IB Math AI HL — Topic 3

Vector Properties

Introduction to Vectors · Scalar & Vector Product · Angles & Perpendicularity · Areas · Components · Geometric Proof

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Marks each	3	4	5
Total Marks	30	40	50

Scan me for help

Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated. Calculator permitted throughout.

Key Formulae — Vector Properties

Magnitude: $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Unit vector: $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Scalar (dot) product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Angle between vectors: $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

Perpendicular condition: $\mathbf{a} \cdot \mathbf{b} = 0$

Parallel condition: $\mathbf{a} = k\mathbf{b}$ for some scalar k

Vector (cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Area of triangle OAB: $= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

Area of parallelogram: $= |\mathbf{a} \times \mathbf{b}|$

Displacement: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

GDC Tips

- **Magnitude:** Use `norm([a,b,c])` or compute $\sqrt{a^2+b^2+c^2}$ directly.
- **Dot product:** Use `dotP([a,b,c],[d,e,f])` in the Vector menu or compute manually.
- **Cross product:** Use `crossP([a,b,c],[d,e,f])` in the Vector menu.
- **Angle:** Use $\cos^{-1}$ after computing the dot product and magnitudes.
- **Exact answers:** Where “exact” or “in surd form” is required, avoid rounding intermediate steps.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)

Scan me for help

1. **EASY** **Calculator** **[3 marks]**

A drone is positioned at point A, with coordinates $(1, 3, 2)$, and flies to point B with coordinates $(5, -1, 6)$, where distances are in metres.

(a) Write down the displacement vector $\overrightarrow{AB}$. [1]

(b) Find $|\overrightarrow{AB}|$, the distance flown by the drone. [2]

2. **EASY** **Calculator** **[3 marks]**

The vectors $\mathbf{p} = \begin{pmatrix} 6 \\ -4 \\ 10 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 3 \\ k \\ 5 \end{pmatrix}$ are parallel.

(a) Write down the value of k. [1]

(b) Hence find a unit vector in the direction of $\mathbf{q}$. [2]

3. **EASY** **Calculator** **[3 marks]**

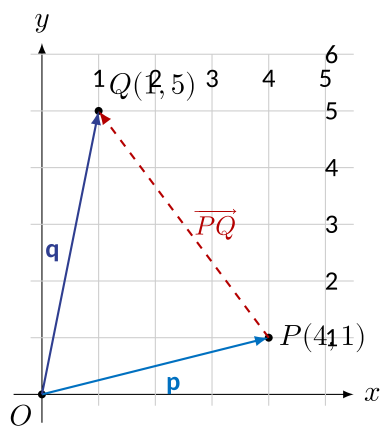
Vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$.

(a) Find $2\mathbf{a} - \mathbf{b}$. [2]

(b) Write down $|2\mathbf{a} - \mathbf{b}|$, correct to 3 significant figures. [1]

4. **EASY** **Calculator** **[3 marks]**

The diagram shows points O, P, and Q in a coordinate plane. The position vectors of P and Q relative to origin O are $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.



(a) Write down $\overrightarrow{PQ}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. [1]

(b) Find the column vector $\overrightarrow{PQ}$. [1]

(c) Find $|\overrightarrow{PQ}|$. [1]

5. EASY Calculator **[3 marks]**

Given $\mathbf{u} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$:

(a) Find $\mathbf{u} \cdot \mathbf{v}$. [2]

(b) Write down whether **u** and **v** are perpendicular. Justify your answer. [1]

6. EASY Calculator **[3 marks]**

A force vector is given by $\mathbf{F} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ (in newtons).

(a) Find $|\mathbf{F}|$. [1]

(b) Find the unit vector $\hat{\mathbf{F}}$ in the direction of **F**. [2]

7. EASY Calculator **[3 marks]**

Find the angle, in degrees, between the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$. Give your answer correct to 1 decimal place.

8. EASY Calculator **[3 marks]**

Point M is the midpoint of segment AB. The position vectors of A and B relative to origin O are $\mathbf{a} = \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 8 \\ 2 \\ -2 \end{pmatrix}$.

(a) Find the position vector of M. [2]

(b) Write down the coordinates of M. [1]

9. EASY Calculator **[3 marks]**

A boat travels from harbour H at $(0,\ 0,\ 0)$ to buoy A at $(3,\ 4,\ 0)$, then continues from A to buoy B at $(7,\ -2,\ 0)$, where distances are in kilometres.

(a) Write down the vector $\overrightarrow{HA}$. [1]

(b) Write down the vector $\overrightarrow{AB}$. [1]

(c) Find the total distance $HA + AB$. Give your answer correct to 3 significant figures. [1]

10. EASY Calculator **[3 marks]**

Given $\mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 1 \\ t \\ 4 \end{pmatrix}$, where t is a real constant. The vectors $\mathbf{r}$ and $\mathbf{s}$ are perpendicular.

(a) Write down an equation in t using the perpendicularity condition. [1]

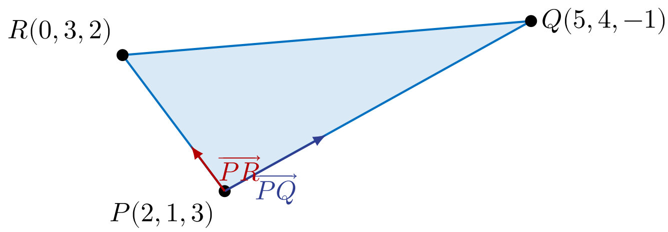
(b) Find the value of t. [2]

Section B — Medium **MEDIUM** (10 questions $\times$ 4 marks = 40 marks)

Scan me for help

11. **MEDIUM** **Calculator** **[4 marks]**

Three points P, Q, and R have coordinates $P(2,\ 1,\ 3)$, $Q(5,\ 4,\ -1)$, and $R(0,\ 3,\ 2)$.
The diagram shows the positions of P, Q, and R in 3D space. (Diagram not to scale.)



(a) Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. [2]

(b) Find the angle $\widehat{QPR}$ at vertex P. Give your answer correct to 1 decimal place. [2]

12. **MEDIUM** **Calculator** **[4 marks]**

Triangle ABC has vertices $A(1,\ 0,\ -1)$, $B(4,\ 2,\ 1)$, and $C(-1,\ 3,\ 0)$.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [2]

(b) Hence find the area of triangle ABC. Give your answer correct to 3 significant figures. [2]

13. MEDIUM Calculator **[4 marks]**

Vectors $\mathbf{m} = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, where $k > 0$. The angle between **m** and **n** is $90°$.

(a) Show that $6 - k - 2 = 0$. [2]

(b) Hence find the value of k. [1]

(c) Find $|\mathbf{m}|$ when k takes this value. [1]

14. MEDIUM Calculator **[4 marks]**

Vectors $\mathbf{a} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(a) Find the scalar projection of **a** onto **b**, defined as $\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$. [2]

(b) Find the vector projection of **a** onto **b**, defined as $\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}$. [2]

15. MEDIUM Calculator **[4 marks]**

In triangle OAB, M is the midpoint of OA and N is the midpoint of OB. The position vectors of A and B relative to O are **a** and **b** respectively.

(a) Write down the position vectors of M and N in terms of **a** and **b**. [1]

(b) Show that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$. [2]

(c) State what this result tells you about the line segment MN and the side AB of the triangle. [1]

16. MEDIUM Calculator **[4 marks]**

A land surveyor records the vertices of a quadrilateral plot $PQRS$ as shown in the table below, where distances are in metres. It is known that $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ are adjacent sides of a parallelogram.

Vertex	x (m)	y (m)	z (m)
P	2	1	0
Q	6	3	0
S	2	5	0

(a) Write down $\overrightarrow{PQ}$ and $\overrightarrow{PS}$. [1]

(b) Find $\overrightarrow{PQ} \times \overrightarrow{PS}$. [2]

(c) Hence find the area of the parallelogram $PQRS$. [1]

17. MEDIUM Calculator **[4 marks]**

Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$ relative to origin O.

Point T divides AB in the ratio $2 : 1$ from A.

(a) Find the position vector of T. [2]

(b) Find $|\overrightarrow{OT}|$. Give your answer in exact surd form. [2]

18. MEDIUM Calculator **[4 marks]**

A laser beam travels in the direction of vector $\mathbf{d} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

(a) Find the unit vector $\hat{\mathbf{d}}$. [1]

(b) Find the angle that $\mathbf{d}$ makes with the positive x-axis. Give your answer correct to 1 decimal place. [1]

(c) A second beam travels in the direction $\mathbf{e} = \begin{pmatrix} 4 \\ 1 \\ k \end{pmatrix}$, where k is real. Given that $\mathbf{d}$ and $\mathbf{e}$ are perpendicular, find k. [2]

19. MEDIUM Calculator **[4 marks]**

Given $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$:

(a) Find $\mathbf{a} \times \mathbf{b}$. [2]

(b) Show that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$. [2]

20. MEDIUM Calculator **[4 marks]**

$OABC$ is a parallelogram where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Write down $\overrightarrow{OB}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. [1]

(b) Show that the diagonals OB and AC bisect each other. [3]

Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)

Scan me for help

21. **HARD** **Calculator** **[5 marks]**

Vectors $\mathbf{u} = \begin{pmatrix} k \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} k \\ -3 \\ k+1 \end{pmatrix}$, where k is a real constant.

(a) Given that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, show that $k^2 - k - 7 = 0$. [3]

(b) Find both values of k, giving your answers in exact form. [1]

(c) For the positive value of k, use the relation $k^2 = k + 7$ to find $|\mathbf{u}|$ in exact surd form. [1]

22. **HARD** **Calculator** **[5 marks]**

Points $A(1,\ 2,\ 0)$, $B(3,\ -1,\ 1)$, and $C(c,\ 0,\ 2)$ are the vertices of a triangle, where c is a positive real number. The area of triangle ABC is $\sqrt{38}$ square units.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of c. [1]

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$ in terms of c. [2]

(c) Find the value of c. [2]

23. HARD Calculator **[5 marks]**

Triangle OAB has position vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ relative to origin O. Let M be the midpoint of AB, and let G be the point on OM such that $OG : GM = 2 : 1$.

(a) Write $\overrightarrow{OM}$ in terms of **a** and **b**. [1]

(b) Find the position vector of G. [2]

(c) Show that G also lies on the median from A to the midpoint of OB, and state what point G represents in the triangle. [2]

24. HARD Calculator **[5 marks]**

Vectors $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

(a) Find $\mathbf{p} \times \mathbf{q}$. [2]

(b) Using $|\mathbf{p} \times \mathbf{q}| = |\mathbf{p}||\mathbf{q}| \sin\theta$, where θ is the angle between **p** and **q**, find θ in degrees. [2]

(c) Verify your answer to (b) using the scalar product formula. [1]

25. HARD Calculator **[5 marks]**

Vector $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and vector $\mathbf{b} = \begin{pmatrix} m \\ n \\ p \end{pmatrix}$, where m, n, p are real constants.

Given that **b** is parallel to **a**, that the third component of **b** satisfies $p = 2$, and that $|\mathbf{b}| = 3$:

(a) Write $\mathbf{b} = k\mathbf{a}$ and use the condition $p = 2$ to find both possible values of k. [2]

(b) Hence write down the two possible vectors **b**. [2]

(c) State the geometric relationship between the two vectors found in (b). [1]

26. HARD Calculator **[5 marks]**

Two points $A(4,\ 1,\ -3)$ and $B(0,\ 3,\ 1)$ define a line segment. Point C has coordinates $(6,\ 5,\ -1)$.

The foot of the perpendicular from C to line AB is point F, where $\overrightarrow{OF} = \overrightarrow{OA} + t\,\overrightarrow{AB}$ for some scalar t.

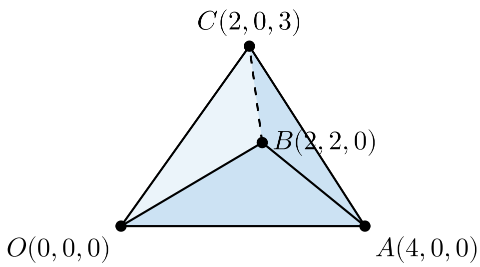
(a) Write down $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [1]

(b) Use the condition $\overrightarrow{CF} \cdot \overrightarrow{AB} = 0$ to find the value of t. [3]

(c) Hence find the shortest distance from C to line AB. Give your answer correct to 3 significant figures. [1]

27. **HARD** **Calculator** **[5 marks]**

The diagram shows a tetrahedron $OABC$ with vertices $O(0,0,0)$, $A(4,0,0)$, $B(2,2,0)$, and $C(2,0,3)$. (Diagram not to scale.)



(a) Find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [1]

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [2]

(c) Find the area of face ABC. Give your answer correct to 3 significant figures. [1]

(d) Find the angle $\widehat{BAC}$ at vertex A. Give your answer correct to 1 decimal place. [1]

28. HARD Calculator **[5 marks]**

Vectors $\mathbf{f} = \begin{pmatrix} t \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} 2 \\ t \\ -1 \end{pmatrix}$, where t is a real number.
The angle between **f** and **g** is $60°$.

(a) Show that $\mathbf{f} \cdot \mathbf{g} = \frac{1}{2}|\mathbf{f}||\mathbf{g}|$ leads to the equation $t^2 - 6t + 9 = 0$. [3]

(b) Hence find the value of t. [1]

(c) For this value of t, find the angle that **f** makes with the positive x-axis. Give your answer correct to 1 decimal place. [1]

29. HARD Calculator **[5 marks]**

In a right-angled triangle OAB, the right angle is at O. The position vectors of A and B relative to O are **a** and **b** respectively.

(a) Write down $\overrightarrow{AB}$ in terms of **a** and **b**. [1]

(b) Show that $|AB|^2 = |OA|^2 + |OB|^2$, given that the angle at O is $90°$. [3]

(c) State the name of the theorem you have proved in part (b). [1]

30. HARD Calculator **[5 marks]**

Triangle PQR has vertices $P(1,\,-1,\,2)$, $Q(3,\,1,\,-1)$, and $R(-1,\,2,\,1)$.

(a) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$. [2]

(b) Hence find a vector that is perpendicular to the plane containing PQR. [1]

(c) Find the area of triangle PQR. Give your answer correct to 3 significant figures. [1]

(d) Find the angle $\widehat{QPR}$. Give your answer correct to 1 decimal place. [1]

Worked Solutions — Section A (Easy)

Q1 — Displacement Vector & Magnitude [EASY

] **(a)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix}$ **A1**

(b) $|\overrightarrow{AB}| = \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{16 + 16 + 16} = \sqrt{48} = 4\sqrt{3} \approx 6.93$ m **M1 A1**

Note: Accept 6.93 *m (3 s.f.).*

Q2 — Parallel Vectors & Unit Vector [EASY

] **(a)** Since $\mathbf{p} = 2\mathbf{q}$, comparing second components: $-4 = 2k \Rightarrow k = -2$ **A1**

(b) With $k = -2$: $\mathbf{q} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$, $\quad |\mathbf{q}| = \sqrt{9 + 4 + 25} = \sqrt{38}$ **(M1)**

$\hat{\mathbf{q}} = \dfrac{1}{\sqrt{38}} \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ **A1**

Note: Accept equivalent decimal form to 3 s.f.

Q3 — Adding & Subtracting Vectors [EASY

] **(a)** $2\mathbf{a} - \mathbf{b} = 2\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ 7 \end{pmatrix}$ **M1 A1**

(b) $|2\mathbf{a} - \mathbf{b}| = \sqrt{64 + 49 + 49} = \sqrt{162} \approx 12.7$ **A1**

Q4 — Position & Displacement Vectors [EASY

] **(a)** $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$ **A1**

(b) $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **A1**

(c) $|\overrightarrow{PQ}| = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ **A1**

Q5 — Scalar Product [EASY

] **(a)** $\mathbf{u} \cdot \mathbf{v} = (2)(4) + (-3)(1) + (1)(-2) = 8 - 3 - 2 = 3$ **M1 A1**

(b) Since $\mathbf{u} \cdot \mathbf{v} = 3 \neq 0$, the vectors are **not perpendicular**. **A1**

Q6 — Magnitude & Unit Vector [EASY

] **(a)** $|\mathbf{F}| = \sqrt{3^2 + (-4)^2 + 0^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ N **A1**

(b) $\hat{\mathbf{F}} = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ -0.8 \\ 0 \end{pmatrix}$ **M1 A1**

Q7 — Angle Between Vectors [EASY

] $\mathbf{a} \cdot \mathbf{b} = (1)(2) + (2)(1) + (2)(-2) = 2 + 2 - 4 = 0$ **M1**

$|\mathbf{a}| = \sqrt{1 + 4 + 4} = 3, \quad |\mathbf{b}| = \sqrt{4 + 1 + 4} = 3$ **(M1)**

$\cos\theta = \frac{0}{3 \times 3} = 0 \Rightarrow \theta = 90.0°$ **A1**

Note: The vectors are perpendicular.

Q8 — Midpoint Position Vector [EASY

] **(a)** $\overrightarrow{OM} = \frac{\mathbf{a} + \mathbf{b}}{2} = \frac{1}{2}\left(\begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 8 \\ 2 \\ -2 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$ **M1 A1**

(b) $M = (3, 3, 2)$ **A1**

Q9 — Introduction to Vectors [EASY

] **(a)** $\overrightarrow{HA} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ **A1**

(b) $\overrightarrow{AB} = \begin{pmatrix} 7 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix}$ **A1**

(c) $|\overrightarrow{HA}| = \sqrt{9 + 16} = 5; \quad |\overrightarrow{AB}| = \sqrt{16 + 36} = \sqrt{52} \approx 7.21$

Total $= 5 + 7.21\ldots = 12.2$ km (3 s.f.) **A1**

Note: Full precision: $5 + \sqrt{52} = 5 + 2\sqrt{13} \approx 12.211\ldots$ *Accept* 12.2 *km.*

Q10 — Perpendicularity Condition [EASY

] **(a)** $\mathbf{r} \cdot \mathbf{s} = 0 \Rightarrow 5(1) + (-3)(t) + 2(4) = 0$ **A1**
i.e. $5 - 3t + 8 = 0$

(b) $13 - 3t = 0 \Rightarrow t = \frac{13}{3}$ **M1 A1**

Worked Solutions — Section B (Medium)

Q11 — 3D Geometry: Vectors & Angle [MEDIUM

] **(a)** $\overrightarrow{PQ} = Q - P = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$ **A1**

$\overrightarrow{PR} = R - P = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$ **A1**

(b) $\overrightarrow{PQ} \cdot \overrightarrow{PR} = (3)(-2) + (3)(2) + (-4)(-1) = -6 + 6 + 4 = 4$ **(M1)**

$|\overrightarrow{PQ}| = \sqrt{9 + 9 + 16} = \sqrt{34}, \quad |\overrightarrow{PR}| = \sqrt{4 + 4 + 1} = 3$ **(M1)**

$\cos\theta = \dfrac{4}{3\sqrt{34}}; \quad \theta = \cos^{-1}\left(\dfrac{4}{3\sqrt{34}}\right) = 76.9°$ **A1**

Note: Full precision: $76.867\ldots°$

Q12 — Cross Product & Triangle Area [MEDIUM

] **(a)** $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ **(A1)(A1) implied**

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ -2 & 3 & 1 \end{vmatrix}$ **M1**

$= \mathbf{i}(2 \cdot 1 - 2 \cdot 3) - \mathbf{j}(3 \cdot 1 - 2 \cdot (-2)) + \mathbf{k}(3 \cdot 3 - 2 \cdot (-2))$

$= \mathbf{i}(2 - 6) - \mathbf{j}(3 + 4) + \mathbf{k}(9 + 4) = \begin{pmatrix} -4 \\ -7 \\ 13 \end{pmatrix}$ **A1**

(b) Area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{16 + 49 + 169} = \frac{1}{2}\sqrt{234} \approx 7.65$ sq. units **M1 A1**

Note: Full precision: $\frac{1}{2}\sqrt{234} = \frac{3\sqrt{26}}{2} \approx 7.6485\ldots$

Q13 — Scalar Product & Unknown Angle [MEDIUM

] **(a)** $\mathbf{m} \cdot \mathbf{n} = (2)(3) + (k)(-1) + (-1)(2) = 6 - k - 2$ **M1**

For perpendicularity, $\mathbf{m} \cdot \mathbf{n} = 0$, so $6 - k - 2 = 0$ **AG**

(No mark for the AG line; the preceding step must show the product.)

(b) $6 - k - 2 = 0 \Rightarrow k = 4$ **A1**

(c) $|\mathbf{m}| = \sqrt{4 + 16 + 1} = \sqrt{21}$ **A1**

Q14 — Components / Projection [MEDIUM

] **(a)** $\mathbf{a}\cdot\mathbf{b} = (4)(1)+(-1)(2)+(2)(3) = 4-2+6 = 8;\quad |\mathbf{b}| = \sqrt{1+4+9} = \sqrt{14}$ **M1**

Scalar projection $= \dfrac{8}{\sqrt{14}} = \dfrac{8\sqrt{14}}{14} = \dfrac{4\sqrt{14}}{7} \approx 2.14$ **A1**

(b) Vector projection $= \dfrac{8}{14}\begin{pmatrix}1\\2\\3\end{pmatrix} = \dfrac{4}{7}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}4/7\\8/7\\12/7\end{pmatrix}$ **M1 A1**

Note: Accept decimal equivalents $\approx (0.571,\ 1.14,\ 1.71)$.

Q15 — Geometric Proof: Midpoint Theorem [MEDIUM

] **(a)** $\overrightarrow{OM} = \frac{1}{2}\mathbf{a};\quad \overrightarrow{ON} = \frac{1}{2}\mathbf{b}$ **A1**

(b) $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{b}-\mathbf{a})$ **M1**

$\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$, so $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$ **AG**

(c) MN is parallel to AB and half its length (the Midpoint Theorem / mid-segment theorem). **R1**

Q16 — Parallelogram Area (Cross Product) [MEDIUM

] **(a)** $\overrightarrow{PQ} = \begin{pmatrix}4\\2\\0\end{pmatrix};\quad \overrightarrow{PS} = \begin{pmatrix}0\\4\\0\end{pmatrix}$ **A1**

(b) $\overrightarrow{PQ}\times\overrightarrow{PS} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 2 & 0\\ 0 & 4 & 0\end{vmatrix}$ **M1**

$= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(16-0) = \begin{pmatrix}0\\0\\16\end{pmatrix}$ **A1**

(c) Area $= |\overrightarrow{PQ}\times\overrightarrow{PS}| = 16\ \text{m}^2$ **A1**

Q17 — Division of a Line Segment [MEDIUM

] **(a)** $\overrightarrow{OT} = \mathbf{a} + \frac{2}{3}\overrightarrow{AB}$ **M1**

$\overrightarrow{AB} = \mathbf{b}-\mathbf{a} = \begin{pmatrix}5\\1\\-4\end{pmatrix} - \begin{pmatrix}-1\\4\\2\end{pmatrix} = \begin{pmatrix}6\\-3\\-6\end{pmatrix}$

$\overrightarrow{OT} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$ **A1**

(b) $|\overrightarrow{OT}| = \sqrt{9+4+4} = \sqrt{17}$ **M1 A1**

Note: Exact form required; $\sqrt{17}$ *is accepted.*

Q18 — Unit Vector / Direction Angle / Perpendicularity [MEDIUM]

(a) $|\mathbf{d}| = \sqrt{1+4+4} = 3; \quad \hat{\mathbf{d}} = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ **A1**

(b) Angle with positive x-axis: $\cos\alpha = \dfrac{\mathbf{d}\cdot\mathbf{i}}{|\mathbf{d}||\mathbf{i}|} = \dfrac{1}{3}; \quad \alpha = \cos^{-1}\frac{1}{3} \approx 70.5°$ **A1**

(c) $\mathbf{d}\cdot\mathbf{e} = 0$: $(1)(4) + (-2)(1) + (2)(k) = 0 \Rightarrow 4 - 2 + 2k = 0 \Rightarrow 2k = -2 \Rightarrow k = -1$ **M1 A1**

Q19 — Cross Product & Perpendicularity Proof [MEDIUM

] **(a)** $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$ **M1**

$= \mathbf{i}((-1)(-1) - (3)(4)) - \mathbf{j}((2)(-1) - (3)(1)) + \mathbf{k}((2)(4) - (-1)(1))$

$= \mathbf{i}(1-12) - \mathbf{j}(-2-3) + \mathbf{k}(8+1) = \begin{pmatrix} -11 \\ 5 \\ 9 \end{pmatrix}$ **A1**

(b) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (-11)(2) + (5)(-1) + (9)(3) = -22 - 5 + 27 = 0$ **M1 A1**

Since the dot product equals zero, $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$. **AG**

Q20 — Geometric Proof: Diagonals of Parallelogram [MEDIUM

] **(a)** $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$ **A1**

(b) Midpoint of diagonal OB: $\overrightarrow{OM_1} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$ **M1**

Position vector of $C = \mathbf{c}$; Position vector of $A = \mathbf{a}$.

Midpoint of diagonal AC: $\overrightarrow{OM_2} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$ **M1**

Since $\overrightarrow{OM_1} = \overrightarrow{OM_2}$, both diagonals share the same midpoint, so they bisect each other. **R1**

Note: Full credit requires explicit identification of both midpoints and the conclusion.

Worked Solutions — Section C (Hard)

Q21 — Unknown Parameter & Perpendicularity [HARD

] **(a)** $\mathbf{u} \cdot \mathbf{v} = (k)(k) + (2)(-3) + (-1)(k+1)$ **M1**

$= k^2 - 6 - k - 1 = k^2 - k - 7$ **A1**

Setting equal to zero (perpendicular condition): $k^2 - k - 7 = 0$ **AG**

Note: M1 for attempting all three component products; A1 for correct expansion; AG carries no mark.

(b) $k = \dfrac{1 \pm \sqrt{1+28}}{2} = \dfrac{1 \pm \sqrt{29}}{2}$ **A1**

Accept exact form only; $k = \frac{1+\sqrt{29}}{2}$ (positive) or $k = \frac{1-\sqrt{29}}{2}$ (negative).

(c) $|\mathbf{u}|^2 = k^2 + 4 + 1 = k^2 + 5$. Using $k^2 = k + 7$: $|\mathbf{u}|^2 = k + 7 + 5 = k + 12$ **M1**

$= \dfrac{1+\sqrt{29}}{2} + 12 = \dfrac{25+\sqrt{29}}{2}$; $\quad |\mathbf{u}| = \sqrt{\dfrac{25+\sqrt{29}}{2}}$ **A1**

Note: Accept equivalent exact forms; numerical value ≈ 3.88.

Q22 — Area of Triangle with Unknown Vertex [HARD

] **(a)** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$; $\quad \overrightarrow{AC} = \begin{pmatrix} c-1 \\ -2 \\ 2 \end{pmatrix}$ **A1**

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ c-1 & -2 & 2 \end{vmatrix}$ **M1**

$= \mathbf{i}((-3)(2) - (1)(-2)) - \mathbf{j}((2)(2) - (1)(c-1)) + \mathbf{k}((2)(-2) - (-3)(c-1))$

$= \mathbf{i}(-6+2) - \mathbf{j}(4 - c + 1) + \mathbf{k}(-4 + 3c - 3)$

$= \begin{pmatrix} -4 \\ -(5-c) \\ 3c-7 \end{pmatrix} = \begin{pmatrix} -4 \\ c-5 \\ 3c-7 \end{pmatrix}$ **A1**

(c) $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{38}$, so $|\overrightarrow{AB} \times \overrightarrow{AC}|^2 = 4 \times 38 = 152$ **M1**

$16 + (c-5)^2 + (3c-7)^2 = 152$

$16 + c^2 - 10c + 25 + 9c^2 - 42c + 49 = 152$

$10c^2 - 52c + 90 = 152 \Rightarrow 10c^2 - 52c - 62 = 0 \Rightarrow 5c^2 - 26c - 31 = 0$

$(5c - 31)(c + 1) = 0 \Rightarrow c = \dfrac{31}{5} = 6.2$ (taking positive value, since $c > 0$) **A1**

Note: The other root $c = -1$ is rejected since $c > 0$.

Q23 — Centroid via Vectors [HARD

] **(a)** M is midpoint of AB: $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ **A1**

(b) G divides OM in ratio $2:1$ from O:

$\overrightarrow{OG} = \overrightarrow{OO} + \frac{2}{3}\overrightarrow{OM} = \frac{2}{3} \cdot \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{3}(\mathbf{a} + \mathbf{b})$ **M1 A1**

(c) Let N' be the midpoint of OB: $\overrightarrow{ON'} = \frac{1}{2}\mathbf{b}$. **M1**

Points on median AN': $\overrightarrow{OP} = \mathbf{a} + s(\frac{1}{2}\mathbf{b} - \mathbf{a}) = (1-s)\mathbf{a} + \frac{s}{2}\mathbf{b}$.

Set equal to $\overrightarrow{OG} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$:

$1 - s = \frac{1}{3}$ and $\frac{s}{2} = \frac{1}{3} \Rightarrow s = \frac{2}{3}$, consistent. **R1**

G is the **centroid** (intersection of all three medians). **A1 implied in R1**

Q24 — Cross Product + Angle Synthesis [HARD

] **(a)** $\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix}$ **M1**

$= \mathbf{i}((2)(2) - (-2)(-1)) - \mathbf{j}((1)(2) - (-2)(2)) + \mathbf{k}((1)(-1) - (2)(2))$

$= \mathbf{i}(4-2) - \mathbf{j}(2+4) + \mathbf{k}(-1-4) = \begin{pmatrix} 2 \\ -6 \\ -5 \end{pmatrix}$ **A1**

(b) $|\mathbf{p} \times \mathbf{q}| = \sqrt{4 + 36 + 25} = \sqrt{65}$ **(M1)**

$|\mathbf{p}| = \sqrt{1+4+4} = 3; \quad |\mathbf{q}| = \sqrt{4+1+4} = 3$

$\sin\theta = \dfrac{\sqrt{65}}{9}$; since $|\mathbf{p} \times \mathbf{q}| = |\mathbf{p}||\mathbf{q}|\sin\theta$ and $0 \le \theta \le 180°$,

$\theta = \sin^{-1}\left(\dfrac{\sqrt{65}}{9}\right) \approx 64.6°$ or $180° - 64.6° = 115.4°$ **A1 A1**

(c) $\mathbf{p} \cdot \mathbf{q} = 2 - 2 - 4 = -4$; $\cos\theta = \dfrac{-4}{9} < 0$, so θ is obtuse.

Therefore $\theta \approx 115.4°$ (consistent with $\cos^{-1}(-\frac{4}{9}) \approx 116.4°$). **A1**

Note: The cross product formula with $\sin\theta$ *gives the value in* $[0°, 90°]$; *the sign of* $\cos\theta$ *resolves the ambiguity to* $\approx 115.4°$.

Q25 — Parallel Vectors with Constraints [HARD

] **(a)** $\mathbf{b} = k\mathbf{a} = k\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, so the third component is $p = 2k$.

Given $p = 2$: $2k = 2 \Rightarrow k = 1$. **M1**

Also $|\mathbf{a}| = \sqrt{1+4+4} = 3$, so $|\mathbf{b}| = |k| \cdot 3 = 3 \Rightarrow |k| = 1 \Rightarrow k = \pm 1$.

Both conditions give $k = 1$ (consistent) and $k = -1$ (gives $p = -2$, inconsistent with $p = 2$).

From the constraint $p = 2$: $k = 1$ only. **A1**

Note: Award M1 for writing $p = 2k$ from **b** $= k$**a**; *A1 for $k = 1$ with correct reasoning.*

(b) For $k = 1$: $\mathbf{b}_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ **A1**

For $k = -1$: $\mathbf{b}_2 = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ (this satisfies $|\mathbf{b}| = 3$ and parallelism but has $p = -2$) **A1**

Note: Award A1A1 for both vectors; accept the candidate presenting both $k = \pm 1$ solutions as the two parallel vectors of magnitude 3.

(c) The two vectors are **anti-parallel** (parallel but in opposite directions; one is the negative of the other). **A1**

Q26 — Foot of Perpendicular (Shortest Distance) [HARD

] **(a)** $\overrightarrow{AB} = B - A = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$; $\quad \overrightarrow{AC} = C - A = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ **A1**

(b) $\overrightarrow{OF} = A + t\,\overrightarrow{AB}$; $\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC}$

$\overrightarrow{CF} = (A + t\,\overrightarrow{AB}) - C = \overrightarrow{AC} \cdot (-1) + t\,\overrightarrow{AB}$

Actually: $\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC}$. Let $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 5 \\ -1 \end{pmatrix}$.

$$\overrightarrow{OF} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4-4t \\ 1+2t \\ -3+4t \end{pmatrix}$$ **M1**

$$\overrightarrow{CF} = \begin{pmatrix} 4-4t-6 \\ 1+2t-5 \\ -3+4t+1 \end{pmatrix} = \begin{pmatrix} -2-4t \\ -4+2t \\ -2+4t \end{pmatrix}$$ **M1**

$\overrightarrow{CF} \cdot \overrightarrow{AB} = 0$:

$(-2-4t)(-4) + (-4+2t)(2) + (-2+4t)(4) = 0$

$8 + 16t - 8 + 4t - 8 + 16t = 0 \Rightarrow 36t - 8 = 0 \Rightarrow t = \dfrac{2}{9}$ **A1**

(c) $\overrightarrow{CF}$ at $t = \frac{2}{9}$:

$$\overrightarrow{CF} = \begin{pmatrix} -2 - \frac{8}{9} \\ -4 + \frac{4}{9} \\ -2 + \frac{8}{9} \end{pmatrix} = \begin{pmatrix} -\frac{26}{9} \\ -\frac{32}{9} \\ -\frac{10}{9} \end{pmatrix}$$

$|\overrightarrow{CF}| = \frac{1}{9}\sqrt{676 + 1024 + 100} = \frac{1}{9}\sqrt{1800} = \frac{30\sqrt{2}}{9} = \frac{10\sqrt{2}}{3} \approx 4.71$ **A1**

Note: Full precision $\frac{10\sqrt{2}}{3} \approx 4.714$*. Accept 4.71 (3 s.f.).*

Q27 — Tetrahedron: Cross Product / Area / Angle [HARD]

(a) $\overrightarrow{AB} = B - A = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$; $\overrightarrow{AC} = C - A = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ **A1**

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 0 \\ -2 & 0 & 3 \end{vmatrix}$ **M1**

$= \mathbf{i}(6 - 0) - \mathbf{j}(-6 - 0) + \mathbf{k}(0 + 4) = \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix}$ **A1**

(c) Area $= \frac{1}{2}\left|\begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix}\right| = \frac{1}{2}\sqrt{36 + 36 + 16} = \frac{1}{2}\sqrt{88} = \frac{1}{2} \cdot 2\sqrt{22} = \sqrt{22} \approx 4.69$ sq. units **A1**

(d) $\overrightarrow{AB} \cdot \overrightarrow{AC} = (-2)(-2) + (2)(0) + (0)(3) = 4$

$|\overrightarrow{AB}| = \sqrt{4 + 4} = 2\sqrt{2}$; $|\overrightarrow{AC}| = \sqrt{4 + 9} = \sqrt{13}$

$\cos\theta = \dfrac{4}{2\sqrt{2} \cdot \sqrt{13}} = \dfrac{4}{2\sqrt{26}} = \dfrac{2}{\sqrt{26}}$; $\theta = \cos^{-1}\left(\dfrac{2}{\sqrt{26}}\right) \approx 66.8°$ **A1**

Q28 — Unknown Parameter & Angle Condition [HARD

] **(a)** $\mathbf{f} \cdot \mathbf{g} = (t)(2) + (1)(t) + (2)(-1) = 3t - 2$ **M1**

$|\mathbf{f}| = \sqrt{t^2 + 1 + 4} = \sqrt{t^2 + 5}$; $|\mathbf{g}| = \sqrt{4 + t^2 + 1} = \sqrt{t^2 + 5}$

$|\mathbf{f}||\mathbf{g}| = t^2 + 5$ **M1**

Setting $3t - 2 = \frac{1}{2}(t^2 + 5)$:

$6t - 4 = t^2 + 5 \Rightarrow t^2 - 6t + 9 = 0$ **A1 AG**

(b) $(t - 3)^2 = 0 \Rightarrow t = 3$ (unique solution) **A1**

(c) For $t = 3$: $\mathbf{f} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$; $|\mathbf{f}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

Angle with positive x-axis: $\cos\alpha = \frac{3}{\sqrt{14}}$; $\alpha = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 36.7°$ **A1**

Q29 — Vector Proof of Pythagoras' Theorem [HARD

] **(a)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **A1**

(b) $|AB|^2 = |\overrightarrow{AB}|^2 = |\mathbf{b} - \mathbf{a}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$ **M1**

$= \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} = |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2$ **M1**

Since angle at O is $90°$: $\mathbf{a} \cdot \mathbf{b} = 0$

$\therefore |AB|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 = |OA|^2 + |OB|^2$ **A1 AG**

(c) Pythagoras' Theorem. **A1**

Q30 — Triangle: Cross Product / Normal / Area / Angle [HARD]

(a) $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$; $\overrightarrow{PR} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$

$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -3 \\ -2 & 3 & -1 \end{vmatrix}$ **M1**

$= \mathbf{i}((2)(-1) - (-3)(3)) - \mathbf{j}((2)(-1) - (-3)(-2)) + \mathbf{k}((2)(3) - (2)(-2))$

$= \mathbf{i}(-2 + 9) - \mathbf{j}(-2 - 6) + \mathbf{k}(6 + 4) = \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix}$ **A1**

(b) A vector perpendicular to plane PQR is $\begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix}$ (or any scalar multiple). **A1**

(c) Area $= \frac{1}{2}\left|\begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix}\right| = \frac{1}{2}\sqrt{49 + 64 + 100} = \frac{1}{2}\sqrt{213} \approx 7.29$ sq. units **A1**

Note: Full precision $\frac{\sqrt{213}}{2} \approx 7.2938\ldots$

(d) $\overrightarrow{PQ} \cdot \overrightarrow{PR} = (2)(-2) + (2)(3) + (-3)(-1) = -4 + 6 + 3 = 5$

$|\overrightarrow{PQ}| = \sqrt{4 + 4 + 9} = \sqrt{17}$; $|\overrightarrow{PR}| = \sqrt{4 + 9 + 1} = \sqrt{14}$

$\cos\theta = \frac{5}{\sqrt{17} \cdot \sqrt{14}} = \frac{5}{\sqrt{238}}$; $\theta = \cos^{-1}\left(\frac{5}{\sqrt{238}}\right) \approx 71.1°$ **A1**

Full precision: $\cos^{-1}\left(\frac{5}{\sqrt{238}}\right) = 71.100\ldots°$

Common Mistakes to Avoid

1. **Displacement vs position vector.** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, not $\mathbf{a} - \mathbf{b}$. Subtracting in the wrong order is the single most common sign error.
2. **Cross product sign in the j component.** The $\mathbf{j}$ term carries a minus sign: $\overrightarrow{AB} \times \overrightarrow{AC} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$. Omitting the minus sign is extremely common.
3. **Area of triangle vs parallelogram.** The cross product gives the area of the *parallelogram*. The area of a triangle is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.
4. **Perpendicularity requires dot product $= 0$, not parallel.** Do not confuse: $\mathbf{a} \cdot \mathbf{b} = 0$ (perpendicular) vs $\mathbf{a} = k\mathbf{b}$ (parallel).
5. **Angle formula uses magnitudes of *both* vectors.** $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$. Dividing by only one magnitude is a common error.
6. **Unit vector.** Divide by the *magnitude*, not any individual component: $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$.