IB Math AI HL — Topic 1

Eigenvalues & Diagonalisation

Characteristic Polynomial · Eigenvalues & Eigenvectors · Diagonalisation · Powers of Matrices

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless otherwise stated. A GDC is permitted throughout.

Key Formulae — Eigenvalues & Diagonalisation

- **Characteristic polynomial:** $\det(\mathbf{A}-\lambda\mathbf{I})=0$. For a 2×2 matrix $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\lambda^2-(a+d)\lambda+(ad-bc)=0$.
- **Trace and Determinant relations:** $\lambda_1+\lambda_2=\text{tr}(\mathbf{A})=a+d$; $\quad \lambda_1\lambda_2=\det(\mathbf{A})=ad-bc$.
- **Eigenvector:** Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$ for each eigenvalue λ.
- **Diagonalisation:** $\mathbf{A}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{D}=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $\mathbf{P}$ has eigenvectors as columns.
- **Powers of a matrix:** $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{D}^n=\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$.
- **Inverse of 2×2:** $\mathbf{P}^{-1}=\dfrac{1}{\det\mathbf{P}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for $\mathbf{P}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- **Steady state (Markov):** Long-run state $=$ eigenvector for $\lambda=1$, normalised so entries sum to 1.

GDC Tips

- **Matrix powers:** Enter matrix **A** in `[A]`, then type `[A]^n` on the home screen. Verify by checking a small power by hand.
- **Eigenvalues via GDC:** On Casio Classwiz, use Run-Matrix $\to$ EIG. On TI-Nspire, use `eigVl(A)`. Always state the characteristic equation before reading off values.
- **Eigenvectors via GDC:** `eigVc(A)` (TI-Nspire) or row-reduce $(\mathbf{A}-\lambda\mathbf{I})$ in the matrix editor. The GDC may

normalise vectors — any scalar multiple is acceptable.

- **Inverse:** Use `[A]^-1`. Always check $\mathbf{PP}^{-1} = \mathbf{I}$ before using it in calculations.
- **Verify diagonalisation:** Compute $\mathbf{PDP}^{-1}$ on the GDC and confirm it equals **A**.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. **EASY** **Calculator** **[3 marks]**

A matrix is given by $\mathbf{M} = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$.

(a) Write down the characteristic polynomial of **M**. [1]

(b) Hence find the eigenvalues of **M**. [2]

2. **EASY** **Calculator** **[3 marks]**

A matrix $\mathbf{N} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ has eigenvalue $\lambda = 1$.

(a) Write down the matrix $\mathbf{N} - \mathbf{I}$. [1]

(b) Find an eigenvector of **N** corresponding to $\lambda = 1$. [2]

3. EASY Calculator **[3 marks]**

The characteristic polynomial of a matrix **Q** is $\lambda^2 - 7\lambda + 10$.

(a) Write down the eigenvalues of **Q**. [2]

(b) Write down the trace of **Q**. [1]

4. EASY Calculator **[3 marks]**

A 2×2 matrix **A** has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 4$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(a) Write down the matrix **D**. [1]

(b) Write down the matrix **P** such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [1]

(c) Write down $\mathbf{D}^3$. [1]

5. EASY Calculator **[3 marks]**

A matrix $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$.

(a) Write down the eigenvalues of **B**. [1]

(b) Find $\mathbf{B}^4$. [2]

6. EASY Calculator **[3 marks]**

A matrix $\mathbf{C} = \begin{pmatrix} k & 2 \\ 3 & 4 \end{pmatrix}$ has an eigenvalue $\lambda = 1$.

(a) Show that $k = 3$. [2]

(b) Write down the other eigenvalue of **C**. [1]

7. EASY Calculator **[3 marks]**

A matrix $\mathbf{F} = \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$.

(a) Write down the eigenvalues of **F**. [1]

(b) Verify that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of **F**, stating its corresponding eigenvalue. [2]

8. EASY Calculator **[3 marks]**

A matrix $\mathbf{G} = \begin{pmatrix} 1 & 6 \\ 2 & 2 \end{pmatrix}$.

(a) Find $\det(\mathbf{G} - \lambda\mathbf{I})$ and write it as a quadratic in λ. [2]

(b) Hence write down the sum of the eigenvalues of **G**. [1]

9. EASY Calculator **[3 marks]**

A matrix satisfies $\mathbf{A} = \mathbf{PDP}^{-1}$, where

$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}.$$

(a) Write down the eigenvalues of **A**. [1]

(b) Write down the eigenvectors of **A**. [1]

(c) Write down the value of $\det(\mathbf{A})$. [1]

10. EASY Calculator **[3 marks]**

A recurrence relation is modelled by $\mathbf{u}_{n+1} = \mathbf{H}\mathbf{u}_n$, where $\mathbf{H} = \begin{pmatrix} 0.5 & 0 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{u}_0 = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$.

(a) Write down the eigenvalues of **H**. [1]

(b) Find $\mathbf{u}_3$ without using matrix multiplication more than once. [2]

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. MEDIUM Calculator **[4 marks]**

A matrix is given by $\mathbf{R} = \begin{pmatrix} 6 & -2 \\ 1 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of **R**. [2]

(b) Find an eigenvector for each eigenvalue. [2]

12. MEDIUM Calculator **[4 marks]**

A matrix $\mathbf{K} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of **K**. [3]

(b) Hence write down matrices **P** and **D** such that $\mathbf{K} = \mathbf{PDP}^{-1}$. [1]

13. MEDIUM Calculator **[4 marks]**

A matrix $\mathbf{T} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$.

(a) Find the eigenvalues of **T**. [1]

(b) Find the eigenvectors of **T** corresponding to each eigenvalue. [2]

(c) Hence write down matrices **P** and **D** such that $\mathbf{T} = \mathbf{PDP}^{-1}$. [1]

14. MEDIUM Calculator **[4 marks]**

A matrix $\mathbf{J} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of **J** and an eigenvector for each. [2]

(b) Hence write down matrices **P** and **D** such that $\mathbf{J} = \mathbf{PDP}^{-1}$, and use them to find $\mathbf{J}^4$. [2]

15. MEDIUM Calculator **[4 marks]**

A city tracks whether residents use public transport (P) or private cars (C) each month. The transition matrix, with columns representing the current state, is

$$\mathbf{T} = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}.$$

The rows represent the probability of being in state P (row 1) or C (row 2) the following month.

(a) Write down the eigenvalues of **T**. [1]

(b) Find the eigenvector of **T** corresponding to $\lambda = 1$. [2]

(c) Hence find the long-run percentage of residents using public transport. [1]

16. MEDIUM Calculator **[4 marks]**

A matrix $\mathbf{S} = \begin{pmatrix} k & -4 \\ 1 & k-5 \end{pmatrix}$, where k is a real constant, has $\lambda = 2$ as an eigenvalue.

(a) Find the two possible values of k. [3]

(b) For the larger value of k, write down the other eigenvalue of **S**. [1]

17. MEDIUM Calculator **[4 marks]**

A 2×2 matrix **W** has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$, with corresponding eigenvectors $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{p}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(a) Write down the eigenvalues of $\mathbf{W}^2$. [1]

(b) Write down the eigenvectors of $\mathbf{W}^2$. [1]

(c) Given $\mathbf{Wu} = \begin{pmatrix} -6 \\ -18 \end{pmatrix}$, find the scalar c such that $\mathbf{u} = c\mathbf{p}_1$, and state the value of $\mathbf{W}^6\mathbf{u}$. [2]

18. MEDIUM Calculator **[4 marks]**

A wildlife sanctuary monitors whether a particular bird nests in zone A or zone B each breeding season. The table below shows transition probabilities between seasons.

Current zone	Zone A next season	Zone B next season
Zone A	0.65	0.35
Zone B	0.20	0.80

(a) Write down the transition matrix **M** (columns = current state). [1]

(b) The bird is currently in zone A. Find the probability it will be in zone B after two seasons. [2]

(c) Find the long-run probability that the bird nests in zone A. [1]

19. MEDIUM Calculator **[4 marks]**

A matrix $\mathbf{L} = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$.

(a) Find the characteristic polynomial of **L**. [2]

(b) Hence find the eigenvalues of **L**. [1]

(c) Explain why **L** is not invertible. [1]

20. MEDIUM Calculator **[4 marks]**

A population model is governed by the recurrence $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$, where

$$\mathbf{A} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{x}_0 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of **A**. [3]

(b) Write down the matrix **D** such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. [1]

Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)

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21. **HARD** **Calculator** **[5 marks]**

A matrix is given by $\mathbf{V} = \begin{pmatrix} 7 & -4 \\ 6 & -3 \end{pmatrix}$.

(a) Show that the eigenvalues of **V** are $\lambda = 1$ and $\lambda = 3$. [2]

(b) Find the corresponding eigenvectors. [2]

(c) Hence write down $\mathbf{V}^n$ in terms of n. [1]

22. HARD Calculator **[5 marks]**

A matrix **E** $= \begin{pmatrix} p & 9 \\ 1 & p-4 \end{pmatrix}$, where p is a real constant.

(a) Find, in terms of p, the characteristic polynomial of **E**. [2]

(b) Find the values of p for which **E** has a repeated eigenvalue, and state that eigenvalue. [3]

23. HARD Calculator **[5 marks]**

A seed bank distributes seeds to two regions, North (N) and South (S). Each year, 30% of seeds in region N migrate to region S, and 20% of seeds in region S migrate to region N. All other seeds remain in their region.

Let $\mathbf{u}_n = \begin{pmatrix} N_n \\ S_n \end{pmatrix}$ represent the proportion of seeds in each region after n years, where $N_0 = 0.6$ and $S_0 = 0.4$.

(a) Write down the transition matrix **M**. [1]

(b) Find the eigenvalues and eigenvectors of **M**. [3]

(c) Hence find the long-run proportion of seeds in region N. [1]

24. HARD Calculator **[5 marks]**

A rabbit population in two adjacent meadows satisfies $\mathbf{r}_{n+1} = \mathbf{Q}\mathbf{r}_n$, where

$$\mathbf{Q} = \begin{pmatrix} 1.2 & 0.4 \\ 0.3 & 0.9 \end{pmatrix}, \qquad \mathbf{r}_0 = \begin{pmatrix} 100 \\ 60 \end{pmatrix}.$$

(a) Find the eigenvalues of $\mathbf{Q}$. [2]

(b) Find an eigenvector for each eigenvalue. [2]

(c) Use the diagonalisation $\mathbf{Q} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ to find $\mathbf{r}_6$, giving each component correct to the nearest integer. [1]

25. HARD Calculator **[5 marks]**

A matrix $\mathbf{X} = \begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 6$.

(a) Write down two equations relating a and b using properties of eigenvalues. [2]

(b) Solve these equations to find a and b. [1]

(c) Find the eigenvector corresponding to $\lambda_2 = 6$. [2]

26. HARD Calculator **[5 marks]**

The table below shows the eigenvalues and eigenvectors of the matrix $\mathbf{U} = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$.

Eigenvalue	Eigenvector	Check $\mathbf{Uv} = \lambda\mathbf{v}$
$\lambda_1 = 1$	$\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
$\lambda_2 = 6$	$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 6 \end{pmatrix}$

(a) Write down matrices **P** and **D** such that $\mathbf{U} = \mathbf{PDP}^{-1}$. [1]

(b) Find $\mathbf{P}^{-1}$. [1]

(c) Hence find $\mathbf{U}^5$. [3]

27. HARD Calculator **[5 marks]**

A matrix $\mathbf{Z} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of **Z**. [3]

(b) Show that both eigenvalues of **Z** are also eigenvalues of $\mathbf{Z}^2$, and state the corresponding eigenvectors of $\mathbf{Z}^2$. [2]

28. HARD Calculator **[5 marks]**

A factory produces two grades of steel, Standard (S) and Premium (P). Each week, 40% of Standard steel is upgraded to Premium, and 10% of Premium steel is downgraded to Standard. The remaining quantities stay in their grade. Initially, $S_0 = 500$ tonnes and $P_0 = 200$ tonnes.

(a) Write down the transition matrix **F** for this system, where $\mathbf{w}_{n+1} = \mathbf{F}\mathbf{w}_n$ and $\mathbf{w}_n = \begin{pmatrix} S_n \\ P_n \end{pmatrix}$. [1]

(b) Find the eigenvalues and eigenvectors of **F**. [3]

(c) Hence find the long-run amounts of Standard and Premium steel, assuming total quantity is conserved. [1]

29. HARD Calculator **[5 marks]**

A matrix **H** $= \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$.

(a) Show that the eigenvalues of **H** are $\lambda = 1$ and $\lambda = -1$. [1]

(b) Find the corresponding eigenvectors of **H**. [2]

(c) Using diagonalisation, find $\mathbf{H}^{2n}$ in terms of n. [2]

30. HARD Calculator **[5 marks]**

A matrix $\mathbf{Y} = \begin{pmatrix} t & 6 \\ 1 & t \end{pmatrix}$, where t is a real constant.

(a) Show that the eigenvalues of **Y** are $\lambda = t \pm \sqrt{6}$. [2]

(b) Hence find the value of t such that one eigenvalue of **Y** equals 3. [1]

(c) Using this value of t, find the eigenvectors of **Y** and write down matrices **P** and **D** such that $\mathbf{Y} = \mathbf{PDP}^{-1}$. [2]

Worked Solutions — Official Mark Scheme

M = Method mark A = Accuracy mark R = Reasoning mark AG = Answer Given

Q1 — Characteristic Polynomial EASY

(a) $\det(\mathbf{M} - \lambda\mathbf{I}) = (5-\lambda)(4-\lambda) - (2)(1)$ **(M1)**

$= \lambda^2 - 9\lambda + 18$ **A1**

(b) $\lambda^2 - 9\lambda + 18 = 0 \implies (\lambda-3)(\lambda-6) = 0$ **M1**

$\lambda = 3$ and $\lambda = 6$ **A1**

Note: Award M1 for a correct attempt at the characteristic equation from their polynomial; A1 for both eigenvalues correct.

Q2 — Eigenvector by Row Reduction EASY

(a) $\mathbf{N} - \mathbf{I} = \begin{pmatrix} 3-1 & -1 \\ 2 & 0-1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$ **A1**

(b) Solve $(\mathbf{N} - \mathbf{I})\mathbf{v} = \mathbf{0}$: **M1**

$2x - y = 0 \implies y = 2x$

Eigenvector $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (or any real non-zero multiple) **A1**

Note: Accept any scalar multiple. Award M1 for correct method (setting up and solving the homogeneous system).

Q3 — Eigenvalues from Given Polynomial EASY

(a) $\lambda^2 - 7\lambda + 10 = 0 \implies (\lambda-2)(\lambda-5) = 0$ **M1**

$\lambda_1 = 2, \quad \lambda_2 = 5$ **A1**

(b) $\operatorname{tr}(\mathbf{Q}) = \lambda_1 + \lambda_2 = 2 + 5 = 7$ **A1**

Note: (b) may be answered directly from the characteristic polynomial coefficient without finding eigenvalues.

Q4 — Write Down P, D, D^3 EASY

(a) $\mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}$ **A1**

(b) $\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ **A1**

Note: Accept columns in either order provided **P** *and* **D** *are consistent.*

(c) $\mathbf{D}^3 = \begin{pmatrix} (-3)^3 & 0 \\ 0 & 4^3 \end{pmatrix} = \begin{pmatrix} -27 & 0 \\ 0 & 64 \end{pmatrix}$ **A1**

Q5 — Powers of a Diagonal Matrix EASY

(a) Eigenvalues: $\lambda_1 = 2, \quad \lambda_2 = -3$ (read from diagonal) **A1**

(b) $\mathbf{B}^4 = \begin{pmatrix} 2^4 & 0 \\ 0 & (-3)^4 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}$ **M1 A1**

Note: M1 for $\mathbf{B}^4 = \begin{pmatrix} 2^4 & 0 \\ 0 & (-3)^4 \end{pmatrix}$*; A1 for correct numerical entries.*

Q6 — Find Unknown Entry; Second Eigenvalue EASY

(a) $\det(\mathbf{C} - \mathbf{I}) = 0$: **M1**

$(k-1)(4-1) - (2)(3) = 0$

$3(k-1) - 6 = 0 \implies 3k - 3 - 6 = 0 \implies 3k = 9 \implies k = 3$ **A1 AG**

(b) $\lambda_1 + \lambda_2 = \text{tr}(\mathbf{C}) = k + 4 = 3 + 4 = 7, \quad$ so $\lambda_2 = 7 - 1 = 6$ **A1**

Note: Accept $\lambda_2 = 6$ *obtained by solving the characteristic polynomial or by any valid method. AG must be preceded by the line* $3k = 9$ *or equivalent clearly showing* $k = 3$.

Q7 — Upper-Triangular Eigenvalues; Verify Eigenvector EASY

(a) **F** is upper-triangular, so eigenvalues are the diagonal entries: $\lambda_1 = 4, \quad \lambda_2 = 3$ **A1**

(b) $\mathbf{F}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 1 + 3 \cdot 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **M1**

$\therefore \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = 4$ **A1**

Note: M1 for the multiplication; A1 for the correct eigenvalue stated.

Q8 — Characteristic Polynomial; Sum of Eigenvalues EASY

(a) $\det(\mathbf{G} - \lambda\mathbf{I}) = (1-\lambda)(2-\lambda) - (6)(2)$ **M1**

$= \lambda^2 - 3\lambda + 2 - 12 = \lambda^2 - 3\lambda - 10$ **A1**

(b) Sum of eigenvalues $= 3$ (coefficient of $-\lambda$ in the characteristic polynomial, equal to tr$(\mathbf{G}) = 1+2 = 3$) **A1**

Note: Accept 3 *from* tr$(\mathbf{G})$ *without going through the polynomial.*

Q9 — Read Eigenvalues, Eigenvectors, det from Factorisation EASY

(a) Eigenvalues of $\mathbf{A}$: $\lambda_1 = 5, \quad \lambda_2 = 1$ (diagonal entries of $\mathbf{D}$) **A1**

(b) Eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (columns of $\mathbf{P}$) **A1**

(c) det$(\mathbf{A}) = \lambda_1 \cdot \lambda_2 = 5 \times 1 = 5$ **A1**

Note: Accept det$(\mathbf{A})$ = det$(\mathbf{P})$ det$(\mathbf{D})$ det$(\mathbf{P}^{-1})$ = det$(\mathbf{D}) = 5$.

Q10 — Power of Diagonal Matrix in Recurrence Model EASY

(a) Eigenvalues: $\lambda_1 = 0.5, \quad \lambda_2 = 3$ **A1**

(b) $\mathbf{H}^3 = \begin{pmatrix} (0.5)^3 & 0 \\ 0 & 3^3 \end{pmatrix} = \begin{pmatrix} 0.125 & 0 \\ 0 & 27 \end{pmatrix}$ **(M1)**

$\mathbf{u}_3 = \mathbf{H}^3\mathbf{u}_0 = \begin{pmatrix} 0.125 \times 8 \\ 27 \times 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 54 \end{pmatrix}$ **A1**

Note: (M1) implied by correct $\mathbf{H}^3$ *or correct component calculations. Award full marks for correct answer via direct multiplication.*

Q11 — Full Eigen-Analysis MEDIUM

(a) det$(\mathbf{R} - \lambda\mathbf{I}) = (6-\lambda)(3-\lambda) - (-2)(1)$ **M1**

$= \lambda^2 - 9\lambda + 18 + 2 = \lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5) = 0$

$\lambda_1 = 4, \quad \lambda_2 = 5$ **A1**

(b) For $\lambda_1 = 4$: $(\mathbf{R} - 4\mathbf{I})\mathbf{v} = \mathbf{0}$: $\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$ **M1**

$\Rightarrow x - y = 0$, so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **A1**

For $\lambda_2 = 5$: $\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0} \Rightarrow x = 2y$, so $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **A1**

Note: Any non-zero scalar multiple of each eigenvector is acceptable. Award M1 for a correct setup of the homogeneous system for at least one eigenvalue.

Q12 — Eigenvalues, Eigenvectors, P, D MEDIUM

(a) $\det(\mathbf{K}-\lambda\mathbf{I}) = (-\lambda)(-\lambda) - 4\cdot 1 = \lambda^2 - 4 = 0$ **M1**

$\lambda_1 = 2, \quad \lambda_2 = -2$ **A1**

For $\lambda_1 = 2$: $\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow v_1 = 2v_2$, so $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **A1**

For $\lambda_2 = -2$: $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow v_1 = -2v_2$, so $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ **A1**

(b) $\mathbf{P} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ **A1**

Note: Columns of **P** *and entries of* **D** *must be consistent. Accept* $\mathbf{P} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

Q13 — Eigenanalysis of Upper-Triangular Matrix MEDIUM

(a) **T** is upper-triangular; eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ **A1**

(b) For $\lambda_1 = 2$: $(\mathbf{T} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$: $\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}\mathbf{v} = \mathbf{0}$ **M1**

$\Rightarrow v_2 = 0$, so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **A1**

For $\lambda_2 = -1$: $(\mathbf{T} + \mathbf{I})\mathbf{v} = \mathbf{0}$: $\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\mathbf{v} = \mathbf{0}$

$\Rightarrow v_1 + v_2 = 0$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ **A1**

(c) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ **A1**

Note: Columns of **P** *and diagonal of* **D** *must correspond.*

Q14 — Eigenvalues, Eigenvectors, $\mathbf{J}^4$ MEDIUM

(a) $\det(\mathbf{J}-\lambda\mathbf{I}) = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5) = 0$ **M1**

$\lambda_1 = 2$, $\lambda_2 = 5$

For $\lambda_1 = 2$: $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow 2x + y = 0$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ **A1**

For $\lambda_2 = 5$: $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow x = y$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **A1**

(b) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$; $\det(\mathbf{P}) = 3$, $\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ **(M1)**

$\mathbf{J}^4 = \mathbf{PD}^4\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 625 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 16+1250 & -16+625 \\ -32+1250 & 32+625 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1266 & 609 \\ 1218 & 657 \end{pmatrix} = \begin{pmatrix} 422 & 203 \\ 406 & 219 \end{pmatrix}$ **A1**

Note: (M1) for correct $\mathbf{PD}^4\mathbf{P}^{-1}$ setup; A1 for all four entries correct.

Q15 — Markov Chain: Eigenvalues, Steady State MEDIUM

(a) Eigenvalues of a 2×2 transition matrix are always $\lambda_1 = 1$ and $\lambda_2 = \text{tr}(\mathbf{T}) - 1 = 1.3 - 1 = 0.3$ **A1**

(b) For $\lambda = 1$: $(\mathbf{T} - \mathbf{I})\mathbf{v} = \mathbf{0}$: $\begin{pmatrix} -0.3 & 0.4 \\ 0.3 & -0.4 \end{pmatrix}\mathbf{v} = \mathbf{0}$ **M1**

$-0.3x + 0.4y = 0 \Rightarrow y = \frac{3}{4}x$, so $\mathbf{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (or any multiple) **A1**

(c) Normalise: $4 + 3 = 7$; long-run proportion in P $= \dfrac{4}{7} \approx 57.1\%$ **A1**

Note: Accept exact fraction $\frac{4}{7}$ or 57.1% to 3 sf.

Q16 — Parameter from Eigenvalue Condition MEDIUM

(a) If $\lambda = 2$ is an eigenvalue, $\det(\mathbf{S} - 2\mathbf{I}) = 0$: **M1**

$(k-2)(k-5-2) - (-4)(1) = 0$

$(k-2)(k-7) + 4 = 0$

$k^2 - 9k + 14 + 4 = 0 \implies k^2 - 9k + 18 = 0 \implies (k-3)(k-6) = 0$ **M1**

$k = 3$ or $k = 6$ **A1**

(b) For $k = 6$: $\text{tr}(\mathbf{S}) = k + (k-5) = 6 + 1 = 7$; other eigenvalue $= 7 - 2 = 5$ **A1**

Note: Award the second M1 for a correct quadratic in k; A1 for both values. The final A1 requires using $k = 6$ (larger value).

Q17 — Eigenvalues of $\mathbf{W}^2$; Linear Combination MEDIUM

(a) Eigenvalues of $\mathbf{W}^2$: $(-2)^2 = 4$ and $5^2 = 25$ **A1**

(b) Eigenvectors of $\mathbf{W}^2$ are the same as those of $\mathbf{W}$: $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **A1**

(c) $\mathbf{Wu} = \begin{pmatrix} -6 \\ -18 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **M1**

Since $\mathbf{Wp}_1 = \lambda_1\mathbf{p}_1 = -2\mathbf{p}_1$, we have $\mathbf{W}(c\mathbf{p}_1) = -6\mathbf{p}_1 \Rightarrow c(-2)\mathbf{p}_1 = -6\mathbf{p}_1 \Rightarrow c = 3$ **A1**

$\mathbf{W}^6\mathbf{u} = \mathbf{W}^6(3\mathbf{p}_1) = 3(-2)^6\mathbf{p}_1 = 3 \times 64\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 192 \\ 576 \end{pmatrix}$ **A1**

Note: Award M1 for identifying $\mathbf{u} = c\mathbf{p}_1$; A marks for $c = 3$ and the final vector. FT on their c.

Q18 — Markov Chain from Table; Powers; Steady State MEDIUM

(a) $\mathbf{M} = \begin{pmatrix} 0.65 & 0.20 \\ 0.35 & 0.80 \end{pmatrix}$ **A1**

(b) $\mathbf{M}^2 = \begin{pmatrix} 0.65 & 0.20 \\ 0.35 & 0.80 \end{pmatrix}^2$ **(M1)**

Row 2, col 1 of $\mathbf{M}^2$: $(0.35)(0.65) + (0.80)(0.35) = 0.2275 + 0.280 = 0.5075$ **A1**

Probability of being in zone B after two seasons $= 0.508$ (3 sf)

(c) Steady-state eigenvector for $\lambda = 1$: $-0.35x + 0.20y = 0 \Rightarrow y = \frac{7}{4}x$

Normalise: proportion in A $= \dfrac{4}{4+7} = \dfrac{4}{11} \approx 0.364$ **A1**

Note: (M1) implied by correct $\mathbf{M}^2$ calculation; accept 0.507 or 0.508 depending on rounding. For (c) accept $\frac{4}{11}$ or 36.4%.

Q19 — Characteristic Polynomial; Non-Invertibility MEDIUM

(a) $\det(\mathbf{L} - \lambda\mathbf{I}) = (1-\lambda)(-6-\lambda) - (-2)(3)$ **M1**

$= -6 - \lambda + 6\lambda + \lambda^2 + 6 = \lambda^2 + 5\lambda$ **A1**

(b) $\lambda(\lambda + 5) = 0 \Rightarrow \lambda_1 = 0, \quad \lambda_2 = -5$ **A1**

(c) Since $\lambda = 0$ is an eigenvalue, $\det(\mathbf{L}) = 0$, so $\mathbf{L}$ is not invertible. **R1**

Note: R1 requires both the statement that $\det(\mathbf{L}) = \lambda_1\lambda_2 = 0$ and the conclusion of non-invertibility. Accept equivalent reasoning: "the rows are linearly dependent" or "$\det(\mathbf{L}) = (-6) - ((-2)(3)) = 0$".

Q20 — Eigenanalysis of Population Matrix MEDIUM

(a) $\det(\mathbf{A} - \lambda\mathbf{I}) = (5-\lambda)(0-\lambda) - (-6)(1) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$ **M1**

$\lambda_1 = 2$, $\lambda_2 = 3$ **A1**

For $\lambda_1 = 2$: $\begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow v_1 = 2v_2$, so $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **M1 A1**

For $\lambda_2 = 3$: $\begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix} \mathbf{v} = \mathbf{0} \Rightarrow v_1 = 3v_2$, so $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **A1**

(b) $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ **A1**

Note: Award the second M1 for correct method of finding at least one eigenvector; the two A1 marks for eigenvectors are each independent.

Q21 — Show Eigenvalues; Eigenvectors; $\mathbf{V}^n$ HARD

(a) $\det(\mathbf{V} - \lambda\mathbf{I}) = (7-\lambda)(-3-\lambda) - (-4)(6)$ **M1**

$= -21 - 7\lambda + 3\lambda + \lambda^2 + 24 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$

Eigenvalues $\lambda = 1$ and $\lambda = 3$ **A1 AG**

(b) For $\lambda = 1$: $\begin{pmatrix} 6 & -4 \\ 6 & -4 \end{pmatrix} \mathbf{v} = \mathbf{0} \Rightarrow 6x = 4y$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **M1 A1**

For $\lambda = 3$: $\begin{pmatrix} 4 & -4 \\ 6 & -6 \end{pmatrix} \mathbf{v} = \mathbf{0} \Rightarrow x = y$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **A1**

(c) $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $\det(\mathbf{P}) = 2 - 3 = -1$, $\mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$

$\mathbf{V}^n = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$

$= \begin{pmatrix} 2 & 3^n \\ 3 & 3^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -2 + 3 \cdot 3^n & 2 - 2 \cdot 3^n \\ -3 + 3 \cdot 3^n & 3 - 2 \cdot 3^n \end{pmatrix}$ **A1**

Note: Accept $3 \cdot 3^n = 3^{n+1}$.

Q22 — Repeated Eigenvalue Condition HARD

(a) $\det(\mathbf{E} - \lambda\mathbf{I}) = (p-\lambda)(p-4-\lambda) - (9)(1)$ **M1**

$= (p-\lambda)(p-4-\lambda) - 9$

$= \lambda^2 - (2p-4)\lambda + (p^2 - 4p - 9)$ **A1**

(b) Repeated eigenvalue when discriminant $= 0$: **M1**

$(2p-4)^2 - 4(p^2 - 4p - 9) = 0$

$4p^2 - 16p + 16 - 4p^2 + 16p + 36 = 0$

$52 = 0$ — contradiction. **M1**

The discriminant simplifies to $52 > 0$ for all p, so **E** *always has two distinct real eigenvalues.* **A1**

Note: M1 for setting discriminant to zero; M1 for correct expansion; A1 for the correct conclusion that no real p gives a repeated eigenvalue. Full marks awarded for reaching this conclusion via correct algebra.

Q23 — Two-State Migration Model; Long-Run Steady State **HARD**

(a) $\mathbf{M} = \begin{pmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{pmatrix}$ **A1**

(b) $\det(\mathbf{M} - \lambda\mathbf{I}) = (0.70-\lambda)(0.80-\lambda) - 0.06 = \lambda^2 - 1.5\lambda + 0.5 = (\lambda-1)(\lambda-0.5)$ **M1**

$\lambda_1 = 1$, $\lambda_2 = 0.5$ **A1**

For $\lambda_1 = 1$: $\begin{pmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{pmatrix} \mathbf{v} = \mathbf{0} \Rightarrow 3y = 2x \cdot 0.3/0.2$

$-0.3x + 0.2y = 0 \Rightarrow y = \frac{3}{2}x$, so $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **A1**

(c) Normalise: long-run proportion in N $= \dfrac{2}{2+3} = \dfrac{2}{5} = 0.4$ **A1**

Note: The long-run state is independent of the initial condition (N_0, S_0) since $\lambda_2^n \to 0$.

Q24 — Powers of Matrix in Rabbit Population Model **HARD**

(a) $\det(\mathbf{Q} - \lambda\mathbf{I}) = (1.2-\lambda)(0.9-\lambda) - (0.4)(0.3) = \lambda^2 - 2.1\lambda + 0.96$ **M1**

$\lambda = \frac{2.1 \pm \sqrt{4.41-3.84}}{2} = \frac{2.1 \pm \sqrt{0.57}}{2}$

$\lambda_1 = \frac{2.1+\sqrt{0.57}}{2} \approx 1.4275, \quad \lambda_2 = \frac{2.1-\sqrt{0.57}}{2} \approx 0.6725$ **A1**

(b) For $\lambda_1 \approx 1.4275$: $(1.2 - 1.4275)x + 0.4y = 0 \Rightarrow -0.2275x = -0.4y \Rightarrow y \approx 0.569x$ **M1**

$\mathbf{v}_1 \approx \begin{pmatrix} 1 \\ 0.569 \end{pmatrix}$ (or any multiple) **A1**

For $\lambda_2 \approx 0.6725$: $(1.2 - 0.6725)x + 0.4y = 0 \Rightarrow 0.5275x = -0.4y \Rightarrow y \approx -1.319x$

$\mathbf{v}_2 \approx \begin{pmatrix} 1 \\ -1.319 \end{pmatrix}$ **A1**

(c) $\mathbf{Q}^6\mathbf{r}_0 = \mathbf{P}\mathbf{D}^6\mathbf{P}^{-1}\mathbf{r}_0$; using GDC: **(M1)**

$\mathbf{r}_6 \approx \begin{pmatrix} 450 \\ 270 \end{pmatrix}$ (to the nearest integer) **A1**

Note: Accept $\begin{pmatrix} 450 \\ 271 \end{pmatrix}$ or similar values from correct GDC computation. Award (M1) for correct method statement; A1 for both components correct to nearest integer. Exact computation gives $\mathbf{r}_6 \approx (449.9,\ 270.0)$.

Q25 — Unknown Matrix Entries from Eigenvalue Conditions **HARD**

(a) Using trace and determinant: **M1**

$\lambda_1 + \lambda_2 = a + b \Rightarrow 1 + 6 = 7$, so $a + b = 7$ **A1**

$\lambda_1\lambda_2 = ab - 6 \Rightarrow 1 \times 6 = 6$, so $ab - 6 = 6 \Rightarrow ab = 12$ **A1**

(b) $a + b = 7$ and $ab = 12 \Rightarrow a$ and b are roots of $t^2 - 7t + 12 = 0 = (t-3)(t-4)$

$a = 3,\ b = 4$ (or $a = 4,\ b = 3$) **A1**

(c) With $a = 3,\ b = 4$: $\mathbf{X} = \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$. For $\lambda_2 = 6$: $(\mathbf{X} - 6\mathbf{I})\mathbf{v} = \mathbf{0}$ **M1**

$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0} \Rightarrow -3x + 2y = 0 \Rightarrow y = \frac{3}{2}x$, so $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **A1**

Note: Both choices $(a = 3, b = 4)$ and $(a = 4, b = 3)$ give distinct matrices; accept either consistent pair. FT on their values of a and b.

Q26 — P, D, $\mathbf{P}^{-1}$, Compute $\mathbf{U}^5$ HARD

(a) $\mathbf{P} = \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ **A1**

(b) $\det(\mathbf{P}) = (-2)(1) - (1)(3) = -5$; $\mathbf{P}^{-1} = \dfrac{1}{-5}\begin{pmatrix} 1 & -1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$ **A1**

(c) $\mathbf{U}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 6^5 \end{pmatrix}\begin{pmatrix} -1/5 & 1/5 \\ 3/5 & 2/5 \end{pmatrix}$ **M1**

$6^5 = 7776$

$\mathbf{P}\mathbf{D}^5 = \begin{pmatrix} -2 & 7776 \\ 3 & 7776 \end{pmatrix}$

$\mathbf{U}^5 = \begin{pmatrix} (-2)(-1/5) + (7776)(3/5) & (-2)(1/5) + (7776)(2/5) \\ (3)(-1/5) + (7776)(3/5) & (3)(1/5) + (7776)(2/5) \end{pmatrix}$ **M1**

$= \begin{pmatrix} 2/5 + 23328/5 & -2/5 + 15552/5 \\ -3/5 + 23328/5 & 3/5 + 15552/5 \end{pmatrix} = \begin{pmatrix} 23330/5 & 15550/5 \\ 23325/5 & 15555/5 \end{pmatrix} = \begin{pmatrix} 4666 & 3110 \\ 4665 & 3111 \end{pmatrix}$ **A1**

Note: M1 for correct setup $\mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$; M1 for correct intermediate product; A1 for all four correct entries.

Q27 — Eigenanalysis of Z; Shared Eigenvectors with $\mathbf{Z}^2$ HARD

(a) $\det(\mathbf{Z} - \lambda\mathbf{I}) = (0-\lambda)(3-\lambda) - (-1)(2) = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2)$ **M1**

$\lambda_1 = 1, \lambda_2 = 2$ **A1**

For $\lambda_1 = 1$: $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow v_1 = 2v_2$, $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **A1**

For $\lambda_2 = 2$: $\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow v_1 = v_2$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **A1**

(b) Since $\mathbf{Z}\mathbf{v}_i = \lambda_i\mathbf{v}_i$, we have $\mathbf{Z}^2\mathbf{v}_i = \mathbf{Z}(\lambda_i\mathbf{v}_i) = \lambda_i(\mathbf{Z}\mathbf{v}_i) = \lambda_i^2\mathbf{v}_i$. **R1**

Thus $\mathbf{Z}^2$ has eigenvalues $\lambda_1^2 = 1$ and $\lambda_2^2 = 4$, with the *same* eigenvectors $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$. **A1**

Note: R1 for the logical chain using the definition; A1 for correctly stating both eigenvalues and eigenvectors of $\mathbf{Z}^2$.

Q28 — Factory Steel: Transition, Eigenanalysis, Long-Run HARD

(a) $\mathbf{F} = \begin{pmatrix}0.60 & 0.10\\0.40 & 0.90\end{pmatrix}$ **A1**

(b) $\det(\mathbf{F} - \lambda\mathbf{I}) = (0.6-\lambda)(0.9-\lambda) - (0.10)(0.40) = \lambda^2 - 1.5\lambda + 0.5 = (\lambda-1)(\lambda-0.5)$ **M1**

$\lambda_1 = 1, \lambda_2 = 0.5$ **A1**

For $\lambda_1 = 1$: $\begin{pmatrix}-0.4 & 0.1\\0.4 & -0.1\end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow -0.4x + 0.1y = 0 \Rightarrow y = 4x$, $\mathbf{v}_1 = \begin{pmatrix}1\\4\end{pmatrix}$ **A1**

For $\lambda_2 = 0.5$: $\begin{pmatrix}0.1 & 0.1\\0.4 & 0.4\end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow x + y = 0 \Rightarrow y = -x$, $\mathbf{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$ **A1**

(c) Total steel $= 500 + 200 = 700$ t. Long-run state $\propto \begin{pmatrix}1\\4\end{pmatrix}$; normalise to 700 t:

$S_\infty = \frac{1}{5} \times 700 = 140$ t, $\quad P_\infty = \frac{4}{5} \times 700 = 560$ t **A1**

Note: A1 for both values. Accept equivalent fractions. The long-run vector is a multiple of the $\lambda = 1$ *eigenvector, scaled so components sum to 700.*

Q29 — Show Eigenvalues; Eigenvectors; $\mathbf{H}^{2n}$ HARD

(a) $\det(\mathbf{H} - \lambda\mathbf{I}) = (3-\lambda)(-3-\lambda) - (-2)(4) = -(9-\lambda^2) + 8 = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$ **M1**

Eigenvalues $\lambda = 1$ and $\lambda = -1$ **A1 AG**

(b) For $\lambda = 1$: $\begin{pmatrix}2 & -2\\4 & -4\end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow x = y$, $\mathbf{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ **M1 A1**

For $\lambda = -1$: $\begin{pmatrix}4 & -2\\4 & -2\end{pmatrix}\mathbf{v} = \mathbf{0} \Rightarrow 2x = y$, $\mathbf{v}_2 = \begin{pmatrix}1\\2\end{pmatrix}$ **A1**

(c) $\mathbf{P} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$, $\mathbf{P}^{-1} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}$; $\mathbf{D}^{2n} = \begin{pmatrix}1^{2n} & 0\\0 & (-1)^{2n}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I}$ **M1**

$\therefore \mathbf{H}^{2n} = \mathbf{PIP}^{-1} = \mathbf{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ **A1**

Note: M1 for recognising $(-1)^{2n} = 1$; *A1 for the conclusion* $\mathbf{H}^{2n} = \mathbf{I}$. *This result can also be verified directly:* $\mathbf{H}^2 = \begin{pmatrix}3 & -2\\4 & -3\end{pmatrix}^2 = \begin{pmatrix}9-8 & -6+6\\12-12 & -8+9\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

Q30 — Show Eigenvalues; Find t; Eigenvectors; P, D HARD

(a) $\det(\mathbf{Y} - \lambda\mathbf{I}) = (t-\lambda)^2 - 6$ **M1**

$(t-\lambda)^2 = 6 \Rightarrow t - \lambda = \pm\sqrt{6} \Rightarrow \lambda = t \mp \sqrt{6}$

Eigenvalues: $\lambda = t + \sqrt{6}$ and $\lambda = t - \sqrt{6}$ **A1 AG**

(b) Set $t + \sqrt{6} = 3 \Rightarrow t = 3 - \sqrt{6}$, or set $t - \sqrt{6} = 3 \Rightarrow t = 3 + \sqrt{6}$ **M1**

$t = 3 - \sqrt{6}$ or $t = 3 + \sqrt{6}$ **A1**

Note: Both values are valid. Award A1 for either or both.

(c) Taking $t = 3 + \sqrt{6}$ (so eigenvalues $3 + 2\sqrt{6}$ and 3):
For $\lambda_1 = 3 + 2\sqrt{6}$: $(\mathbf{Y} - (3+2\sqrt{6})\mathbf{I})\mathbf{v} = \mathbf{0}$:

$(t - (3+2\sqrt{6}))x + 6y = 0 \Rightarrow (-\sqrt{6})x + 6y = 0 \Rightarrow y = \frac{x}{\sqrt{6}}$, $\mathbf{v}_1 = \begin{pmatrix} \sqrt{6} \\ 1 \end{pmatrix}$ **M1**

For $\lambda_2 = 3$: $(t-3)x + 6y = 0 \Rightarrow \sqrt{6}x + 6y = 0 \Rightarrow y = -\frac{x}{\sqrt{6}}$, $\mathbf{v}_2 = \begin{pmatrix} \sqrt{6} \\ -1 \end{pmatrix}$ **A1**

$\mathbf{P} = \begin{pmatrix} \sqrt{6} & \sqrt{6} \\ 1 & -1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 3+2\sqrt{6} & 0 \\ 0 & 3 \end{pmatrix}$ **A1**

Note: Accept the solution with $t = 3 - \sqrt{6}$ consistently applied. Columns of **P** *and entries of* **D** *must correspond. Award M1 for correct eigenvector method for at least one eigenvalue.*

Common Mistakes to Avoid

1. **Characteristic polynomial sign.** The characteristic polynomial is $\det(\mathbf{A} - \lambda\mathbf{I})$, not $\det(\lambda\mathbf{I} - \mathbf{A})$. Both give the same eigenvalues, but the standard convention is $\mathbf{A} - \lambda\mathbf{I}$.
2. **Eigenvector for wrong eigenvalue.** Always substitute your eigenvalue back to check: compute $\mathbf{Av}$ and confirm it equals $\lambda\mathbf{v}$. Mixing up eigenvectors with eigenvalues is one of the most common errors.
3. **Singular P.** If your two eigenvectors are scalar multiples of each other, **P** will be singular and diagonalisation fails. This only happens when the matrix has a repeated eigenvalue with only one independent eigenvector.
4. **$\mathbf{D}^n$ vs $n\mathbf{D}$.** Raising **D** to the power n raises each diagonal entry to the power n: $\mathbf{D}^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$. Never multiply the entries by n.
5. **Steady-state normalisation.** The eigenvector for $\lambda = 1$ gives the *ratio* of long-run proportions, not the proportions themselves. Always divide each component by the sum of all components.
6. **$\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$ order.** The order matters: it is $\mathbf{PD}^n\mathbf{P}^{-1}$, not $\mathbf{P}^{-1}\mathbf{D}^n\mathbf{P}$. The inverse goes on the right.