IB Math AI HL — Topic 3 [Set 1]

Matrix Transformations

Geometric Transformations · Composite Transformations · Determinant & Area Scale Factor

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________ Date:____________

Score:________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated. GDC permitted throughout.

Key Formulae

Rotation anticlockwise by θ:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Reflection in $y = x\tan\alpha$ (i.e. line at angle α to x-axis):

$$M_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

Reflection in x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **in y-axis:** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflection in $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **in $y = -x$:** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Enlargement, scale factor k, centre O:

$$E(k) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Composite transformation (apply A first, then B):

$$\text{Single matrix} = B \cdot A$$

Image of point P: $\mathbf{P}' = M\,\mathbf{P}$

Determinant: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Area scale factor: $\text{Area}' = |\det M| \times \text{Area}$

Pure rotation/reflection: $|\det M| = 1$

Enlargement factor k: $|\det M| = k^2$

GDC Tips

1. **Matrix multiply:** On most GDCs, store matrices in `[A]`, `[B]` then enter `[B][A]` to get the composite (right-to-left order).

2. **Matrix powers:** Use `[A]^n` to find A^n quickly for checking steady-state or repeated transformations.
3. **Determinant:** On Casio, `MENU` $\to$ `Matrix` $\to$ `det(`; on TI, `MATRIX` $\to$ `MATH` $\to$ `det(`.
4. **Inverse:** Enter `[A]^-1` to find A^{-1} for reverse-transformation problems.
5. **Angle from matrix entry:** If a rotation entry gives cos $\theta = 0.5$, type `cos^-1(0.5)` to recover $\theta = 60°$.

Section A — Easy **EASY** (Q1–10, 3 marks each = 30 marks)

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1. **EASY** **GDC** **[3 marks]**

A point $P(4,\ -3)$ is rotated anticlockwise by $90°$ about the origin O.

(a) Write down the 2×2 matrix R that represents this rotation. [1]

(b) Find the coordinates of the image P'. [2]

2. **EASY** **GDC** **[3 marks]**

A graphic designer reflects the point $Q(5,\ 2)$ in the x-axis.

(a) Write down the matrix M that represents a reflection in the x-axis. [1]

(b) Find the coordinates of the image Q'. [2]

3. EASY GDC **[3 marks]**

An enlargement centred at the origin has scale factor 3.

(a) Write down the matrix E that represents this enlargement. [1]

(b) Write down the value of $\det(E)$. [1]

(c) A triangle has area 7 cm^2. Find the area of its image under E. [1]

4. EASY GDC **[3 marks]**

The matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a geometric transformation.

(a) Describe fully the transformation represented by T. [1]

(b) Find the image of the point $A(3,\,-7)$ under T. [1]

(c) Write down the value of $\det(T)$. [1]

5. EASY GDC **[3 marks]**

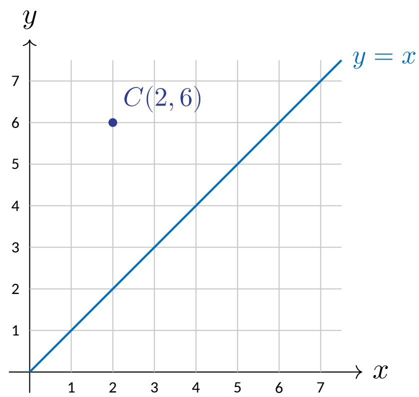
A point $B(0,\,5)$ undergoes a rotation of $60°$ anticlockwise about the origin.

(a) Write down the matrix R that represents this rotation. [1]

(b) Find the exact coordinates of the image B'. [2]

6. EASY GDC **[3 marks]**

The diagram shows the point $C(2,\ 6)$ and the line $y = x$.



(a) Write down the matrix M that represents a reflection in the line $y = x$. [1]

(b) Find the coordinates of the image C'. [1]

(c) Mark and label C' on the diagram above. [1]

7. EASY GDC **[3 marks]**

The matrix $N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a geometric transformation of the plane.

(a) Write down the value of det(N). [1]

(b) Describe fully the transformation represented by N. [2]

8. EASY GDC **[3 marks]**

A point $D(-4,\,3)$ is reflected in the y-axis.

(a) Write down the matrix F that represents a reflection in the y-axis. [1]

(b) Find the coordinates of the image D'. [1]

(c) Explain why the area of any shape is unchanged by this transformation. [1]

9. EASY GDC **[3 marks]**

A transformation is represented by the matrix $K = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

(a) Write down the value of det(K). [1]

(b) A square has area 5 cm^2. Find the area of its image under K. [1]

(c) Describe the transformation represented by K. [1]

10. EASY GDC **[3 marks]**

A point $E(6,\,2)$ undergoes a rotation of $270°$ anticlockwise about the origin.

(a) Write down the matrix R that represents a $270°$ anticlockwise rotation. [1]

(b) Find the coordinates of the image E'. [2]

Section B — Medium MEDIUM (Q11–20, 4 marks each = 40 marks)

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11. MEDIUM GDC **[4 marks]**

A robot arm applies two transformations to a tool, in order:

I. a rotation of $90°$ anticlockwise about the origin, represented by matrix A;
II. a reflection in the x-axis, represented by matrix B.

(a) Write down matrices A and B. [2]

(b) Find the single matrix $C = BA$ that represents the composite transformation. [1]

(c) The tool is initially at position $P(3,\,1)$. Find the final position P'. [1]

12. MEDIUM GDC **[4 marks]**

An animation sequence applies a rotation of $45°$ anticlockwise, then a further rotation of $45°$ anticlockwise, both about the origin O.

(a) Write down the single matrix R_{45} for a $45°$ anticlockwise rotation. [1]

(b) Find the matrix $M = R_{45}^2$ representing the composite of both rotations. [1]

(c) Describe the single transformation equivalent to M. [1]

(d) A triangle with area 12 cm^2 undergoes M. Find the area of its image. [1]

13. MEDIUM GDC **[4 marks]**

A transformation maps:

$$A(1,\,0) \mapsto A'(0,\,-1) \qquad \text{and} \qquad B(0,\,1) \mapsto B'(1,\,0).$$

(a) Find the 2×2 matrix T representing this transformation. [2]

(b) Calculate det(T). [1]

(c) Describe fully the transformation represented by T. [1]

14. MEDIUM GDC **[4 marks]**

A design is first enlarged by scale factor 2 (centre O), then rotated $180°$ about O.

(a) Write down the matrix E for the enlargement and the matrix H for the $180°$ rotation. [2]

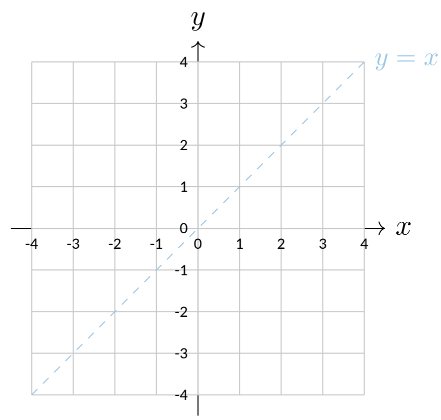
(b) Find the single matrix $S = HE$ for the composite transformation. [1]

(c) A parallelogram has area 9 cm^2. Find the area of its image under S. [1]

15. MEDIUM GDC **[4 marks]**

The matrix $F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ represents a reflection in a line ℓ through the origin.

The diagram shows the line $y = x$ for reference.



(a) Use the formula $M_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ to find the angle α that the line ℓ makes with the positive x-axis. [2]

(b) Hence state the equation of line ℓ. [1]

(c) Find the image of point $G(3, -1)$ under F. [1]

16. MEDIUM GDC **[4 marks]**

An architectural drawing is transformed by two operations in order:

I. a reflection in the y-axis, represented by matrix V;

II. an enlargement of scale factor 5, centre O, represented by matrix W.

(a) Write down matrices V and W. [2]

(b) Find the single matrix $L = WV$ for the composite transformation. [1]

(c) A rectangle of area 6 m^2 undergoes the composite transformation. Find the area of its image. [1]

17. MEDIUM GDC **[4 marks]**

Under a linear transformation T, the images of $(1,\ 0)$ and $(0,\ 1)$ are $(-3,\ 0)$ and $(0,\ -3)$ respectively.

(a) Write down the matrix representing T. [1]

(b) Find det(T) and hence state the area scale factor of T. [2]

(c) Describe fully the transformation represented by T. [1]

18. MEDIUM GDC **[4 marks]**

The matrix $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents a rotation of angle θ anticlockwise about O.

(a) Write down R^{-1} in terms of θ. [1]

(b) A rotation of $\theta = 120°$ anticlockwise is followed by a rotation of $120°$ clockwise (i.e. $\theta = -120°$). Show that the composite matrix equals the identity matrix I. [2]

(c) Describe what this result means geometrically. [1]

19. MEDIUM GDC **[4 marks]**

A transformation is represented by the matrix $T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

(a) Calculate det(T). [1]

(b) Find T^{-1}. [2]

(c) The image of a point P under T is $P'(7, 6)$. Find the coordinates of P. [1]

20. MEDIUM GDC **[4 marks]**

A transformation U consists of first reflecting in the y-axis and then reflecting in the x-axis.

(a) Write down the matrix for each individual reflection. [1]

(b) Find the single matrix U for the composite transformation. [1]

(c) Describe fully the single transformation equivalent to U. [1]

(d) A region of area 15 cm^2 undergoes U. Find the area of its image. [1]

Section C — Hard **HARD** (Q21–30, 5 marks each = 50 marks)

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21. **HARD** **GDC** **[5 marks]**

The transformation matrix $P = \begin{pmatrix} 3 & -3\sqrt{3} \\ 3\sqrt{3} & 3 \end{pmatrix}$ can be written as $P = R(\theta) \cdot E(k)$, where $R(\theta)$ is a rotation matrix and $E(k)$ is an enlargement matrix.

(a) Find the scale factor k of the enlargement. [2]

(b) Hence find the angle θ of the rotation. [2]

(c) A quadrilateral has area 2 cm^2. Find the area of its image under P. [1]

22. HARD GDC **[5 marks]**

Three transformations are applied in order to a shape:

I. Rotation of $60°$ anticlockwise about O (matrix A);
II. Reflection in the x-axis (matrix B);
III. Rotation of $30°$ clockwise about O (matrix C).

(a) Write down matrices A, B, and C. [2]

(b) Find the single matrix $Q = CBA$. [2]

(c) Describe fully the single transformation equivalent to Q. [1]

23. HARD GDC **[5 marks]**

A transformation is represented by the matrix $M = \begin{pmatrix} k & 2 \\ 3 & k+1 \end{pmatrix}$, where $k \in \mathbb{R}$.

(a) Find det(M) in terms of k. [2]

(b) Given that the transformation maps a unit square to a region of area 4, find the possible values of k. [3]

24. HARD GDC **[5 marks]**

A composite transformation T is defined by: first, an enlargement of scale factor 3 about O; then, a reflection in the line $y = x$.

(a) Find the single matrix T representing the composite transformation. [2]

(b) The image of a point X under T is $X'(12,\ -6)$. Find the coordinates of X. [2]

(c) A triangle with vertices at the origin, $(1,\ 0)$, and $(0,\ 1)$ undergoes T. Find the area of the image triangle. [1]

25. HARD GDC **[5 marks]**

The line ℓ has equation $y = \sqrt{3}\,x$.

The transformation matrix $N = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ is applied to every point in the plane.

(a) A general point on ℓ may be written as $\begin{pmatrix} t \\ \sqrt{3}\,t \end{pmatrix}$. Show that the image of this general point under N also lies on ℓ. [3]

(b) State the name of the transformation represented by N, and write down the equation of its mirror line. [1]

(c) Calculate det(N) and explain, with reference to area, what this value means. [1]

26. HARD GDC **[5 marks]**

A transformation matrix has the form $G = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a, b > 0$.

Under G, the point $(2,\ 1)$ maps to $(1,\ 7)$.

(a) Write down a system of two equations in a and b. [2]

(b) Solve for a and b. [1]

(c) Hence find det(G) and state the area scale factor of G. [2]

27. HARD GDC **[5 marks]**

The matrix for a rotation of θ anticlockwise is $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Compute the product $R(\theta) \cdot R(\theta)$, giving each entry in simplified form. [2]

(b) Write down the matrix for a rotation of 2θ anticlockwise. [1]

(c) By equating your answers to (a) and (b), write down two trigonometric identities. [2]

28. HARD GDC **[5 marks]**

A point $P(4,\,0)$ is rotated anticlockwise by angle θ about the origin. The image P' lies on the line $y = \sqrt{3}\,x$ in the first quadrant.

(a) Write down the coordinates of P' in terms of θ. [1]

(b) Hence find the value of θ, where $0 < \theta < 90°$. [2]

(c) A second point $Q(0,\,4)$ undergoes the same rotation. Find the exact coordinates of the image Q'. [2]

29. HARD GDC **[5 marks]**

Two reflections are applied in order:

I. Reflection in the x-axis (matrix A);

II. Reflection in the line $y = x\tan(75°)$, i.e. the line at $75°$ to the positive x-axis (matrix B).

(a) Write down matrix A. [1]

(b) Write down matrix B, giving entries as exact values. [2]

(c) Find the composite matrix $C = BA$ and hence describe the single equivalent transformation. [2]

30. HARD GDC **[5 marks]**

The matrix $J = \begin{pmatrix} k & -\sqrt{3}k \\ \sqrt{3}k & k \end{pmatrix}$, where $k > 0$, represents a composite of an enlargement and a rotation about O.

(a) Find $\det(J)$ in terms of k. [1]

(b) Given that a unit square maps to a square of area 16, find the value of k. [2]

(c) Hence find the angle of rotation, and state the scale factor of the enlargement. [2]

Worked Solutions

Section A — Easy

Q1 | Rotation $90°$ anticlockwise [EASY]

(a) $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **A1**

(b) $P' = R\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} (0)(4)+(-1)(-3) \\ (1)(4)+(0)(-3) \end{pmatrix}$ **M1**

$= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, so $P'(3,\,4)$ **A1**

Note: Method mark awarded for correct matrix-vector multiplication attempt.

Q2 | Reflection in x-axis [EASY]

(a) $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **A1**

(b) $Q' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ **M1**

So $Q'(5,\,-2)$ **A1**

Q3 | Enlargement scale factor 3 [EASY]

(a) $E = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **A1**

(b) $\det(E) = (3)(3) - (0)(0) = 9$ **A1**

(c) Area of image $= |\det(E)| \times 7 = 9 \times 7 = 63$ cm^2 **A1**

Q4 | Identify transformation from matrix [EASY]

(a) Reflection in the line $y = x$ **A1**

(b) $A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$, so $A'(-7,\,3)$ **A1**

(c) $\det(T) = (0)(0) - (1)(1) = -1$ **A1**

Q5 | Rotation 60° anticlockwise [EASY]

(a) $R = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ **A1**

(b) $B' = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ **M1**

$= \begin{pmatrix} 0 \cdot \frac{1}{2} + 5 \cdot (-\frac{\sqrt{3}}{2}) \\ 0 \cdot \frac{\sqrt{3}}{2} + 5 \cdot \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{5\sqrt{3}}{2} \\ \frac{5}{2} \end{pmatrix}$

So $B'\left(-\dfrac{5\sqrt{3}}{2}, \dfrac{5}{2}\right)$ **A1**

Q6 | Reflection in y = x [EASY]

(a) $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **A1**

(b) $C' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, so $C'(6,\, 2)$ **A1**

(c) $C'(6,\, 2)$ correctly plotted and labelled on the diagram. **A1**

Note: Award A1 for point plotted at $(6, 2)$ with label. FT from (b).

Q7 | Identify rotation 180 degrees [EASY]

(a) $\det(N) = (-1)(-1) - (0)(0) = 1$ **A1**

(b) Rotation of 180° about the origin O. **A1A1**

Note: Award A1 for "rotation" and A1 for "180° about the origin". Do not accept "reflection".

Q8 | Reflection in y-axis [EASY]

(a) $F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **A1**

(b) $D' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, so $D'(4,\, 3)$ **A1**

(c) $|\det(F)| = |(-1)(1) - (0)(0)| = |-1| = 1$, so the area scale factor is 1, meaning the area is unchanged. **R1**

Note: Accept equivalent reasoning referencing that reflections are isometries.

Q9 | Enlargement scale factor 4 [EASY]

(a) $\det(K) = (4)(4) - (0)(0) = 16$ **A1**
(b) Area of image $= 16 \times 5 = 80$ cm^2 **A1**
(c) Enlargement (stretch) of scale factor 4 centred at the origin. **A1**

Q10 | Rotation $270°$ anticlockwise [EASY]

(a) $270°$ anticlockwise $= 90°$ clockwise: $R = \begin{pmatrix} \cos 270° & -\sin 270° \\ \sin 270° & \cos 270° \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **A1**

(b) $E' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ **M1**

$= \begin{pmatrix} 0 \cdot 6 + 1 \cdot 2 \\ -1 \cdot 6 + 0 \cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$, so $E'(2, -6)$ **A1**

Section B — Medium

Q11 — Composite: rotation then reflection [MEDIUM]

(a) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (90° anticlockwise) **A1**

$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (reflection in x-axis) **A1**

(b) $C = BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **M1**

$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ **A1**

(c) $P' = C \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} (0)(3) + (-1)(1) \\ (-1)(3) + (0)(1) \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, so $P'(-1, -3)$ **A1**

Q12 — Composite: two 45° rotations [MEDIUM]

(a) $R_{45} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ **A1**

(b) $M = R_{45}^2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **M1A1**

Note: M1 for attempting to square the matrix; A1 for correct result.

(c) Rotation of $90°$ anticlockwise about the origin. **A1**
(d) $\det(M) = 1$; area scale factor $= 1$; area of image $= 12$ cm^2. **A1**

Q13 — Find matrix from point-image pairs [MEDIUM]

(a) Image of $(1,0)$ gives first column; image of $(0,1)$ gives second column:

$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **M1A1**

Note: M1 for recognising column method; A1 for correct matrix.

(b) $\det(T) = (0)(0) - (1)(-1) = 1$ **A1**

(c) Rotation of $90°$ *clockwise* about the origin (equivalently $270°$ anticlockwise). **A1**

Q14 — Composite: enlargement then 180° rotation [MEDIUM]

(a) $E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad H = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **A1A1**

(b) $S = HE = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ **M1A1**

(c) $\det(S) = (-2)(-2) - (0)(0) = 4$; area of image $= 4 \times 9 = 36$ cm^2 **A1**

Q15 — Identify reflection line from matrix entries [MEDIUM]

(a) From $M_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$: **M1**

$\cos 2\alpha = 0 \Rightarrow 2\alpha = 90° \Rightarrow \alpha = 45°$; check: $\sin 2\alpha = \sin 90° = 1 \neq -1$

Re-reading: $\sin 2\alpha = -1$ and $\cos 2\alpha = 0$, so $2\alpha = 270°$ (or $-90°$), giving $\alpha = 135°$. **A1**

Note: Award A1 for $\alpha = 135°$ (or equivalent $-45°$). The line is at $135°$ to positive x-axis.

(b) The line ℓ has equation $y = -x$ (gradient $\tan 135° = -1$). **A1**

(c) $G' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 3 + (-1)(-1) \\ (-1)(3) + 0(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, so $G'(1, -3)$ **A1**

Q16 — Composite: reflection then enlargement [MEDIUM]

(a) $V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ **A1A1**

(b) $L = WV = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix}$ **M1A1**

(c) $\det(L) = (-5)(5) - (0)(0) = -25$; area scale factor $= |\det L| = 25$; area $= 25 \times 6 = 150$ m^2 **A1**

Q17 — Matrix from column images, det, describe [MEDIUM]

(a) $T = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ **A1**

(b) $\det(T) = (-3)(-3) - (0)(0) = 9$; area scale factor $= 9$. **A1A1**

(c) Enlargement of scale factor 3 centred at the origin combined with a rotation of $180°$ (equivalently, enlargement of scale factor -3, or enlargement of scale factor 3 with a $180°$ rotation). **A1**

Note: Also accept: "enlargement of scale factor 3, centre O, followed by rotation 180° about O". Do not accept scale factor 9.

Q18 — Inverse rotation matrix, composite gives I [MEDIUM]

(a) $R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ (i.e. $R(-\theta)$) **A1**

(b) Using the rotation addition property: $R(120°) \cdot R(-120°) = R(120° + (-120°)) = R(0°)$ **M1**

$R(0°) = \begin{pmatrix} \cos 0° & -\sin 0° \\ \sin 0° & \cos 0° \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ **M1**

Therefore the composite matrix equals I. **AG**

(c) The two rotations exactly cancel each other; every point is returned to its original position. **R1**

Note: A fully expanded matrix multiplication is also acceptable for M1M1.

Q19 — Inverse matrix, find pre-image [MEDIUM]

(a) $\det(T) = (2)(3) - (1)(0) = 6$ **A1**

(b) $T^{-1} = \frac{1}{6}\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix}$ **M1A1**

(c) $P = T^{-1}P' = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 7 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} - 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ 2 \end{pmatrix}$

So $P\left(\frac{5}{2}, 2\right)$ **A1**

Q20 — Composite of two reflections [MEDIUM]

(a) Reflection in y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; reflection in x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **A1**

(b) $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **M1A1**

(c) Rotation of $180°$ about the origin O. **A1**

(d) $|\det(U)| = |(-1)(-1)| = 1$; area of image $= 1 \times 15 = 15$ cm^2. **A1**

Section C — Hard

Q21 — Decompose matrix into rotation $\times$ enlargement [HARD]

(a) $P = R(\theta) \cdot E(k) = \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix}$

Comparing with $P = \begin{pmatrix} 3 & -3\sqrt{3} \\ 3\sqrt{3} & 3 \end{pmatrix}$:
$k\cos\theta = 3$ and $k\sin\theta = 3\sqrt{3}$
$k^2 = (k\cos\theta)^2 + (k\sin\theta)^2 = 9 + 27 = 36$ **M1**
$k = 6$ **A1**
(b) $\cos\theta = \dfrac{3}{6} = \dfrac{1}{2}$ and $\sin\theta = \dfrac{3\sqrt{3}}{6} = \dfrac{\sqrt{3}}{2}$ **M1**
$\theta = 60°$ **A1**
(c) $\det(P) = k^2 = 36$; area of image $= 36 \times 2 = 72$ cm^2 **A1**

Q22 — Composite of three transformations [HARD]

(a) $A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ **A1A1**

(b) First compute BA: $BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ **M1**

$Q = CBA = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & -\frac{3}{4} - \frac{1}{4} \\ -\frac{1}{4} - \frac{3}{4} & \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ **A1**

(c) $\det(Q) = -1$; entries match M_α with $\cos 2\alpha = 0$, $\sin 2\alpha = -1$, so $2\alpha = -90°$ (or $270°$), $\alpha = -45°$ (or $135°$).
Reflection in the line $y = -x$ (the line at $135°$ to the positive x-axis). **A1**

Q23 — Unknown parameter from area condition [HARD]

(a) $\det(M) = k(k+1) - 2 \cdot 3 = k^2 + k - 6$ **M1A1**
(b) Area of image $= |\det(M)| \times$ Area of unit square $= |k^2 + k - 6| \times 1 = 4$ **M1**
$k^2 + k - 6 = 4 \;\Rightarrow\; k^2 + k - 10 = 0 \;\Rightarrow\; k = \dfrac{-1 \pm \sqrt{41}}{2}$ **A1**
or $k^2 + k - 6 = -4 \;\Rightarrow\; k^2 + k - 2 = 0 \;\Rightarrow\; (k+2)(k-1) = 0 \;\Rightarrow\; k = -2$ or $k = 1$ **A1**
So $k = \dfrac{-1 + \sqrt{41}}{2} \approx 2.70$, or $k = \dfrac{-1 - \sqrt{41}}{2} \approx -3.70$, or $k = 1$, or $k = -2$.
Note: Award A1 for the case $k^2 + k - 6 = 4$ and A1 for the case $k^2 + k - 6 = -4$; both sets of roots required for full marks.

Q24 — Composite enlargement then reflection; find pre-image [HARD]

(a) Enlargement by 3: $E = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$; Reflection in $y = x$: $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$T = FE = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ **M1A1**

(b) $\det(T) = (0)(0) - (3)(3) = -9$

$T^{-1} = \dfrac{1}{-9}\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$ **M1**

$X = T^{-1}X' = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}\begin{pmatrix} 12 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

So $X(-2,\,4)$ **A1**

(c) Area of unit triangle (with vertices $(0,0)$, $(1,0)$, $(0,1)$) $= \frac{1}{2}$.

Area of image $= |\det(T)| \times \frac{1}{2} = 9 \times \frac{1}{2} = 4.5$ units^2 **A1**

Q25 — Invariant line under reflection matrix [HARD]

(a) $N\begin{pmatrix} t \\ \sqrt{3}\,t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} t \\ \sqrt{3}\,t \end{pmatrix}$ **M1**

$= \begin{pmatrix} -\frac{t}{2} + \frac{3t}{2} \\ \frac{\sqrt{3}\,t}{2} + \frac{\sqrt{3}\,t}{2} \end{pmatrix} = \begin{pmatrix} t \\ \sqrt{3}\,t \end{pmatrix}$ **M1**

The image is $\begin{pmatrix} t \\ \sqrt{3}\,t \end{pmatrix}$, and since the y-coordinate equals $\sqrt{3}$ times the x-coordinate, this point lies on the line $y = \sqrt{3}\,x$. **AG**

(b) N is a **reflection**; mirror line is $y = \sqrt{3}\,x$ (the line ℓ itself). **A1**

(c) $\det(N) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{4} - \frac{3}{4} = -1$ **A1**

$|\det(N)| = 1$: the area of any shape is unchanged by this transformation (reflections are isometries).

Note: Award A1 for det $= -1$ *and a correct area interpretation. The negative sign confirms orientation reversal.*

Q26 — Matrix parameters and determinant [HARD]

(a) $G\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$:

$\begin{cases} 2a - b = 1 \\ 2b + a = 7 \end{cases}$ **M1A1**

(b) From the first equation: $b = 2a - 1$. Substituting into the second:

$2(2a-1) + a = 7 \Rightarrow 4a - 2 + a = 7 \Rightarrow 5a = 9 \Rightarrow a = \dfrac{9}{5}$

$b = 2 \cdot \dfrac{9}{5} - 1 = \dfrac{18}{5} - 1 = \dfrac{13}{5}$ **A1**

(c) $\det(G) = a^2 + b^2 = \left(\dfrac{9}{5}\right)^2 + \left(\dfrac{13}{5}\right)^2 = \dfrac{81 + 169}{25} = \dfrac{250}{25} = 10$ **M1A1**

Area scale factor $= |\det(G)| = 10$.

Note: The matrix $G = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *always has* $\det G = a^2 + b^2$*, which is also the square of the scale factor of the enlargement component.*

Q27 — Rotation matrix squared; derive double-angle identities [HARD]

(a) $R(\theta)^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ **M1**

$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$ **A1**

(b) $R(2\theta) = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$ **A1**

(c) Since $R(\theta)^2 = R(2\theta)$ (two rotations of θ equal one rotation of 2θ), equating entries: **M1**

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ **A1**

$\sin 2\theta = 2\sin\theta\cos\theta$ **(A1)**

Note: Award A1 for each identity. The R1 for justifying $R(\theta)^2 = R(2\theta)$ *may be implied by the argument.*

Q28 — Rotation angle from external-point condition [HARD]

(a) $P(4,0)$ rotated by θ: $P' = \begin{pmatrix} 4\cos\theta \\ 4\sin\theta \end{pmatrix}$ **A1**

(b) P' lies on $y = \sqrt{3}x$ (first quadrant), so $4\sin\theta = \sqrt{3}(4\cos\theta)$ **M1**

$\tan\theta = \sqrt{3} \Rightarrow \theta = 60°$ **A1**

(c) $Q(0,4)$ rotated $60°$ anticlockwise:

$Q' = \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \cdot 4 \\ \frac{1}{2} \cdot 4 \end{pmatrix}$ **M1**

$= \begin{pmatrix} -2\sqrt{3} \\ 2 \end{pmatrix}$

$Q'(-2\sqrt{3},\ 2)$ **A1**

Q29 — Composite of two reflections; identify net rotation [HARD]

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **A1**

(b) Line at $75°$ to x-axis: use M_α with $\alpha = 75°$:

$\cos 150° = -\frac{\sqrt{3}}{2}, \quad \sin 150° = \frac{1}{2}$

$B = \begin{pmatrix} \cos 150° & \sin 150° \\ \sin 150° & -\cos 150° \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ **M1A1**

(c) $C = BA = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ **M1**

Compare with $R(\theta)$: $\cos\theta = -\dfrac{\sqrt{3}}{2}$, $\sin\theta = \dfrac{1}{2} \Rightarrow \theta = 150°$.

Single transformation: rotation of $150°$ anticlockwise about the origin. **A1**

Note: The composite of two reflections in lines through the origin, making angle $75°$ between them (since the x-axis is at $0°$ and the second line is at $75°$), gives a rotation of $2 \times 75° = 150°$.

Q30 — Unknown parameter from area; rotation angle and scale factor [HARD]

(a) $\det(J) = k \cdot k - (-\sqrt{3}k)(\sqrt{3}k) = k^2 + 3k^2 = 4k^2$ **A1**

(b) Area of unit square $= 1$; image area $= |\det(J)| \times 1 = 4k^2 = 16$ **M1**

$k^2 = 4 \Rightarrow k = 2$ (since $k > 0$) **A1**

(c) With $k = 2$: $J = \begin{pmatrix} 2 & -2\sqrt{3} \\ 2\sqrt{3} & 2 \end{pmatrix}$

This equals $E(k_s) \cdot R(\theta)$ form: $J = k_s \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$k_s \cos\theta = 2$, $k_s \sin\theta = 2\sqrt{3}$; $k_s^2 = 4 + 12 = 16 \Rightarrow k_s = 4$ **M1**

$\cos\theta = \dfrac{2}{4} = \dfrac{1}{2}$, $\sin\theta = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = 60°$

Angle of rotation: $60°$ anticlockwise; scale factor of enlargement: 4. **A1**

Common Mistakes to Avoid

1. **Matrix multiplication order.** For a composite "apply A first, then B", the single matrix is $B \cdot A$ (not $A \cdot B$). Matrix multiplication is not commutative.

2. **Rotation direction.** Anticlockwise is the standard positive direction. The rotation matrix is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$; for clockwise, use $-\theta$.

3. **Determinant and area.** The area scale factor is $|\det(M)|$, *not* $\det(M)$. A negative determinant means the orientation is reversed, but area is always positive.

4. **Reflection matrix formula.** For reflection in $y = x\tan\alpha$, use α (not 2α) in the formula $M_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$.

5. **Column method for finding a matrix.** If you know $(1,0) \mapsto (a,b)$ and $(0,1) \mapsto (c,d)$, the matrix is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ (images become *columns*).

6. **Inverse transformation for pre-image.** To find the original point P given image $P' = MP$, compute

$P = M^{-1}P'$. Always check det$(M) \neq 0$ first.

7. **Pure rotations and reflections have** $|$det$| = 1$. An enlargement of scale factor k has det $= k^2$. Confusing these leads to wrong area calculations.