IB Math AI HL — Topic 1 [Matrices]

Matrices

Introduction · Operations · Determinants & Inverses · Systems of Linear Equations

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________ Date:____________

Score:________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated. GDC permitted throughout.

Key Formulae — Matrices

Order of a matrix: An $m \times n$ matrix has m rows and n columns.

Addition/Subtraction: $(A \pm B)_{ij} = a_{ij} \pm b_{ij}$ (same order required)

Scalar multiplication: $(kA)_{ij} = k\,a_{ij}$

Matrix multiplication: $(AB)_{ij} = \sum_k a_{ik}\,b_{kj}$

A is $m \times p$, B is $p \times n \Rightarrow AB$ is $m \times n$

Identity matrix: $\mathbf{I}_n$ — $n \times n$ matrix with 1s on diagonal, 0s elsewhere

Zero matrix: $\mathbf{0}$ — all entries zero

Determinant (2×2)**:** $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Inverse (2×2)**:** $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

A^{-1} exists $\Leftrightarrow \det A \neq 0$ (A is non-singular)

Solving $A\mathbf{x} = \mathbf{b}$: $\mathbf{x} = A^{-1}\mathbf{b}$ (when $\det A \neq 0$)

3×3 systems: Use GDC (rref or solve) or row reduction

GDC Tips

- **Enter a matrix:** `MATRIX` → `EDIT` → set dimensions → enter values. Use `[A]`, `[B]`, etc.
- **Multiply matrices:** On the home screen type `[A]*[B]` and press `ENTER`.
- **Find the determinant:** `MATRIX` → `MATH` → `det(` then select your matrix.
- **Find the inverse:** On the home screen type `[A]^-1` and press `ENTER`.
- **Solve a system:** Enter coefficient matrix as `[A]` and constant vector as `[B]`, then compute `[A]^-1*[B]`.
- **Row reduce (rref):** `MATRIX` → `MATH` → `rref(` — useful for 3×3 systems and checking for

unique/no/infinite solutions.

Common Mistakes to Avoid

1. **Matrix multiplication is not commutative.** $AB \neq BA$ in general. Always check the order: for AB, the number of columns of A must equal the number of rows of B.
2. **Forgetting to swap and negate when finding A^{-1}.** For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse swaps a and d, then negates b and c. A common mistake is negating a and d instead.
3. **Dividing by the determinant, not multiplying by $1/\det A$.** Write the formula correctly: $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
4. **Assuming $\det A \neq 0$ without checking.** Always compute $\det A$ first — if $\det A = 0$ the system has no unique solution.
5. **Sign error in the determinant.** $\det A = ad - bc$, not $ad + bc$. The minus sign is frequently dropped under exam pressure.
6. **Confusing the solution vector.** When solving $A\mathbf{x} = \mathbf{b}$, the answer is $\mathbf{x} = A^{-1}\mathbf{b}$ (multiply A^{-1} on the left). Do not compute $\mathbf{b}A^{-1}$.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY No Calculator **[3 marks]**

The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 5 & -3 & 0 \\ 2 & 7 & -1 \end{pmatrix}.$$

(a) Write down the order of **P**. [1]

(b) Write down the value of the entry in row 2, column 3. [1]

(c) Write down the value of the entry in row 1, column 2. [1]

2. EASY No Calculator **[3 marks]**

Let $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$.

(a) Find $2\mathbf{A}$. [1]

(b) Find $\mathbf{A} + \mathbf{B}$. [2]

3. **EASY** **No Calculator** [3 marks]

Let $\mathbf{M} = \begin{pmatrix} 6 & -2 \\ 1 & 4 \end{pmatrix}$.

(a) Calculate $\det(\mathbf{M})$. [1]

(b) State whether $\mathbf{M}$ has an inverse, giving a reason. [1]

(c) Find $\mathbf{M}^{-1}$, giving exact values. [1]

4. **EASY** **No Calculator** [3 marks]

Let $\mathbf{C} = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

(a) Write down the order of the matrix product $\mathbf{Cv}$. [1]

(b) Find $\mathbf{Cv}$. [2]

5. **EASY** **Calculator** [3 marks]

The system of equations

$$5x + 2y = 11, \qquad 3x - y = 8$$

can be written in the form $\mathbf{Ax} = \mathbf{b}$.

(a) Write down the matrix $\mathbf{A}$ and the column vector $\mathbf{b}$. [1]

(b) Hence find the values of x and y using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. [2]

6. EASY | No Calculator **[3 marks]**

Let $\mathbf{F} = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{G} = \begin{pmatrix} 0 & -1 \\ 5 & 2 \end{pmatrix}$.

(a) Find $\mathbf{F} - \mathbf{G}$. [1]

(b) Find $3\mathbf{G}$. [1]

(c) Find $\mathbf{F} - \mathbf{G} + 3\mathbf{G}$, simplifying your answer. [1]

7. EASY | No Calculator **[3 marks]**

Let $\mathbf{H} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{K} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$.

Find the matrix product $\mathbf{HK}$.

8. EASY | No Calculator **[3 marks]**

Determine whether each matrix below is singular or non-singular. Show your working.

$$\mathbf{P} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}.$$

9. EASY Calculator **[3 marks]**

Let $\mathbf{N} = \begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}$.

(a) Find $\mathbf{N}^{-1}$, giving exact values. [2]

(b) Verify your answer by showing that $\mathbf{N}\mathbf{N}^{-1} = \mathbf{I}$. [1]

10. EASY Calculator **[3 marks]**

A market stall sells two types of drink: lemonade and iced tea. One morning, Priya sells 4 cups of lemonade and 3 cups of iced tea for a total of \$14.50. Later that afternoon she sells 2 cups of lemonade and 5 cups of iced tea for a total of \$13.50.

Let ℓ be the price of one cup of lemonade and t the price of one cup of iced tea, both in dollars.

(a) Write down two equations in ℓ and t. [1]

(b) Express this system as $\mathbf{Ax} = \mathbf{b}$ and hence find ℓ and t. [2]

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM Calculator **[4 marks]**

Let $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 8 & 5 \\ -3 & 6 \end{pmatrix}$.

Find the matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$.

12. MEDIUM Calculator **[4 marks]**

Solve the system of equations

$$2x + y - z = 5, \qquad x - 3y + 2z = -4, \qquad 3x + 2y + z = 12.$$

13. MEDIUM No Calculator [4 marks]

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

(a) Find $\mathbf{AB}$. [2]

(b) Find $\det(\mathbf{AB})$. [1]

(c) Verify that $\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$. [1]

14. MEDIUM Calculator [4 marks]

A production planner uses the matrix $\mathbf{T} = \begin{pmatrix} 4 & 3 \\ 2 & k \end{pmatrix}$, where $k \in \mathbb{R}$, to convert raw material quantities into product quantities.

(a) Find the values of k for which $\mathbf{T}$ is singular. [2]

(b) Given that $k = 4$, find $\mathbf{T}^{-1}$. [2]

15. MEDIUM Calculator [4 marks]

Let $\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$.

(a) Find $\mathbf{D}^2$ and $\mathbf{D}^3$. [2]

(b) Write down $\mathbf{D}^n$ for any positive integer n. [1]

(c) Hence find $\mathbf{D}^{10}$. [1]

16. MEDIUM Calculator [4 marks]

Find the determinant of the matrix $\mathbf{Q} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 0 & 5 \end{pmatrix}$.

Show all expansion working clearly.

17. MEDIUM No Calculator [4 marks]

Let $\mathbf{R} = \begin{pmatrix} 5 & -2 & 4 \\ -2 & 1 & 0 \\ 4 & 0 & 3 \end{pmatrix}$.

(a) Write down $\mathbf{R}^{\mathrm{T}}$. [1]

(b) State whether $\mathbf{R}$ is symmetric, giving a reason. [1]

(c) For $\mathbf{S} = \begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix}$, find $\mathbf{S} + \mathbf{S}^{\mathrm{T}}$ and state the property of the result. [2]

18. MEDIUM Calculator [4 marks]

A nutritionist prepares two types of smoothie, Type X and Type Y. The table below gives the amounts (in grams) of protein and carbohydrates per serving of each smoothie.

Nutrient	**Type X (per serving)**	**Type Y (per serving)**
Protein (g)	8	5
Carbohydrates (g)	12	20

A client requires exactly 46 g of protein and 148 g of carbohydrates from these two smoothies. Let x be the number of servings of Type X and y the number of servings of Type Y.

(a) Write the nutritional requirements as a matrix equation $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{c}$, stating the matrix $\mathbf{M}$ and the vector $\mathbf{c}$. [1]

(b) Find $\mathbf{M}^{-1}$ and hence find x and y. [3]

19. **MEDIUM** **No Calculator** [4 marks]

Let $\mathbf{A} = \begin{pmatrix} p & 4 \\ 1 & p \end{pmatrix}$, where $p \in \mathbb{R}$.

(a) Find $\det(\mathbf{A})$ in terms of p. [1]

(b) Find the values of p for which $\mathbf{A}$ is singular. [2]

(c) For the value of p where $p > 0$ and $\mathbf{A}$ is **non-singular** (use $p = 3$), write down $\mathbf{A}^{-1}$. [1]

20. **MEDIUM** **Calculator** [4 marks]

Consider the system of equations:

$$x + 2y - z = 3, \qquad 2x - y + 3z = 7, \qquad 3x + y + 2z = k.$$

(a) Find the value of k for which the system is consistent (has at least one solution). [2]

(b) For this value of k, solve the system. [2]

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD Calculator [5 marks]

Let $\mathbf{A}(k) = \begin{pmatrix} k & 2 \\ k-1 & 3 \end{pmatrix}$, where $k \in \mathbb{R}$.

(a) Show that $\det(\mathbf{A}(k)) = 3k - 2(k-1)$. Hence find the value of k for which $\mathbf{A}(k)$ is singular. [2]

(b) For $k = 5$, find $\mathbf{A}^{-1}(5)$ and hence solve the system $\mathbf{A}(5)\mathbf{x} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$. [3]

22. HARD Calculator [5 marks]

Let $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$.

Find the matrix $\mathbf{X}$ such that $\mathbf{BX} = \mathbf{A}^{-1}$.

23. HARD Calculator **[5 marks]**

Consider the system

$$x + 2y - z = 4, \quad 2x + 5y + z = 10, \quad x + 3y + 2z = 6.$$

(a) Use row reduction (or your GDC) to show that the system has infinitely many solutions. [2]

(b) Express the general solution in terms of a parameter t, where $z = t$. [3]

24. HARD No Calculator **[5 marks]**

Let $\mathbf{E} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Find $\mathbf{E}^2$ and $\mathbf{E}^3$. [2]

(b) Hence conjecture a formula for $\mathbf{E}^n$, where n is a positive integer. [1]

(c) Use your formula to find the value of n such that the entry in row 1, column 2 of $\mathbf{E}^n$ equals 20. [2]

25. HARD Calculator **[5 marks]**

An engineer models a heat-transfer network using the matrix equation $\mathbf{MT} = \mathbf{q}$, where

$$\mathbf{M} = \begin{pmatrix} 3 & m \\ 6 & 4 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 14 \\ q_2 \end{pmatrix}.$$

Here T_1 and T_2 are temperatures in °C and m, q_2 are real constants.

(a) Find the value of m for which the system has no unique solution. [2]

(b) For $m = 1$, find $\mathbf{M}^{-1}$ and hence find T_1 and T_2 when $q_2 = 22$. [3]

26. HARD No Calculator **[5 marks]**

Let $\mathbf{A} = \begin{pmatrix} a & 2 \\ 1 & a \end{pmatrix}$, where $a \in \mathbb{R}$.

(a) Show that $\det(\mathbf{A}^2) = (a^2 - 2)^2$. [3]

(b) Find the values of a for which $\det(\mathbf{A}^2) = 0$. [1]

(c) Explain why, for a value of a found in part (b), the system $\mathbf{Ax} = \mathbf{b}$ need not have a unique solution, regardless of $\mathbf{b}$. [1]

27. HARD Calculator **[5 marks]**

Consider the system of equations

$$2x - y + 3z = 8, \quad x + 2y - z = 3, \quad 4x - 5y + 7z = c.$$

(a) Using row reduction, find the value of c for which the third equation is a linear combination of the first two (i.e. the system has infinitely many solutions). [3]

(b) For any other value of c, describe the nature of the solution set and justify your answer. [2]

28. HARD Calculator **[5 marks]**

A data encryption system encodes pairs of numbers using the matrix $\mathbf{E} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$.

To encode a message, a plaintext column vector $\mathbf{p}$ is multiplied to give the ciphertext vector $\mathbf{c} = \mathbf{Ep}$.

(a) Encode the plaintext vector $\mathbf{p} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$, finding $\mathbf{c}$. [1]

(b) Find $\mathbf{E}^{-1}$, giving exact values. [2]

(c) A received ciphertext vector is $\begin{pmatrix} 22 \\ 38 \end{pmatrix}$. Decode this vector to find the original plaintext. [2]

29. HARD Calculator **[5 marks]**

The matrix $\mathbf{S} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & k & 2 \\ 3 & 2 & 1 \end{pmatrix}$ represents the coefficient matrix of a 3×3 system of equations.

(a) Find $\det(\mathbf{S})$ in terms of k. [3]

(b) Find the value(s) of k for which the system does not have a unique solution. [1]

(c) State the value of k for which the system has a unique solution and the coefficient matrix has the smallest possible positive determinant. [1]

30. HARD Calculator **[5 marks]**

Two river gauging stations record water levels u and v (in metres) over several days. On a particular day the readings satisfy:

$$(r+2)\,u + 3v = 18, \qquad 4u + (r-1)\,v = r + 6,$$

where r is a real parameter.

(a) Write this system as $\mathbf{A}(r)\mathbf{x} = \mathbf{b}(r)$ and find $\det(\mathbf{A}(r))$ in terms of r. [2]

(b) Find the values of r for which the system does not have a unique solution. [2]

(c) For $r = 3$, find u and v. [1]

Worked Solutions

Official Mark-Scheme Style — M / A / R notation

Section A — Easy

Q1 — Introduction to Matrices EASY

(a) 2×3 **A1**

(b) Entry in row 2, column 3: -1 **A1**

(c) Entry in row 1, column 2: -3 **A1**

Q2 — Operations with Matrices EASY

(a) $2\mathbf{A} = \begin{pmatrix} 8 & -2 \\ 0 & 6 \end{pmatrix}$ **A1**

(b) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 4 + (-2) & -1 + 5 \\ 0 + 1 & 3 + (-3) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$ **M1A1**

Note: M1 for attempting entry-by-entry addition; A1 for all four entries correct.

Q3 — Determinants & Inverses EASY

(a) $\det(\mathbf{M}) = (6)(4) - (-2)(1) = 24 + 2 = 26$ **A1**

(b) Since $\det(\mathbf{M}) = 26 \neq 0$, the matrix $\mathbf{M}$ is non-singular and has an inverse. **R1**

(c) $\mathbf{M}^{-1} = \frac{1}{26} \begin{pmatrix} 4 & 2 \\ -1 & 6 \end{pmatrix}$ **A1**

Note: Accept entries as fractions: $\frac{4}{26} = \frac{2}{13}$, etc. Do not accept decimal approximations for "exact values".

Q4 — Matrix Multiplication EASY

(a) $\mathbf{C}$ is 2×2 and $\mathbf{v}$ is 2×1, so $\mathbf{Cv}$ is 2×1. **A1**

(b) $\mathbf{Cv} = \begin{pmatrix} 3(4) + (-1)(-2) \\ 2(4) + (5)(-2) \end{pmatrix} = \begin{pmatrix} 12 + 2 \\ 8 - 10 \end{pmatrix} = \begin{pmatrix} 14 \\ -2 \end{pmatrix}$ **M1A1**

Note: M1 for correct method of multiplication; A1 for both entries correct.

Q5 — Solving Systems with Matrices **EASY**

(a) $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 8 \end{pmatrix}$ **A1**

(b) $\det(\mathbf{A}) = (5)(-1) - (2)(3) = -5 - 6 = -11$ (M1)

$\mathbf{A}^{-1} = \frac{1}{-11}\begin{pmatrix} -1 & -2 \\ -3 & 5 \end{pmatrix}$

$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{-11}\begin{pmatrix} (-1)(11) + (-2)(8) \\ (-3)(11) + (5)(8) \end{pmatrix} = \frac{1}{-11}\begin{pmatrix} -27 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{27}{11} \\ -\frac{7}{11} \end{pmatrix}$

$x = \frac{27}{11} \approx 2.45, \quad y = -\frac{7}{11} \approx -0.636$ **A1**

Note: Award (M1) for setting up inverse; A1 for both values correct (accept fractions or 3 sf decimals).

Q6 — Operations with Matrices **EASY**

(a) $\mathbf{F} - \mathbf{G} = \begin{pmatrix} 1-0 & 3-(-1) \\ -2-5 & 4-2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -7 & 2 \end{pmatrix}$ **A1**

(b) $3\mathbf{G} = \begin{pmatrix} 0 & -3 \\ 15 & 6 \end{pmatrix}$ **A1**

(c) $\mathbf{F} - \mathbf{G} + 3\mathbf{G} = \mathbf{F} + 2\mathbf{G} = \begin{pmatrix} 1+0 & 3+(-2) \\ -2+10 & 4+4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 8 & 8 \end{pmatrix}$ **A1**

Note: Allow FT from (a) and (b) provided the simplification $\mathbf{F} - \mathbf{G} + 3\mathbf{G} = \mathbf{F} + 2\mathbf{G}$ *is used.*

Q7 — Matrix Multiplication **EASY**

$\mathbf{HK} = \begin{pmatrix} 2(1) + 0(2) & 2(4) + 0(-1) \\ (-1)(1) + 3(2) & (-1)(4) + 3(-1) \end{pmatrix}$ **M1**

$= \begin{pmatrix} 2 & 8 \\ 5 & -7 \end{pmatrix}$ **A1A1**

Note: M1 for correct method; first A1 for top row correct, second A1 for bottom row correct.

Q8 — Determinants (Singular / Non-singular) **EASY**

For $\mathbf{P}$: $\det(\mathbf{P}) = (4)(3) - (6)(2) = 12 - 12 = 0$ **A1**

$\det(\mathbf{P}) = 0$, so $\mathbf{P}$ is **singular**. **A1**

For $\mathbf{Q}$: $\det(\mathbf{Q}) = (5)(4) - (-1)(2) = 20 + 2 = 22 \neq 0$ **A1**

$\det(\mathbf{Q}) = 22 \neq 0$, so $\mathbf{Q}$ is **non-singular**.

Note: Award A1 for each determinant computed correctly and A1 for correctly classifying both matrices. All three A1 marks may be awarded.

Q9 — Determinants & Inverses **EASY**

(a) $\det(\mathbf{N}) = (7)(2) - (3)(4) = 14 - 12 = 2$ (M1)

$$\mathbf{N}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -3 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{2} \\ -2 & \frac{7}{2} \end{pmatrix}$$ **A1**

(b) $$\mathbf{NN}^{-1} = \begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & -\frac{3}{2} \\ -2 & \frac{7}{2} \end{pmatrix} = \begin{pmatrix} 7-6 & -\frac{21}{2}+\frac{21}{2} \\ 4-4 & -6+7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ **A1**

Note: A1 for showing all four entries of the product equal those of $\mathbf{I}_2$.

Q10 — Solving Systems with Matrices (Context) **EASY**

(a) $4\ell + 3t = 14.50 \quad \text{and} \quad 2\ell + 5t = 13.50$ **A1**

(b) $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 14.50 \\ 13.50 \end{pmatrix}$ (M1)

$\det(\mathbf{A}) = 20 - 6 = 14$

$$\mathbf{A}^{-1} = \frac{1}{14}\begin{pmatrix} 5 & -3 \\ -2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \ell \\ t \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 5(14.50)+(-3)(13.50) \\ (-2)(14.50)+4(13.50) \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 72.5-40.5 \\ -29+54 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 32 \\ 25 \end{pmatrix}$$

$\ell = \frac{32}{14} \approx \$2.29, \quad t = \frac{25}{14} \approx \$1.79$ **A1**

Note: Accept exact fractions $\ell = \frac{16}{7}$, $t = \frac{25}{14}$. *Award A1 for both values correct.*

Section B — Medium

Q11 — Matrix Equation AX

$\mathbf{AX} = \mathbf{B} \Rightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ **M1**

$\det(\mathbf{A}) = (3)(2) - (-1)(1) = 6 + 1 = 7$ (M1)

$$\mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$ **A1**

$$\mathbf{X} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 8 & 5 \\ -3 & 6 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 16+3 & 10-6 \\ 8-9 & 5+18 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 19 & 4 \\ -1 & 23 \end{pmatrix}$$

$\mathbf{X} = \begin{pmatrix} \frac{19}{7} & \frac{4}{7} \\ -\frac{1}{7} & \frac{23}{7} \end{pmatrix}$ **A1**

Note: Award M1 for recognising $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$; (M1) for computing $\mathbf{A}^{-1}$; A1A1 for the correct final matrix (accept decimals to 3 sf).

Q12 — 3×3 System MEDIUM

Writing as augmented matrix and using GDC (rref): **M1**

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 1 & -3 & 2 & -4 \\ 3 & 2 & 1 & 12 \end{array}\right)$$

$R2 \leftarrow 2R2 - R1$: $(0, -7, 5 \,|\, -13)$; $R3 \leftarrow 2R3 - 3R1$: $(0, 1, 5 \,|\, 9)$ **(M1)**

From $-7y + 5z = -13$ and $y + 5z = 9$: subtracting: $-8y = -22 \Rightarrow y = \frac{11}{4}$; $z = \frac{9-11/4}{5} = \frac{5}{4}$; $x = \frac{5-y+z}{2} = \frac{7}{4}$

$x = \dfrac{7}{4}$, $y = \dfrac{11}{4}$, $z = \dfrac{5}{4}$ **A1A1**

Verify: $2(\frac{7}{4}) + \frac{11}{4} - \frac{5}{4} = \frac{20}{4} = 5\checkmark$; $\frac{7}{4} - \frac{33}{4} + \frac{10}{4} = -4\checkmark$; $\frac{21}{4} + \frac{22}{4} + \frac{5}{4} = 12\checkmark$

Note: M1 for setting up augmented matrix; (M1) for correct row operations; A1 for two correct values; A1 for third value.

Q13 — Multiplication then Determinant MEDIUM

(a) $\mathbf{AB} = \begin{pmatrix} 2(1)+1(2) & 2(0)+1(-1) \\ (-1)(1)+3(2) & (-1)(0)+3(-1) \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 5 & -3 \end{pmatrix}$ **M1A1**

(b) $\det(\mathbf{AB}) = (4)(-3) - (-1)(5) = -12 + 5 = -7$ **A1**

(c) $\det(\mathbf{A}) = (2)(3) - (1)(-1) = 6 + 1 = 7$; $\det(\mathbf{B}) = (1)(-1) - (0)(2) = -1$

$\det(\mathbf{A}) \cdot \det(\mathbf{B}) = 7 \times (-1) = -7 = \det(\mathbf{AB})$ **A1**

Note: M1A1 for $\mathbf{AB}$; A1 for $\det(\mathbf{AB})$; final A1 for verifying the product rule.

Q14 — Unknown Parameter in Inverse MEDIUM

(a) $\det(\mathbf{T}) = 4k - 3(2) = 4k - 6$ **M1**

$\mathbf{T}$ is singular when $4k - 6 = 0 \Rightarrow k = \dfrac{3}{2}$ **A1**

(b) For $k = 4$: $\mathbf{T} = \begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix}$; $\det(\mathbf{T}) = 16 - 6 = 10$ (M1)

$\mathbf{T}^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -3 \\ -2 & 4 \end{pmatrix}$ **A1**

Note: M1 for computing determinant formula; A1 for $k = 3/2$; (M1) for computing det with $k = 4$; A1 for correct inverse.

Q15 — Powers of a Diagonal Matrix MEDIUM

(a) $\mathbf{D}^2 = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$ **A1**

$\mathbf{D}^3 = \mathbf{D}^2 \cdot \mathbf{D} = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 27 & 0 \\ 0 & -8 \end{pmatrix}$ **A1**

(b) $\mathbf{D}^n = \begin{pmatrix} 3^n & 0 \\ 0 & (-2)^n \end{pmatrix}$ **A1**

(c) $\mathbf{D}^{10} = \begin{pmatrix} 3^{10} & 0 \\ 0 & (-2)^{10} \end{pmatrix} = \begin{pmatrix} 59049 & 0 \\ 0 & 1024 \end{pmatrix}$ **A1**

Note: $(-2)^{10} = 1024$ (positive since exponent is even).

Q16 — 3×3 Determinant MEDIUM

Expanding along row 1: **M1**

$\det(\mathbf{Q}) = 2\det\begin{pmatrix} 4 & -2 \\ 0 & 5 \end{pmatrix} - (-1)\det\begin{pmatrix} 0 & -2 \\ 1 & 5 \end{pmatrix} + 3\det\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$

$= 2(20 - 0) + 1(0 - (-2)) + 3(0 - 4)$ **(M1)**

$= 2(20) + 1(2) + 3(-4) = 40 + 2 - 12 = 30$ **A1A1**

Note: M1 for correct expansion structure; (M1) for evaluating 2×2 minors; A1 for correct combination; A1 for final answer 30.

Q17 — Transpose and Symmetric Matrices MEDIUM

(a) $\mathbf{R}^{\mathrm{T}} = \begin{pmatrix} 5 & -2 & 4 \\ -2 & 1 & 0 \\ 4 & 0 & 3 \end{pmatrix}$ **A1**

(b) Since $\mathbf{R}^{\mathrm{T}} = \mathbf{R}$, the matrix $\mathbf{R}$ is **symmetric**. **R1**

(c) $\mathbf{S}^{\mathrm{T}} = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$; $\mathbf{S} + \mathbf{S}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$ **M1**

The result is a **symmetric matrix** (since $(\mathbf{S} + \mathbf{S}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{S} + \mathbf{S}^{\mathrm{T}}$ always holds). **A1**

Q18 — Solving System from Context (Nutrition) MEDIUM

(a) $\mathbf{M} = \begin{pmatrix} 8 & 5 \\ 12 & 20 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 46 \\ 148 \end{pmatrix}$ **A1**

(b) $\det(\mathbf{M}) = (8)(20) - (5)(12) = 160 - 60 = 100$ **M1**

$\mathbf{M}^{-1} = \frac{1}{100}\begin{pmatrix} 20 & -5 \\ -12 & 8 \end{pmatrix}$ **A1**

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 20(46) + (-5)(148) \\ (-12)(46) + (8)(148) \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 920 - 740 \\ -552 + 1184 \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 180 \\ 632 \end{pmatrix}$$

$x = 1.8$ (servings of Type X), $\quad y = 6.32$ (servings of Type Y) **A1**

Note: Award M1 for finding $\det(\mathbf{M})$*; A1 for* $\mathbf{M}^{-1}$*; A1 for both values of* x *and* y*.*

Q19 — Unknown Parameter in Determinant MEDIUM

(a) $\det(\mathbf{A}) = p \cdot p - 4 \cdot 1 = p^2 - 4$ **A1**

(b) $\mathbf{A}$ is singular when $p^2 - 4 = 0 \Rightarrow p^2 = 4 \Rightarrow p = \pm 2$ **M1A1**

(c) For $p = 3$: $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$; $\quad \det(\mathbf{A}) = 9 - 4 = 5$

$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -1 & 3 \end{pmatrix}$ **A1**

Note: A1 for $\det = p^2 - 4$*; M1A1 for* $p = \pm 2$*; A1 for correct inverse at* $p = 3$*.*

Q20 — Consistent/Inconsistent System with Parameter MEDIUM

(a) Coefficient matrix: $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ **M1**

$\det(\mathbf{A}) = 1[(-1)(2) - (3)(1)] - 2[(2)(2) - (3)(3)] + (-1)[(2)(1) - (-1)(3)]$

$= 1(-2 - 3) - 2(4 - 9) + (-1)(2 + 3) = -5 + 10 - 5 = 0$

Since $\det(\mathbf{A}) = 0$, the system is dependent. Forming augmented matrix with k:

$R3 - R2 - R1$: $(3 - 2 - 1)x + (1 + 1 - 2)y + (2 - 3 + 1)z = k - 7 - 3 \Rightarrow 0x + 0y + 0z = k - 10$

For consistency: $k - 10 = 0 \Rightarrow k = 10$ **A1**

(b) With $k = 10$: the system reduces (using rref or back-substitution). From rows 1 and 2:

$R2 - 2R1$: $-5y + 5z = 1 \Rightarrow y - z = -\frac{1}{5}$. Let $z = t$: $y = t - \frac{1}{5}$.

From row 1: $x = 3 - 2y + z = 3 - 2(t - \frac{1}{5}) + t = 3 - 2t + \frac{2}{5} + t = \frac{17}{5} - t$.

$x = \frac{17}{5} - t, \quad y = t - \frac{1}{5}, \quad z = t \quad$ (for any real t) **A1A1**

Note: M1 for computing $\det(\mathbf{A})$*; A1 for* $k = 10$*; A1A1 for expressing general solution with parameter.*

Section C — Hard

Q21 — Parameter + Inverse + System (Synthesis) HARD

(a) $\det(\mathbf{A}(k)) = k(3) - 2(k-1) = 3k - 2k + 2 = k + 2$ **M1**

$\mathbf{A}(k)$ is singular when $k + 2 = 0 \Rightarrow k = -2$ **A1**

(b) For $k = 5$: $\mathbf{A}(5) = \begin{pmatrix} 5 & 2 \\ 4 & 3 \end{pmatrix}$; $\quad \det = 15 - 8 = 7$ (M1)

$\mathbf{A}^{-1}(5) = \frac{1}{7}\begin{pmatrix} 3 & -2 \\ -4 & 5 \end{pmatrix}$ **A1**

$$\mathbf{x} = \frac{1}{7}\begin{pmatrix} 3 & -2 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 6 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 21 - 12 \\ -28 + 30 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

$x = \frac{9}{7} \approx 1.29, \quad y = \frac{2}{7} \approx 0.286$ **A1**

Note: M1 for expanding the determinant correctly; A1 for $k = -2$*; (M1) implied by evaluating inverse; A1 for* $\mathbf{A}^{-1}$*; A1 for both solution values.*

Q22 — Matrix Equation BX

$\det(\mathbf{A}) = (5)(1) - (2)(3) = 5 - 6 = -1$ **(M1)**

$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix}$ **A1**

$\mathbf{BX} = \mathbf{A}^{-1} \Rightarrow \mathbf{X} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ **M1**

$\det(\mathbf{B}) = (2)(3) - (-1)(0) = 6; \quad \mathbf{B}^{-1} = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ **A1**

$$\mathbf{X} = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & -5 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -3 + 3 & 6 - 5 \\ 0 + 6 & 0 - 10 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 & 1 \\ 6 & -10 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{6} \\ 1 & -\frac{5}{3} \end{pmatrix}$$ **A1**

Note: (M1) for $\mathbf{A}^{-1}$*; A1 for correct* $\mathbf{A}^{-1}$*; M1 for* $\mathbf{X} = \mathbf{B}^{-1}\mathbf{A}^{-1}$*; A1 for* $\mathbf{B}^{-1}$*; A1 for final* $\mathbf{X}$*.*

Q23 — 3×3 Parametric Solution (Infinite Solutions) HARD

(a) Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 5 & 1 & 10 \\ 1 & 3 & 2 & 6 \end{array}\right)$

$R2 \leftarrow R2 - 2R1$: $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 1 & 3 & 2 & 6 \end{array}\right)$

$R3 \leftarrow R3 - R1$: $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{array}\right)$

$R3 \leftarrow R3 - R2$: $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ **M1**

The last row is all zeros, so there is no unique solution. Since $0 = 0$ (consistent), the system has **infinitely many solutions**. **R1**

(b) Let $z = t$ (parameter). From row 2: $y + 3t = 2 \Rightarrow y = 2 - 3t$ **M1**

From row 1: $x + 2(2 - 3t) - t = 4 \Rightarrow x + 4 - 6t - t = 4 \Rightarrow x = 7t$ **A1**

General solution: $x = 7t, \quad y = 2 - 3t, \quad z = t$ **A1**

Note: M1 for correct row-reduction showing zero row; R1 for identifying infinite solutions; M1 for expressing y from row 2; A1A1 for correct expressions for x and y in terms of t.

Q24 — Powers of a Matrix (Pattern) HARD

(a) $\mathbf{E}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ **A1**

$\mathbf{E}^3 = \mathbf{E}^2 \cdot \mathbf{E} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ **A1**

(b) $\mathbf{E}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \in \mathbb{Z}^+$ **A1**

(c) Entry in row 1, column 2 of $\mathbf{E}^n$ is n. Setting $n = 20$: **M1**

$n = 20$ **A1**

Note: A1A1 for $\mathbf{E}^2$ and $\mathbf{E}^3$; A1 for correct conjecture (award even if not proved); M1 for setting up $n = 20$; A1 for the answer.

Q25 — Unknown Parameter: Heat-Transfer Network **HARD**

(a) $\det(\mathbf{M}) = 3(4) - m(6) = 12 - 6m$ **M1**

No unique solution when $12 - 6m = 0 \Rightarrow m = 2$ **A1**

(b) For $m = 1$: $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$; $\det(\mathbf{M}) = 12 - 6 = 6$ (M1)

$\mathbf{M}^{-1} = \frac{1}{6}\begin{pmatrix} 4 & -1 \\ -6 & 3 \end{pmatrix}$ **A1**

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 4 & -1 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 14 \\ 22 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 56-22 \\ -84+66 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 34 \\ -18 \end{pmatrix}$$

$T_1 = \frac{17}{3} \approx 5.67\,°\text{C}, \quad T_2 = -3\,°\text{C}$ **A1**

Note: M1 for determinant formula; A1 for $m = 2$; (M1) for $\det$ at $m = 1$; A1 for $\mathbf{M}^{-1}$; A1 for both T_1, T_2.

Q26 — Determinant of A^2 (Synthesis) **HARD**

(a) $\det(\mathbf{A}) = a^2 - 2(1) = a^2 - 2$ **M1**

By the product rule for determinants: $\det(\mathbf{A}^2) = \det(\mathbf{A}\cdot\mathbf{A}) = [\det(\mathbf{A})]^2 = (a^2-2)^2$ **M1A1**

Note: First M1 for $\det(\mathbf{A})$; second M1 for applying $\det(\mathbf{A}^2) = [\det(\mathbf{A})]^2$; A1 for the correct form. Award full marks for computing $\mathbf{A}^2$ directly and taking its determinant if all working is correct.

(b) $(a^2-2)^2 = 0 \Rightarrow a^2 = 2 \Rightarrow a = \pm\sqrt{2}$ **A1**

(c) When $a = \pm\sqrt{2}$, $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is singular. A singular matrix has no inverse, which means the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ cannot be solved uniquely using $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$; the system either has no solution or infinitely many. **R1**

Q27 — 3×3 Inconsistent System **HARD**

(a) Augmented matrix: $\left(\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ 1 & 2 & -1 & 3 \\ 4 & -5 & 7 & c \end{array}\right)$

$R2 \leftarrow 2R2 - R1$: row 2 becomes $(0,\ 5,\ -5\,|\,-2)$ **M1**

$R3 \leftarrow R3 - 2R1$: row 3 becomes $(0,\ -3,\ 1\,|\,c-16)$

$R3 \leftarrow 5R3 + 3R2$: row 3 becomes $(0,\ 0,\ -10\,|\,5c-86)$ **M1**

For the system to be consistent, this final row must not create a contradiction. Since $-10 \neq 0$, this row always gives a valid equation for z, so the system always has a solution for any c.

However, row 3 of the *original* system is row3 $= 2\times$row1 $-$ row2: $(4,-5,7) = 2(2,-1,3) -$

$(1,2,-1) = (3,-4,7) \neq (4,-5,7)$. So row 3 is not a combination of rows 1 and 2; the system generically has a unique solution.

For consistency with infinite solutions (row 3 redundant): $R3 = \alpha R1 + \beta R2$. Column 1: $4 = 2\alpha + \beta$; col 2: $-5 = -\alpha + 2\beta$; col 3: $7 = 3\alpha - \beta$. From cols 1&3: adding gives $5\alpha = 11 \Rightarrow \alpha = 11/5$; $\beta = 4 - 22/5 = -2/5$. Col 2 check: $-11/5 - 4/5 = -3 \neq -5$. So no exact redundancy — the system is always consistent and has a unique solution for any c. The right-hand side constraint: $c = \alpha \cdot 8 + \beta \cdot 3 = (11/5)(8) + (-2/5)(3) = 88/5 - 6/5 = 82/5$. **A1**

The system is consistent with the third equation dependent when $c = \dfrac{82}{5} = 16.4$. **A1**

Verify: row3 rhs $= 5(16.4) - 86 = 82 - 86 = -4$. *Then* $-10z = -4 \Rightarrow z = 0.4$; *from row2:* $5y - 5(0.4) = -2 \Rightarrow y = 0$; *from row1:* $2x - 0 + 1.2 = 8 \Rightarrow x = 3.4$. *Check row3:* $4(3.4) - 5(0) + 7(0.4) = 13.6 + 2.8 = 16.4 = c$✓

(b) For $c \neq \dfrac{82}{5}$: after row reduction, the third row gives a non-zero equation $-10z = 5c - 86$, which is consistent (unique z), so the system has a **unique solution** for all other c. **M1R1**

Q28 — Encode/Decode with Matrix Inverse HARD

(a) $\mathbf{c} = \mathbf{Ep} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 21+6 \\ 35+12 \end{pmatrix} = \begin{pmatrix} 27 \\ 47 \end{pmatrix}$ **A1**

(b) $\det(\mathbf{E}) = (3)(4) - (2)(5) = 12 - 10 = 2$ **M1**

$\mathbf{E}^{-1} = \dfrac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix}$ **A1**

(c) $\mathbf{p} = \mathbf{E}^{-1}\begin{pmatrix} 22 \\ 38 \end{pmatrix} = \begin{pmatrix} 2(22) + (-1)(38) \\ (-\frac{5}{2})(22) + \frac{3}{2}(38) \end{pmatrix} = \begin{pmatrix} 44 - 38 \\ -55 + 57 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ **M1A1**

The original plaintext vector is $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

Note: A1 for encoding; M1A1 for inverse; M1 for applying decode; A1 for correct plaintext.

Q29 — Unknown Parameter: 3×3 Determinant HARD

(a) Expanding $\det(\mathbf{S})$ along row 1: **M1**

$\det(\mathbf{S}) = 2\det\begin{pmatrix} k & 2 \\ 2 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + (-1)\det\begin{pmatrix} 1 & k \\ 3 & 2 \end{pmatrix}$

$= 2(k-4) - 1(1-6) + (-1)(2-3k)$ **M1**

$= 2k - 8 + 5 - 2 + 3k = 5k - 5$ **A1**

(b) No unique solution when $\det(\mathbf{S}) = 0$: $5k - 5 = 0 \Rightarrow k = 1$ **A1**

(c) For a unique solution, $k \neq 1$. The smallest positive determinant corresponds to the value of k closest to 1 while $k > 1$ that gives an integer determinant. Since $\det(\mathbf{S}) = 5k - 5$, the smallest positive integer determinant is 5 when $k = 2$. **A1**

$k = 2 \quad$ (giving $\det = 5$)

Note: M1M1 for expanding determinant; A1 for $\det = 5k - 5$*; A1 for* $k = 1$*; A1 for* $k = 2$.

Q30 — Parametric System: River Gauging **HARD**

(a) $\mathbf{A}(r) = \begin{pmatrix} r+2 & 3 \\ 4 & r-1 \end{pmatrix}, \quad \mathbf{b}(r) = \begin{pmatrix} 18 \\ r+6 \end{pmatrix}$ **A1**

$\det(\mathbf{A}(r)) = (r+2)(r-1) - 3(4) = r^2 + r - 2 - 12 = r^2 + r - 14$ **A1**

(b) No unique solution when $r^2 + r - 14 = 0$: **M1**

$$r = \frac{-1 \pm \sqrt{1+56}}{2} = \frac{-1 \pm \sqrt{57}}{2}$$

$$r \approx \frac{-1 + 7.550}{2} \approx 3.27 \quad \text{or} \quad r \approx \frac{-1 - 7.550}{2} \approx -4.27$$ **A1**

Accept exact forms $r = \frac{-1 \pm \sqrt{57}}{2}$.

(c) For $r = 3$: $\det = 9 + 3 - 14 = -2 \neq 0$, so a unique solution exists. (M1)

$$\mathbf{A}(3) = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}, \quad \mathbf{b}(3) = \begin{pmatrix} 18 \\ 9 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{-2} \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 2(18) - 3(9) \\ -4(18) + 5(9) \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 36 - 27 \\ -72 + 45 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 9 \\ -27 \end{pmatrix} = \begin{pmatrix} -4.5 \\ 13.5 \end{pmatrix}$$

$u = -4.5\,\text{m}, \quad v = 13.5\,\text{m}$ **A1**

Note: A1A1 for $\mathbf{A}(r)$ *and* det*; M1A1 for solving quadratic; (M1)A1 for* $r = 3$ *solution.*

End of Worksheet

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