarrow x = 3$ ✓ ($0 \le 3 \le 4$) **A1**

Branch 2: $-2x+20 = 10 \Rightarrow x = 5$ ✓ ($4 < 5 \le 8$)

Note: Award A1 for either or both values $x = 3$ and $x = 5$.

(c) Area (trapezoids with x-axis):

Branch 1: $\frac{1}{2}(4+12)\times 4 = 32$; Branch 2: $\frac{1}{2}(12+4)\times 4 = 32$; Branch 3: $\frac{1}{2}(4+7)\times 4 = 22$

Total area $= 32+32+22 =$ **86** sq units **A1**

Note: (M1) for trapezoid area method or integration. A1 for correct total.

Q28 — Logistic with Unknown k HARD

(a) $B(0) = \dfrac{3600}{1+k} = 450$ **M1**

$1+k = \dfrac{3600}{450} = 8 \Rightarrow k =$ **7** **A1**

(b) Solve $\dfrac{3600}{1+7e^{-0.25t}} = 3000$: **M1**

$1 + 7e^{-0.25t} = \dfrac{3600}{3000} = 1.2 \Rightarrow e^{-0.25t} = \dfrac{0.2}{7} = 0.02857\ldots$

$t = \dfrac{-\ln(0.02857\ldots)}{0.25} = \dfrac{3.5553\ldots}{0.25} = 14.22\ldots$

First exceeds 3 000 in **year 15**. **A1**

(c) Long-run population: **3 600** seabirds. As the population approaches 3 600, the growth rate decreases towards zero — the population levels off and growth becomes negligible. **R1**

Q29 — Piecewise Log Two Unknowns HARD

(a) $T(0) = 28 - 3\ln(1) = 28 - 0 = \mathbf{28}\,^{\circ}\text{C}$ **A1**

(b) $T(30) = 28 - 3\ln(30+1) = 28 - 3\ln(31)$ **AG**

(c) Continuity at $d = 30$: $a - b\ln(31) = 28 - 3\ln(31) \;\cdots(1)$ **M1**

At $d = 80$, $T = 2$: $a - b\ln(81) = 2 \;\cdots(2)$ **M1**

$(1)-(2)$: $b\ln(81) - b\ln(31) = (28 - 3\ln 31) - 2 = 26 - 3\ln 31$

Numerically: $\ln 31 = 3.4340\ldots,\quad \ln 81 = 4.3944\ldots,\quad \ln 81 - \ln 31 = 0.9604\ldots$

$b(0.9604\ldots) = 26 - 3(3.4340\ldots) = 26 - 10.302\ldots = 15.698\ldots$

$b = \dfrac{15.698\ldots}{0.9604\ldots} = 16.345\ldots \approx \mathbf{16.3}$ (3 sf) **A1**

From (2): $a = 2 + b\ln(81) = 2 + 16.345\ldots \times 4.3944\ldots = 2 + 71.83\ldots = 73.83\ldots \approx \mathbf{73.8}$ (3 sf) **A1**

Check: $T(30) = 73.8 - 16.3\ln 31 = 73.8 - 56.0 = 17.8 \approx 28 - 3\ln 31 = 17.7$ ✓

$T(80) = 73.8 - 16.3\ln 81 = 73.8 - 71.6 = 2.2 \approx 2$ ✓ *(rounding)*

Q30 — Logistic Two Parameters from Data Points HARD

(a) $A(0) = \dfrac{0.95}{1+m} = 0.05 \Rightarrow 1 + m = \dfrac{0.95}{0.05} = 19 \Rightarrow m = 18$ **M1 AG**

(b) $A(3) = \dfrac{0.95}{1 + 18e^{-3n}} = 0.20$ **M1**

$1 + 18e^{-3n} = \dfrac{0.95}{0.20} = 4.75 \Rightarrow 18e^{-3n} = 3.75 \Rightarrow e^{-3n} = \dfrac{3.75}{18} = 0.20833\ldots$ **A1**

$-3n = \ln(0.20833\ldots) = -1.5686\ldots \Rightarrow n = \dfrac{1.5686\ldots}{3} = 0.52287\ldots \approx \mathbf{0.523}$ (3 sf) **A1**

(c) $\dfrac{0.95}{1 + 18e^{-0.523t}} = 0.80$: **(M1)**

$1 + 18e^{-0.523t} = \dfrac{0.95}{0.80} = 1.1875 \Rightarrow e^{-0.523t} = \dfrac{0.1875}{18} = 0.010417\ldots$

$t = \dfrac{-\ln(0.010417\ldots)}{0.523} = \dfrac{4.5643\ldots}{0.523} = 8.727\ldots$

Proportion first exceeds 0.80 in **year 9**. **A1**

Note: FT from their n. Must round up to next integer year.

End of Worksheet

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