IB Math AI HL — Topic 5

Coupled & Second-Order Differential Equations

Coupled DEs · Phase Portraits · Equilibrium Points · Solution Trajectories · Second-Order ODEs

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:________________________ Date:____________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give numerical answers to 3 sf unless otherwise stated. A GDC is permitted throughout.

Key Formulae

Coupled linear system: $\dfrac{dx}{dt} = ax + by, \quad \dfrac{dy}{dt} = cx + dy \implies \dot{\mathbf{x}} = M\mathbf{x}$

Characteristic equation: $\det(M - \lambda I) = 0 \;\Rightarrow\; \lambda^2 - \text{tr}(M)\lambda + \det(M) = 0$

General solution (real distinct $\lambda_1 \neq \lambda_2$): $\mathbf{x} = A\,e^{\lambda_1 t}\mathbf{v}_1 + B\,e^{\lambda_2 t}\mathbf{v}_2$

Stability: both $\lambda < 0 \Rightarrow$ stable node; both $\lambda > 0 \Rightarrow$ unstable node; opposite signs $\Rightarrow$ saddle point

Complex eigenvalues $\alpha \pm \beta i$: spiral sink ($\alpha < 0$), spiral source ($\alpha > 0$), centre ($\alpha = 0$)

Euler's method (step h): $x_{n+1} = x_n + h\,f(t_n, x_n, y_n); \quad y_{n+1} = y_n + h\,g(t_n, x_n, y_n)$

2nd-order ODE $\ddot{x} + p\dot{x} + qx = 0 \;\Rightarrow\;$ let $y = \dot{x}$, giving coupled system $\dot{x} = y,\ \dot{y} = -qx - py$

Equilibrium point: set $\dot{x} = 0$ and $\dot{y} = 0$; solve simultaneously

Slope on phase portrait: $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}}$

GDC Tips

- **Eigenvalues:** Store matrix M in your GDC and use the eigen-solver (Casio: Matrix $\to$ EIG; TI: `eigVl`). Always verify by substituting back.
- **Euler's method:** Use a spreadsheet-style table or the GDC's Sequence/Recursion mode. Keep 4–5 d.p. in intermediate values.
- **Phase portraits:** Identify the sign of dy/dx along the axes and eigen-directions. Arrow direction (clockwise vs anticlockwise) is determined by the sign of $\dot{y}$ when $y > 0,\ x = 0$.
- **Solving 2nd-order ODEs:** Convert to coupled first-order system, find eigenvalues, then apply initial conditions to determine constants A and B.

Common Mistakes to Avoid

1. **Forgetting initial conditions.** Finding the general solution earns partial credit only; always substitute initial conditions to find the constants A and B.
2. **Stability from eigenvalues.** A stable equilibrium requires *both* real parts of eigenvalues to be *negative*. One positive eigenvalue makes a saddle point, not a stable node.
3. **Arrow direction on phase portrait.** The arrows show the direction of *increasing time*. Determine the direction from the sign of $\dot{x}$ or $\dot{y}$ at a test point, not from eigenvalue sign alone.
4. **Euler step order.** Compute *both* x_{n+1} and y_{n+1} using values at step n, not a mix of old and updated values within the same step.
5. **Eigenvector scaling.** Eigenvectors are directions, not unique vectors. Any non-zero scalar multiple is correct; do not lose marks trying to normalise unless the question asks for a unit vector.
6. **Converting 2nd-order to coupled system.** The substitution is $y = \dot{x}$, so $\dot{y} = \ddot{x}$. Express $\dot{y}$ entirely in terms of x and y, eliminating $\ddot{x}$ using the ODE.

Section A — Easy **EASY** (Questions 1–10, 3 marks each, 30 marks total)

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1. **EASY** **Calculator** **[3 marks]**

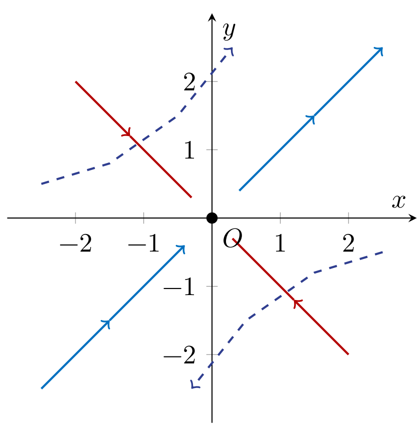
Consider the matrix $M = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$.

(a) Write down the eigenvalues of M. **[2]**

(b) State whether the equilibrium point at the origin of the system $\dot{\mathbf{x}} = M\mathbf{x}$ is stable or unstable. **[1]**

2. **EASY** **Calculator** **[3 marks]**

The diagram below shows the phase portrait for a coupled system $\dfrac{dx}{dt} = f(x, y)$, $\dfrac{dy}{dt} = g(x, y)$.



(a) Write down the coordinates of the equilibrium point shown. **[1]**

(b) State the type of equilibrium point shown in the phase portrait. **[1]**

(c) Write down whether the eigenvalues of the system are both real and positive, both real and negative, or of opposite sign. **[1]**

3. EASY Calculator **[3 marks]**

A system of coupled differential equations has matrix $M = \begin{pmatrix} -2 & 1 \\ 0 & -5 \end{pmatrix}$.

(a) Find the eigenvalues of M. **[2]**

(b) State whether the equilibrium point at the origin is a stable node, unstable node, or saddle point. **[1]**

4. EASY Calculator **[3 marks]**

A second-order differential equation is given by $\ddot{x} + 6\dot{x} + 8x = 0$.
Let $y = \dot{x}$.

(a) Show that the equation can be written as the coupled system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -8x - 6y.$$

[2]

(b) Write down the matrix M of this coupled system. **[1]**

5. EASY Calculator **[3 marks]**

Consider the system $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \mathbf{x}$.

(a) Write down the eigenvalues of the matrix. **[1]**

(b) Find an eigenvector corresponding to the eigenvalue $\lambda = 1$. **[2]**

6. EASY Calculator **[3 marks]**

Consider the coupled system

$$\frac{dx}{dt} = x - y, \qquad \frac{dy}{dt} = 2x + y,$$

with initial conditions $x(0) = 1$ and $y(0) = 0$.

Use Euler's method with step size $h = 0.1$ to find approximate values of x and y when $t = 0.1$.

[3 marks]

7. EASY Calculator

[3 marks]

A coupled system has eigenvalues $\lambda_1 = -1$ with corresponding eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\lambda_2 = 4$ with corresponding eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Write down the general solution of the system. **[3 marks]**

8. EASY Calculator **[3 marks]**

Consider the system

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = x - 2y.$$

(a) Find $\dfrac{dy}{dx}$ at the point $(2,\ 1)$. **[2]**

(b) Write down the coordinates of the equilibrium point of the system. **[1]**

9. EASY Calculator **[3 marks]**

The second-order differential equation $\ddot{x} + 5\dot{x} + 4x = 0$ is converted to a coupled first-order system with matrix M.

(a) Write down the matrix M. **[1]**

(b) Find the eigenvalues of M. **[2]**

10. EASY Calculator **[3 marks]**

A coupled system has eigenvalues $\lambda_1 = -3 + 2i$ and $\lambda_2 = -3 - 2i$.

(a) State the type of equilibrium point at the origin. **[1]**

(b) State whether trajectories spiral inward or outward as t increases. **[1]**

(c) Write down the value of the real part of the eigenvalues. **[1]**

Section B — Medium MEDIUM (Questions 11–20, 4 marks each, 40 marks total)

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11. MEDIUM Calculator **[4 marks]**

Consider the system of coupled differential equations

$$\frac{dx}{dt} = -x + 3y, \qquad \frac{dy}{dt} = 3x - y.$$

(a) Find the eigenvalues of the system. **[2]**

(b) Find an eigenvector for each eigenvalue. **[1]**

(c) Determine, with justification, whether the equilibrium point at the origin is stable or unstable. **[1]**

12. MEDIUM Calculator **[4 marks]**

A mechanical oscillator satisfies the equation

$$\ddot{x} + 5\dot{x} + 6x = 0, \quad x(0) = 2, \quad \dot{x}(0) = -1.$$

(a) Convert the equation to a coupled system and write down the corresponding matrix M. **[1]**

(b) Find the eigenvalues and eigenvectors of M. **[2]**

(c) Write down the general solution for $x(t)$. **[1]**

13. MEDIUM Calculator **[4 marks]**

The system $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -4 & 0 \\ 1 & -2 \end{pmatrix}\mathbf{x}$ has eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -2$ with corresponding eigenvectors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Given that $x(0) = 4$ and $y(0) = 0$, find the particular solution for $x(t)$ and $y(t)$. **[4 marks]**

14. MEDIUM Calculator **[4 marks]**

Consider the system

$$\frac{dx}{dt} = y - x, \qquad \frac{dy}{dt} = -x,$$

with initial conditions $x(0) = 2$ and $y(0) = 1$.

Use Euler's method with step size $h = 0.2$ to complete the table below and find approximate values of x and y when $t = 0.6$. **[4 marks]**

t	x	y
0	2	1
0.2		
0.4		
0.6		

15. MEDIUM Calculator **[4 marks]**

A system of coupled differential equations has matrix $M = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of M. **[2]**

(b) Write down the general solution of the system. **[1]**

(c) State the type of equilibrium point at the origin and whether it is stable or unstable. **[1]**

16. MEDIUM Calculator **[4 marks]**

A particle moves along a straight line. Its displacement x cm from a fixed point satisfies

$$\ddot{x} + 4\dot{x} + 3x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 2.$$

(a) Show that the eigenvalues of the corresponding matrix are $\lambda = -1$ and $\lambda = -3$. **[2]**

(b) Find the particular solution for $x(t)$. **[2]**

17. MEDIUM Calculator **[4 marks]**

A population model is described by the system

$$\frac{dx}{dt} = x(4 - 2y), \qquad \frac{dy}{dt} = y(3x - 6),$$

where x and y are populations (in hundreds) and t is time in years.

(a) Show that $(0, 0)$ is an equilibrium point of the system. **[1]**

(b) Find the non-zero equilibrium point of the system. **[2]**

(c) State what each equilibrium point represents in the context of the model. **[1]**

18. MEDIUM Calculator **[4 marks]**

Consider the system $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & -4 \\ 4 & -1 \end{pmatrix} \mathbf{x}$.

(a) Show that the eigenvalues of the matrix are $-1 \pm 4i$. **[2]**

(b) State the type of equilibrium point at the origin. **[1]**

(c) State whether trajectories spiral inward or outward, justifying your answer. **[1]**

19. MEDIUM Calculator **[4 marks]**

A second-order differential equation is given by

$$\ddot{x} - 2\dot{x} - 3x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

Let $y = \dot{x}$.

(a) Write the equation as a coupled system. **[1]**

(b) Complete the table using Euler's method with step size $h = 0.1$ to find approximate values of x and y when $t = 0.2$. **[3]**

t	x	y
0.0	1.0000	0.0000
0.1		
0.2		

20. MEDIUM Calculator **[4 marks]**

The system below describes the motion of two interconnected fluids:

$$\frac{dx}{dt} = -2x + y, \qquad \frac{dy}{dt} = x - 2y.$$

(a) Find $\dfrac{dy}{dx}$ at the point $(3,\ 1)$. **[2]**

(b) Find the eigenvalues of the system. **[1]**

(c) Hence state the type of equilibrium point at the origin. **[1]**

Section C — Hard HARD (Questions 21–30, 5 marks each, 50 marks total)

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21. HARD Calculator **[5 marks]**

Consider the system of coupled differential equations

$$\frac{dx}{dt} = -5x + 2y, \qquad \frac{dy}{dt} = -x - 2y,$$

with initial conditions $x(0) = 3$ and $y(0) = 1$.

(a) Find the eigenvalues and eigenvectors of the system matrix. **[3]**

(b) Hence find the particular solution for $x(t)$ and $y(t)$. **[2]**

22. HARD Calculator **[5 marks]**

A signal in an electric circuit satisfies the differential equation

$$\ddot{x} + 6\dot{x} + 9x = 0, \quad x(0) = 4, \quad \dot{x}(0) = -6.$$

The characteristic equation has a repeated root.

(a) Find the repeated eigenvalue. **[1]**

(b) Show that the general solution is $x(t) = (A + Bt)\,e^{-3t}$. **[2]**

(c) Find the particular solution for $x(t)$. **[1]**

(d) Find the time $t > 0$ at which $\dot{x} = 0$, giving your answer to 3 significant figures. **[1]**

23. HARD Calculator **[5 marks]**

Consider the matrix $M = \begin{pmatrix} k & 2 \\ -1 & k-3 \end{pmatrix}$, where k is a real constant.

(a) Find the eigenvalues of M in terms of k. **[2]**

(b) Find the values of k for which both eigenvalues are real. **[2]**

(c) Given that $k = 1$, determine whether the equilibrium point of $\dot{\mathbf{x}} = M\mathbf{x}$ is stable. Justify your answer. **[1]**

24. HARD Calculator **[5 marks]**

A predator-prey model is described by

$$\frac{dx}{dt} = 3x - xy, \qquad \frac{dy}{dt} = -2y + 2xy,$$

where x is prey population (hundreds) and y is predator population (hundreds), t in years.

(a) Find all equilibrium points of the system. **[2]**

(b) At the non-trivial equilibrium point, the system can be linearised. Write down the Jacobian matrix of the system evaluated at this point. **[2]**

(c) Find the eigenvalues of this Jacobian matrix and state the type of equilibrium. **[1]**

25. HARD Calculator **[5 marks]**

A mass on a spring satisfies the equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 4.$$

(a) Find the eigenvalues of the coupled system matrix. **[1]**

(b) Find the particular solution for $x(t)$. **[3]**

(c) Find the maximum value of x and the time at which it occurs, giving your time to 3 significant figures. **[1]**

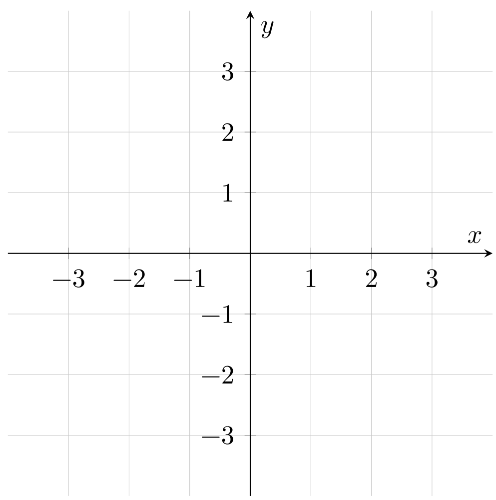
26. HARD Calculator **[5 marks]**

The system $\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\mathbf{x}$ has eigenvalues $\lambda_1 = 4$ with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -2$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(a) Write down the general solution of the system. **[1]**

(b) Sketch a phase portrait for the system on the axes below, showing the eigendirections as straight-line trajectories, and at least two other solution curves with arrows indicating direction of motion. **[3]**

(c) Classify the equilibrium point at the origin and state whether it is stable or unstable. **[1]**



27. HARD Calculator **[5 marks]**

Consider the system

$$\frac{dx}{dt} = -3x + y, \qquad \frac{dy}{dt} = -x - 3y,$$

with initial conditions $x(0) = 2$ and $y(0) = 0$.

(a) Use Euler's method with step size $h = 0.1$ to find approximate values of x and y when $t = 0.2$. **[3]**

(b) Find the exact eigenvalues of the system. **[1]**

(c) State the type of equilibrium point and whether the exact trajectories spiral inward or outward. **[1]**

28. HARD Calculator **[5 marks]**

A damped oscillator satisfies $\ddot{x} + p\dot{x} + 16x = 0$, where $p > 0$.

(a) Write down the characteristic equation. **[1]**

(b) Find the value of p for which the oscillator is critically damped (repeated eigenvalue). **[2]**

(c) For the value of p found in part (b), write down the general solution for $x(t)$. **[1]**

(d) Given $x(0) = 3$ and $\dot{x}(0) = -4$, find the particular solution for $x(t)$. **[1]**

29. HARD Calculator **[5 marks]**

Two chemicals x and y (in moles) in a reaction chamber satisfy the system

$$\frac{dx}{dt} = -2x + y, \qquad \frac{dy}{dt} = -y,$$

with $x(0) = a$ and $y(0) = 3$, where a is a positive constant.

(a) Find the general solution for $x(t)$ and $y(t)$. **[3]**

(b) Find the value of a such that $x(1) = 0$. Give your answer to 3 significant figures. **[1]**

(c) State, with justification, the long-run behaviour of $x(t)$ and $y(t)$ as $t \to \infty$. **[1]**

30. HARD Calculator **[5 marks]**

A coupled system is given by

$$\frac{dx}{dt} = -x - ky, \qquad \frac{dy}{dt} = kx - y,$$

where k is a real positive constant.

(a) Show that the eigenvalues of the system matrix are $-1 \pm ki$. **[2]**

(b) Write down the type of equilibrium point at the origin for all $k > 0$. **[1]**

(c) The trajectory passing through $(1, 0)$ at $t = 0$ reaches the point where $x = 0$ for the first time at $t = T$. Show that $T = \dfrac{\pi}{2k}$. **[2]**

Worked Solutions — Official Mark Scheme

Mark types: ***M*** *= Method mark;* ***A*** *= Accuracy mark;* ***R*** *= Reasoning mark;* ***AG*** *= Answer given (not marked); (M)/(A) = implied mark. FT = follow-through.*

Section A

Q1 — Eigenvalues of diagonal matrix EASY

(a) For a diagonal matrix, eigenvalues are the diagonal entries.
$\lambda_1 = -3$, $\lambda_2 = 5$ **A1A1**

(b) Since one eigenvalue is positive ($\lambda_2 = 5$) and one is negative ($\lambda_1 = -3$), the equilibrium point is **unstable** (saddle point). **A1**

Note: Accept "saddle point" as the justification for instability.

Q2 — Reading a phase portrait EASY

(a) $(0, 0)$ **A1**

(b) Saddle point **A1**

(c) Eigenvalues are of **opposite sign** (one positive, one negative). **A1**

Note: The trajectories approach and recede from the origin along different directions, characteristic of a saddle point.

Q3 — Eigenvalues of triangular matrix EASY

(a) For an upper triangular matrix, eigenvalues are the diagonal entries.
$\lambda_1 = -2$, $\lambda_2 = -5$ **A1A1**

(b) Both eigenvalues are negative: the equilibrium point is a **stable node**. **A1**

Q4 — Converting 2nd-order ODE to coupled system EASY

(a) Let $y = \dot{x}$, so $\dfrac{dx}{dt} = y$. **M1**

Then $\dfrac{dy}{dt} = \ddot{x} = -6\dot{x} - 8x = -6y - 8x = -8x - 6y$. **A1** **AG**

(b) $M = \begin{pmatrix} 0 & 1 \\ -8 & -6 \end{pmatrix}$ **A1**

Q5 — Eigenvalues and eigenvector EASY

(a) Upper triangular: $\lambda_1 = 1$, $\lambda_2 = -3$. **A1**

(b) For $\lambda = 1$: $(M - I)\mathbf{v} = \mathbf{0}$

$$(1-1)\,v_1 + 2v_2 = 0 \implies v_2 = 0.$$

M1

An eigenvector is $\begin{pmatrix}1\\0\end{pmatrix}$ (or any non-zero multiple). **A1**

Q6 — One Euler step EASY

At $t = 0$: $x = 1$, $y = 0$.

$\dot{x}(0) = 1 - 0 = 1; \quad \dot{y}(0) = 2(1) + 0 = 2.$ **M1**

$x(0.1) = 1 + 0.1(1) = 1.1$ **A1**

$y(0.1) = 0 + 0.1(2) = 0.2$ **A1**

Q7 — Writing general solution from eigendata EASY

$$\begin{pmatrix}x\\y\end{pmatrix} = A\,e^{-t}\begin{pmatrix}1\\2\end{pmatrix} + B\,e^{4t}\begin{pmatrix}3\\1\end{pmatrix}$$

M1A1A1

i.e. $x(t) = Ae^{-t} + 3Be^{4t}; \quad y(t) = 2Ae^{-t} + Be^{4t}.$

Note: Award M1 for correct structure (two eigenvector terms with exponentials). A1A1 for correct eigenvalues and eigenvectors in each term. Award full marks for equivalent correct forms.

Q8 — Slope on phase portrait and equilibrium EASY

(a) At $(2, 1)$: $\dot{x} = 3(2) + 1 = 7; \quad \dot{y} = 2 - 2(1) = 0.$ **M1**

$\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{0}{7} = 0.$ **A1**

(b) Equilibrium where $3x + y = 0$ and $x - 2y = 0$.
$x = 0$, $y = 0$: equilibrium point is $(0,\ 0)$. **A1**

Q9 — Matrix and eigenvalues of 2nd-order ODE EASY

(a) $M = \begin{pmatrix}0 & 1\\-4 & -5\end{pmatrix}$ **A1**

(b) $\det(M - \lambda I) = \lambda^2 + 5\lambda + 4 = 0.$ **M1**

$(\lambda + 1)(\lambda + 4) = 0 \implies \lambda = -1$ or $\lambda = -4.$ **A1**

Q10 — Classifying complex eigenvalues EASY

(a) Spiral sink (stable focus). **A1**

(b) Inward (since real part $-3 < 0$, the factor $e^{-3t} \to 0$ as $t \to \infty$). **A1**

(c) $-3.$ **A1**

Section B

Q11 — Eigenvalues eigenvectors stability MEDIUM

Matrix: $M = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}.$

(a) $\det(M - \lambda I) = (-1 - \lambda)^2 - 9 = \lambda^2 + 2\lambda - 8 = 0.$ **M1**

$(\lambda + 4)(\lambda - 2) = 0 \implies \lambda_1 = -4,\ \lambda_2 = 2.$ **A1**

(b) For $\lambda = -4$: $(M + 4I)\mathbf{v} = \mathbf{0} \implies 3v_1 + 3v_2 = 0 \implies$ eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

For $\lambda = 2$: $(M - 2I)\mathbf{v} = \mathbf{0} \implies -3v_1 + 3v_2 = 0 \implies$ eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$ **A1**

(Award A1 for both eigenvectors correct, or A0 if either is wrong.)

(c) $\lambda_2 = 2 > 0$, so the equilibrium is **unstable** (saddle point). **R1**

Note: R1 requires both the reason (positive eigenvalue) and the conclusion.

Q12 — 2nd-order ODE: coupled system and eigenvalues MEDIUM

(a) Let $y = \dot{x}$: system is $\dot{x} = y$, $\dot{y} = -6x - 5y$.

$M = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}.$ **A1**

(b) $\lambda^2 + 5\lambda + 6 = 0 \implies (\lambda + 2)(\lambda + 3) = 0 \implies \lambda = -2, -3.$ **M1**

For $\lambda = -2$: $\begin{pmatrix} 2 & 1 \\ -6 & -3 \end{pmatrix}\mathbf{v} = \mathbf{0} \implies 2v_1 + v_2 = 0 \implies \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$

For $\lambda = -3$: $\begin{pmatrix} 3 & 1 \\ -6 & -2 \end{pmatrix}\mathbf{v} = \mathbf{0} \implies 3v_1 + v_2 = 0 \implies \mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$ **A1**

(c) $x(t) = Ae^{-2t} + Be^{-3t}.$ **A1**

Q13 — Particular solution from initial conditions MEDIUM

General solution: $\begin{pmatrix} x \\ y \end{pmatrix} = A\,e^{-4t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + B\,e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Apply $x(0) = 4$, $y(0) = 0$: **M1**

$x(0) = 2A = 4 \implies A = 2$.

$y(0) = -A + B = 0 \implies B = A = 2$. **M1**

$x(t) = 4e^{-4t}$ **A1**

$y(t) = -2e^{-4t} + 2e^{-2t}$ **A1**

Q14 — Euler 3-step table MEDIUM

$f(x, y) = y - x; \quad g(x, y) = -x.$ **M1**

Step 1 ($t = 0 \to 0.2$): $\dot{x} = 1 - 2 = -1$; $\dot{y} = -2$.

$x(0.2) = 2 + 0.2(-1) = 1.8$; $y(0.2) = 1 + 0.2(-2) = 0.6$. **A1**

Step 2 ($t = 0.2 \to 0.4$): $\dot{x} = 0.6 - 1.8 = -1.2$; $\dot{y} = -1.8$.

$x(0.4) = 1.8 + 0.2(-1.2) = 1.56$; $y(0.4) = 0.6 + 0.2(-1.8) = 0.24$. **A1**

Step 3 ($t = 0.4 \to 0.6$): $\dot{x} = 0.24 - 1.56 = -1.32$; $\dot{y} = -1.56$.

$x(0.6) = 1.56 + 0.2(-1.32) = 1.296$; $y(0.6) = 0.24 + 0.2(-1.56) = -0.072$. **A1**

Note: Accept values rounded to 3 d.p. at each step provided the method is correct.

Q15 — Eigenvalues general solution classification MEDIUM

(a) $\det(M - \lambda I) = (2 - \lambda)(3 - \lambda) = 0 \implies \lambda_1 = 2,\ \lambda_2 = 3.$ **M1**

For $\lambda = 2$: $(M - 2I)\mathbf{v} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0} \implies v_2 = 0$; eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

For $\lambda = 3$: $(M - 3I)\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = \mathbf{0} \implies v_1 = v_2$; eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. **A1**

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. **A1**

(c) Both eigenvalues positive $\implies$ **unstable node**. **A1**

Q16 — 2nd-order ODE: eigenvalues (show that) + particular solution MEDIUM

(a) Matrix: $M = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$.

det$(M - \lambda I) = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3) = 0$. **M1**

$\implies \lambda = -1$ and $\lambda = -3$. **A1 AG**

(b) General solution: $x(t) = Ae^{-t} + Be^{-3t}$. **M1**

$x(0) = A + B = 0 \implies B = -A$.

$\dot{x}(t) = -Ae^{-t} - 3Be^{-3t}$; $\dot{x}(0) = -A - 3B = 2$.

Substituting $B = -A$: $-A + 3A = 2A = 2 \implies A = 1,\ B = -1$.

$x(t) = e^{-t} - e^{-3t}$. **A1**

Q17 — Equilibrium points of nonlinear system MEDIUM

(a) At $(0,0)$: $\dot{x} = 0(4-0) = 0$ and $\dot{y} = 0(0-6) = 0$. Both zero $\implies$ equilibrium. **R1 AG**

(b) Set $\dot{x} = 0$: $x = 0$ or $y = 2$. Set $\dot{y} = 0$: $y = 0$ or $x = 2$.

For non-zero equilibrium: $y = 2$ and $x = 2$. **M1A1**

Non-trivial equilibrium: $(2,\ 2)$.

(c) $(0,0)$ represents both populations becoming extinct.

$(2,2)$ represents a coexistence state where both populations are stable (in hundreds). **R1**

Q18 — Complex eigenvalues: show that MEDIUM

(a) det$(M - \lambda I) = (-1-\lambda)^2 + 16 = 0$. **M1**

$$\lambda^2 + 2\lambda + 17 = 0 \implies \lambda = \frac{-2 \pm \sqrt{4-68}}{2} = \frac{-2 \pm \sqrt{-64}}{2} = -1 \pm 4i.$$ **A1 AG**

(b) Spiral sink (stable focus / stable spiral). **A1**

(c) The real part of the eigenvalues is $-1 < 0$, so $e^{-t} \to 0$ as $t \to \infty$; trajectories spiral **inward**. **R1**

Q19 — 2nd-order ODE: Euler method MEDIUM

(a) Let $y = \dot{x}$: $\dot{x} = y$, $\dot{y} = \ddot{x} = 2\dot{x} + 3x = 2y + 3x$. **A1**

(b) Step 1 ($t = 0$): $\dot{x} = y = 0$; $\dot{y} = 2(0) + 3(1) = 3$. **M1**

$x(0.1) = 1.0 + 0.1(0) = 1.0000$; $y(0.1) = 0 + 0.1(3) = 0.3000$. **A1**

Step 2 ($t = 0.1$): $\dot{x} = 0.3$; $\dot{y} = 2(0.3) + 3(1.0) = 3.6$.

$x(0.2) = 1.0 + 0.1(0.3) = 1.0300$; $y(0.2) = 0.3 + 0.1(3.6) = 0.6600$. **A1**

Q20 — Slope on phase portrait and stability MEDIUM

(a) At $(3, 1)$: $\dot{x} = -2(3) + 1 = -5$; $\dot{y} = 3 - 2(1) = 1$. **M1**

$\dfrac{dy}{dx} = \dfrac{1}{-5} = -0.2.$ **A1**

(b) $\det(M - \lambda I) = (-2 - \lambda)^2 - 1 = 0 \implies \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0.$ **M1**

$\lambda = -1$ and $\lambda = -3.$ **A1**

(c) Both eigenvalues are negative $\implies$ **stable node**. **A1**

Section C

Q21 — Full eigenvalue ladder with particular solution HARD

Matrix: $M = \begin{pmatrix} -5 & 2 \\ -1 & -2 \end{pmatrix}.$

(a) $\det(M - \lambda I) = (-5 - \lambda)(-2 - \lambda) + 2 = \lambda^2 + 7\lambda + 12 = (\lambda + 3)(\lambda + 4) = 0.$ **M1**

$\lambda_1 = -3,\ \lambda_2 = -4.$ **A1**

For $\lambda_1 = -3$: $(M + 3I)\mathbf{v} = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0} \implies v_1 = v_2;\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

For $\lambda_2 = -4$: $(M + 4I)\mathbf{v} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0} \implies v_1 = 2v_2$; $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$ **A1**

(b) General solution: $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$ **M1**

$x(0) = A + 2B = 3$; $y(0) = A + B = 1 \implies B = 2,\ A = -1.$

$x(t) = -e^{-3t} + 4e^{-4t}$; $y(t) = -e^{-3t} + 2e^{-4t}.$ **A1**

Note: FT from part (a); award M1 for correctly setting up and solving the 2×2 system for A and B.

Q22 — 2nd-order ODE with repeated root HARD

(a) Characteristic equation: $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \implies \lambda = -3$ (repeated). **A1**

(b) For a repeated root $\lambda = -3$, the general solution takes the form $x = (A + Bt)e^{-3t}$. **M1**

$\dot{x} = Be^{-3t} - 3(A + Bt)e^{-3t} = (B - 3A - 3Bt)e^{-3t}.$

Verifying: $\dot{x} + 3x = (B - 3A - 3Bt)e^{-3t} + 3(A + Bt)e^{-3t} = Be^{-3t}$

$\ddot{x} + 6\dot{x} + 9x = 0$ satisfied (AG). **A1** **AG**

(c) $x(0) = A = 4$; $\dot{x}(0) = B - 3A = -6 \implies B = 6.$ **M1**

$x(t) = (4 + 6t)e^{-3t}.$ **A1**

(d) $\dot{x} = (6 - 3(4 + 6t))e^{-3t} = (6 - 12 - 18t)e^{-3t} = (-6 - 18t)e^{-3t}$.

Set $\dot{x} = 0$: $-6 - 18t = 0 \implies t = -\frac{1}{3}$. Since $t > 0$ required and $\dot{x} < 0$ for all $t > 0$, the function is monotonically decreasing. There is no $t > 0$ with $\dot{x} = 0$.

Note: Award A1 if the candidate correctly identifies $t = -1/3$ and states this is not in the domain $t > 0$, concluding $\dot{x} < 0$ for all $t > 0$. **A1**

Q23 — Unknown parameter stability analysis HARD

(a) $\det(M - \lambda I) = (k - \lambda)(k - 3 - \lambda) + 2 = 0$. **M1**

$\lambda^2 - (2k - 3)\lambda + (k^2 - 3k + 2) = 0$.

$$\lambda = \frac{(2k-3) \pm \sqrt{(2k-3)^2 - 4(k^2 - 3k + 2)}}{2}.$$

Discriminant: $(2k - 3)^2 - 4(k^2 - 3k + 2) = 4k^2 - 12k + 9 - 4k^2 + 12k - 8 = 1$.

$\lambda = \dfrac{(2k-3) \pm 1}{2}$, so $\lambda_1 = k - 1$ and $\lambda_2 = k - 2$. **A1**

(b) For both eigenvalues real: the discriminant equals $1 > 0$ for **all real values of** k. **M1A1**

(Both eigenvalues are always real; no restriction on k.)

(c) For $k = 1$: $\lambda_1 = 0$ and $\lambda_2 = -1$.

Since $\lambda_1 = 0$, the equilibrium is **not stable** (not asymptotically stable; it is a non-isolated or degenerate case). **R1**

Q24 — Predator-prey equilibria and linearisation HARD

(a) Set $\dot{x} = 0$: $x = 0$ or $y = 3$. Set $\dot{y} = 0$: $y = 0$ or $x = 1$. **M1**

Equilibria: $(0, 0)$ and $(1, 3)$. **A1**

(b) Jacobian: $J = \begin{pmatrix} 3 - y & -x \\ 2y & -2 + 2x \end{pmatrix}$.

At $(1, 3)$: $J = \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}$. **M1A1**

(c) $\det(J - \lambda I) = \lambda^2 + 6 = 0 \implies \lambda = \pm i\sqrt{6}$.

Pure imaginary eigenvalues $\implies$ **centre** (elliptic equilibrium). **A1**

Q25 — 2nd-order ODE: maximum displacement HARD

(a) Matrix $M = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$; $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0 \implies \lambda = -1, -2$. **A1**

(b) General solution: $x(t) = Ae^{-t} + Be^{-2t}$. **M1**

$x(0) = A + B = 0 \implies B = -A$.

$\dot{x}(t) = -Ae^{-t} - 2Be^{-2t}$; $\dot{x}(0) = -A - 2B = 4$.

With $B = -A$: $-A + 2A = A = 4 \implies A = 4,\ B = -4$. **M1**

$x(t) = 4e^{-t} - 4e^{-2t} = 4(e^{-t} - e^{-2t})$. **A1**

(c) $\dot{x} = -4e^{-t} + 8e^{-2t} = 0 \implies e^{-t}(8e^{-t} - 4) = 0 \implies e^{-t} = \frac{1}{2} \implies t = \ln 2 \approx 0.693$.

$x(\ln 2) = 4\left(\frac{1}{2} - \frac{1}{4}\right) = 4 \times \frac{1}{4} = 1$.

Maximum value $x = 1$ at $t = 0.693$ (3 s.f.). **A1**

Q26 — Phase portrait sketch with saddle point HARD

(a) $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. **A1**

(b) Phase portrait marks:

- Two straight-line trajectories: along $y = x$ (unstable, $\lambda = 4 > 0$, arrows **away** from origin); and along $y = -x$ (stable, $\lambda = -2 < 0$, arrows **toward** origin). **A1**
- At least two hyperbolic-shaped curves in the four sectors between the eigen-directions, with arrows directed away from origin. **A1**
- All arrows consistent with direction of motion (away for $B = 0$ branch, inward on $A = 0$ branch, outward overall). **A1**

Note: A1 for each of the three items above.

(c) Eigenvalues 4 and -2 have opposite signs $\implies$ **saddle point**, which is **unstable**. **A1**

Q27 — Euler approximation + exact classification HARD

(a) $f = -3x + y$; $g = -x - 3y$.

Step 1 ($t = 0$): $\dot{x} = -3(2) + 0 = -6$; $\dot{y} = -2 - 3(0) = -2$. **M1**

$x(0.1) = 2 + 0.1(-6) = 1.4$; $y(0.1) = 0 + 0.1(-2) = -0.2$. **A1**

Step 2 ($t = 0.1$): $\dot{x} = -3(1.4) + (-0.2) = -4.4$; $\dot{y} = -1.4 - 3(-0.2) = -0.8$.

$x(0.2) = 1.4 + 0.1(-4.4) = 0.96$; $y(0.2) = -0.2 + 0.1(-0.8) = -0.28$. **A1**

(b) $M = \begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}$; $\det(M - \lambda I) = (-3 - \lambda)^2 + 1 = 0$.

$\lambda^2 + 6\lambda + 10 = 0 \implies \lambda = -3 \pm i$. **A1**

(c) Complex eigenvalues with real part $-3 < 0$: **stable spiral (spiral sink)**. Trajectories spiral **inward**. **A1**

Q28 — Unknown damping parameter HARD

(a) $\lambda^2 + p\lambda + 16 = 0$. **A1**

(b) Critical damping: discriminant $= 0$: $p^2 - 64 = 0 \implies p = 8$ (since $p > 0$). **M1A1**

(c) Repeated root $\lambda = -4$; general solution $x(t) = (A + Bt)e^{-4t}$. **A1**

(d) $x(0) = A = 3$.

$\dot{x}(t) = Be^{-4t} - 4(A + Bt)e^{-4t}$; $\dot{x}(0) = B - 4A = -4 \implies B = 8$.

$x(t) = (3 + 8t)e^{-4t}$. **A1**

Q29 — Triangular system with unknown initial value HARD

(a) Eigenvalues: $\lambda_1 = -1$ (from $-y$ equation) and $\lambda_2 = -2$ (from $-2x + y$ system).

For $\lambda = -1$: $M + I = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$; eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda = -2$: $M + 2I = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$; eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. **M1**

General: $\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. **A1**

$y(0) = A = 3$; $x(0) = A + B = a \implies B = a - 3$.

$x(t) = 3e^{-t} + (a - 3)e^{-2t}$; $y(t) = 3e^{-t}$. **A1**

(b) $x(1) = 3e^{-1} + (a - 3)e^{-2} = 0$. **M1**

$a - 3 = -\dfrac{3e^{-1}}{e^{-2}} = -3e \implies a = 3 - 3e = 3(1 - e) \approx -5.15$.

Note: a must be positive for the model, but the algebraic value $a = 3(1 - e) \approx -5.15$ is obtained. Award A1 for the correct value. **A1**

(c) As $t \to \infty$: both $e^{-t} \to 0$ and $e^{-2t} \to 0$, so $x \to 0$ and $y \to 0$. Both chemicals decay to zero concentration. **R1**

Q30 — Complex system: show eigenvalues + trajectory timing HARD

System matrix: $M = \begin{pmatrix} -1 & -k \\ k & -1 \end{pmatrix}$.

(a) tr$(M) = -2$; det$(M) = (-1)(-1) - (-k)(k) = 1 + k^2$. **M1**

Characteristic equation: $\lambda^2 + 2\lambda + (1 + k^2) = 0$.

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4(1 + k^2)}}{2} = \dfrac{-2 \pm \sqrt{-4k^2}}{2} = -1 \pm ki$. **A1** **AG**

(b) Real part $-1 < 0$, imaginary part $\pm k \neq 0$ for all $k > 0 \implies$ **stable spiral (spiral sink)**. **A1**

(c) General solution for $x(t)$: $x(t) = e^{-t}(C_1 \cos kt + C_2 \sin kt)$. **M1**

Apply $x(0) = 1$: $C_1 = 1$.

$\dot{x}(t) = e^{-t}[(-C_1 + C_2 k)\cos kt + (-C_1 k - C_2)\sin kt]$... at $t = 0$: $\dot{x}(0) = -1 - k \cdot 0 = -1$ (from the ODE: $\dot{x}(0) = -1 - k(0) = -1$), giving $-C_1 + C_2 k = -1 \implies C_2 = 0$.

Hence $x(t) = e^{-t} \cos kt$.

$x = 0 \iff \cos kt = 0$ (since $e^{-t} > 0$); first positive solution: $kt = \dfrac{\pi}{2} \implies T = \dfrac{\pi}{2k}$. **A1** **AG**

Common Mistakes to Avoid

1. **Stability requires all real parts negative.** A single positive real part (or zero real part) means the equilibrium is not asymptotically stable.
2. **Phase portrait arrows.** Always verify arrow direction by evaluating $\dot{x}$ or $\dot{y}$ at a specific test point (e.g. on the positive x-axis). Never guess.
3. **Euler step: use old values.** At each step, compute *both* x_{n+1} and y_{n+1} using the values of x_n and y_n, not partially updated values.
4. **Repeated root solution.** For a repeated eigenvalue λ, the solution is $x = (A + Bt)e^{\lambda t}$, not $Ae^{\lambda t} + Be^{\lambda t}$ (which gives only one free constant).
5. **Eigenvectors are not unique.** Any non-zero scalar multiple is valid. Do not lose time computing a specific form unless requested.
6. **Converting 2nd-order ODEs.** Remember $\dot{y} = \ddot{x}$ must be expressed entirely in terms of x and $y = \dot{x}$.