IB Math AI HL — Topic 3

Vector Equations of Lines

Vector Form · Parametric Form · Angle Between Lines · Shortest Distance (Point to Line) · Shortest Distance (Line to Line)

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________ Date:____________

Score:________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated. GDC permitted.

Key Formulae

1. **Vector equation of a line:** $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, where $\mathbf{a}$ is a position vector of a known point and $\mathbf{d}$ is the direction vector; $\lambda \in \mathbb{R}$.
2. **Parametric form** (3D): $x = a_1 + \lambda d_1, \quad y = a_2 + \lambda d_2, \quad z = a_3 + \lambda d_3$.
3. **Angle between two lines:** $\cos\theta = \dfrac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}$, where θ is the *acute* angle.
4. **Shortest distance from point P to line ℓ: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$:**
   $d = \dfrac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|}$, where A is any point on ℓ.
   Alternatively: find λ_0 so that $(\mathbf{r}(\lambda_0) - \mathbf{p}) \cdot \mathbf{d} = 0$ (foot of perpendicular), then compute $|F - P|$.
5. **Shortest distance between two skew lines:** $d = \dfrac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$.
6. **Scalar product:** $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$.
7. **Vector product:** $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

GDC Tips

- To solve a system of equations for λ and μ (line intersection), use **Equations** on your GDC (Casio: `MENU` $\to$ `Equation` $\to$ `Simultaneous`).

- For angle between lines, use the scalar product formula with **absolute value** of the dot product — this always gives the acute angle.
- To find $|\mathbf{u} \times \mathbf{v}|$, compute the cross product by hand and then find its magnitude; GDC matrix mode can assist.
- Always label your parameter clearly (λ, μ, t) — mixing parameters is the most common algebra error on vector line questions.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY GDC **[3 marks]**

A drone flies along a straight path. Its position (in metres) can be described using a starting point $A(1, -3, 5)$ and a direction vector $\mathbf{d} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

(a) Write down a vector equation $\mathbf{r}$ for the path of the drone. [1]

(b) Write down the parametric equations of the path. [2]

2. EASY GDC **[3 marks]**

A cable is stretched in a straight line. Its path is given by the vector equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

(a) Write down the coordinates of one point that lies on the cable. [1]

(b) Write down the direction vector of the cable. [1]

(c) Find the coordinates of the point on the cable when $\lambda = 2$. [1]

3. EASY GDC **[3 marks]**

A straight tunnel passes through points $P(4, 1, -2)$ and $Q(10, 7, 1)$.

(a) Find $\overrightarrow{PQ}$. [1]

(b) Write down a vector equation for the line through P and Q. [1]

(c) Find the coordinates of the point on the tunnel corresponding to $\lambda = \frac{1}{2}$. [1]

4. EASY GDC **[3 marks]**

A straight power line has vector equation **r** $= \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Determine whether the point $R(14, -1, 11)$ lies on the power line. Justify your answer.

5. EASY GDC **[3 marks]**

Two roads meet at a junction. Road ℓ_1 is given by the parametric equations

$$x = 1 + 3\lambda, \quad y = 4 - 2\lambda, \quad z = 0, \qquad \lambda \in \mathbb{R}.$$

Road ℓ_2 passes through the point $(1, 4, 0)$ and has direction vector $\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$.

Find the acute angle between the two roads. Give your answer in degrees, correct to 3 significant figures.

6. EASY GDC **[3 marks]**

A laser beam travels along the line with parametric equations

$$x = 2 + t, \quad y = -1 + 3t, \quad z = 5 - 2t, \qquad t \in \mathbb{R}.$$

(a) Write down the direction vector of the laser beam. [1]

(b) Find the value of t for which the z-coordinate of the laser beam is zero. [1]

(c) Hence find the coordinates of the point where the beam crosses the plane $z = 0$. [1]

7. EASY GDC **[3 marks]**

The diagram shows two straight pipes, L_1 and L_2, meeting at a point. Their direction vectors are $\mathbf{d}_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ and $\mathbf{d}_2 = \begin{pmatrix} 0 \\ 5 \\ 12 \end{pmatrix}$.

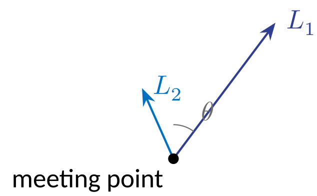

Diagram not to scale

Find the acute angle θ between the two pipes. Give your answer in degrees, correct to 1 decimal place.

8. EASY GDC **[3 marks]**

A straight ski run passes through the point $A(-2, 0, 8)$ with direction vector $\mathbf{d} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$.

The parametric equations of the run are $x = -2 + 4\lambda$, $y = 2\lambda$, $z = 8 - 3\lambda$.

(a) Find the value of λ when the run reaches the level $z = 2$. [1]

(b) Hence find the x- and y-coordinates at that point. [2]

9. EASY GDC **[3 marks]**

A straight overhead cable has vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ 6 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, where coordinates are in metres.

(a) Write down the y-coordinate of the starting point of the cable. [1]

(b) Find the value of λ for which the x-coordinate of a point on the cable equals 7. [1]

(c) Find the distance of this point from the starting point $(-1, 6, 4)$. [1]

10. EASY GDC **[3 marks]**

A conveyor belt moves along the line ℓ: $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$. A sensor is located at point $S(5, 3, 1)$.

(a) Show that the vector from a general point on ℓ to S is $\begin{pmatrix} 3-\lambda \\ 2-2\lambda \\ -2-2\lambda \end{pmatrix}$. [1]

(b) Find the value of λ such that this vector is perpendicular to the direction of ℓ. [2]

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. **MEDIUM** **GDC** **[4 marks]**

A hiking trail follows the line $\ell\colon\ \mathbf{r} = \begin{pmatrix}0\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$, $\lambda \in \mathbb{R}$, where coordinates are in kilometres. A rescue hut is at point $H(5, 2, 7)$.

(a) Find the position vector of the foot of the perpendicular from H to ℓ. [2]

(b) Hence find the shortest distance from H to the trail. [2]

12. MEDIUM GDC **[4 marks]**

Two straight pipelines in a refinery are modelled by

$$\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \qquad \ell_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{R}$.

(a) Find the acute angle between the two pipelines. Give your answer in degrees, correct to 1 decimal place. [2]

(b) Determine whether the two pipelines intersect or are skew. Justify your answer. [2]

13. MEDIUM GDC **[4 marks]**

A fault line on a rock face passes through $A(3, 0, 2)$ with direction vector $\mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, where units are in metres.

A sensor probe is placed at point $P(6, 5, 1)$.

Find the shortest distance from the probe P to the fault line. Give your answer correct to 3 significant figures.

14. MEDIUM GDC **[4 marks]**

Two tightropes at an acrobatics venue are modelled by

$$\ell_1 : \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}, \qquad \ell_2 : \mathbf{r} = \begin{pmatrix}3\\0\\4\end{pmatrix} + \mu\begin{pmatrix}1\\1\\-1\end{pmatrix},$$

where $\lambda, \mu \in \mathbb{R}$ and coordinates are in metres.

Determine whether the two tightropes intersect. If they do, state the coordinates of the intersection point.

15. MEDIUM GDC **[4 marks]**

Two model trains move along straight tracks. At time t seconds, their position vectors (in cm) are

$$\mathbf{r}_A = \begin{pmatrix}4\\1\\0\end{pmatrix} + t\begin{pmatrix}3\\2\\0\end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix}14\\-3\\0\end{pmatrix} + t\begin{pmatrix}1\\4\\0\end{pmatrix}.$$

(a) Find $\mathbf{r}_A - \mathbf{r}_B$ in terms of t. [1]

(b) Find the time t at which the trains are closest together, and find this minimum distance. [3]

16. MEDIUM GDC **[4 marks]**

Two straight ziplines at an adventure park are given by

$$\ell_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \qquad \ell_2 : \mathbf{r} = \begin{pmatrix} 9 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{R}$ and coordinates are in metres.

(a) Show that ℓ_2 is parallel to ℓ_1. [1]

(b) Find the shortest distance between the two ziplines. [3]

17. MEDIUM GDC **[4 marks]**

A straight pipe ℓ has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, where k is a constant. A second pipe m has direction vector $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$.

Find the value of k such that ℓ and m are perpendicular to each other. Justify your answer.

18. MEDIUM GDC **[4 marks]**

A straight road L passes through $A(2,4,1)$ and $B(5,1,4)$. A cyclist leaves the road at the point B and travels to point $C(8,-1,3)$ along a straight path.

(a) Find a vector equation for road L. [1]

(b) Find $\overrightarrow{BC}$. [1]

(c) Find the angle between the road L and the cyclist's path BC. Give your answer in degrees correct to 3 significant figures. [2]

19. MEDIUM GDC **[4 marks]**

Two skew beams in a building structure are modelled by

$$\ell_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \ell_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{R}$ and coordinates are in metres.

Find the shortest distance between the two beams.

20. MEDIUM GDC **[4 marks]**

An unmanned aircraft flies in a straight path given by $\mathbf{r} = \begin{pmatrix} 10 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, where coordinates are in kilometres. A radar station is at the origin $O(0, 0, 0)$.

(a) Find the foot of the perpendicular from O to the aircraft's path. [2]

(b) Hence find the closest distance the aircraft comes to the radar station. Give your answer correct to 3 significant figures. [2]

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD GDC **[5 marks]**

Line ℓ_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} k \\ 2 \\ 3 \end{pmatrix}$ and line ℓ_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ k \\ -1 \end{pmatrix}$, where k is a real constant and $\lambda, \mu \in \mathbb{R}$.

(a) Find the value of k such that ℓ_1 and ℓ_2 are perpendicular. [3]

(b) Using this value of k, find the acute angle between ℓ_1 and the line ℓ_3 with direction vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Give your answer in degrees correct to 3 significant figures. [2]

22. HARD GDC **[5 marks]**

Two ski gondola cables are modelled by skew lines

$$\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \ell_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{R}$ and units are in metres.

(a) Show that $\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}$. [2]

(b) Hence find the shortest distance between the two cables. Give your answer in exact form. [3]

23. HARD GDC **[5 marks]**

Three straight fences on a farm are modelled by

$$\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \ell_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \ell_3 : \mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix},$$

where $\lambda, \mu, \nu \in \mathbb{R}$.

Determine which two of these three fences intersect, and find their intersection point.

24. HARD GDC **[5 marks]**

A straight water pipe ℓ passes through $A(0, 2, 1)$ with direction vector $\mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. A maintenance worker stands at point $W(4, 3, -1)$.

(a) Find the foot of the perpendicular from W to ℓ. [3]

(b) A second worker stands at $V(7, 1, 5)$. Find the area of triangle FWV, where F is the foot of the perpendicular found in part (a). [2]

25. HARD GDC **[5 marks]**

Two ships are moving in the open sea. At time t hours after noon, their position vectors (in km) relative to a lighthouse at the origin are

$$\mathbf{r}_P = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_Q = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A maritime safety rule requires the ships to be at least 3 km apart at all times.

(a) Write an expression for $|\mathbf{r}_P - \mathbf{r}_Q|^2$ in terms of t. [2]

(b) Find the minimum distance between the ships and the time at which it occurs. [2]

(c) Determine whether the safety rule is violated. Justify your answer. [1]

26. HARD GDC **[5 marks]**

Line ℓ_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and line ℓ_2 has equation $\mathbf{r} = \begin{pmatrix} m \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, where m is a real constant and $\lambda, \mu \in \mathbb{R}$.

(a) Find the value of m such that ℓ_1 and ℓ_2 intersect. [3]

(b) Using this value of m, find the acute angle between ℓ_1 and ℓ_2. Give your answer in degrees correct to 3 significant figures. [2]

27. HARD GDC **[5 marks]**

A straight zip-line is modelled by $\ell\colon \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, where units are in metres. Two attachment anchors are at $B(3, -1, 5)$ and $C(5, 9, 2)$.

(a) Find the foot of the perpendicular from B to ℓ. [3]

(b) Hence find the area of triangle FBC, where F is this foot of perpendicular. [2]

28. HARD GDC **[5 marks]**

Two support cables in a bridge structure are modelled by skew lines

$$m_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad m_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

where coordinates are in metres.

(a) Find the point F_1 on m_1 and the point F_2 on m_2 that are closest to each other. [3]

(b) Find the length of the shortest connecting segment F_1F_2. [2]

29. HARD GDC **[5 marks]**

A line ℓ has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$, and a point $T(6, -3, 9)$ is given.

(a) Show that T does not lie on ℓ. [2]

(b) Find the shortest distance from T to ℓ using the cross product formula. Give your answer correct to 3 significant figures. [3]

30. HARD GDC **[5 marks]**

Line ℓ_1 has equation $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\k\\-2\end{pmatrix}$ and line ℓ_2 has equation $\mathbf{r} = \begin{pmatrix}0\\4\\1\end{pmatrix} + \mu\begin{pmatrix}k\\-1\\3\end{pmatrix}$, where k is a real constant and $\lambda, \mu \in \mathbb{R}$.

(a) Find the value(s) of k such that $\ell_1 \perp \ell_2$. [3]

(b) For the positive value of k found in part (a), find the shortest distance from the point $P(4, 2, 0)$ to ℓ_1. Give your answer correct to 3 significant figures. [2]

Common Mistakes to Avoid

1. **Using the dot product without the absolute value.** The angle formula gives $\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}$. The absolute value is mandatory to ensure an *acute* angle. Forgetting it gives the obtuse supplement.

2. **Confusing two parameters.** When checking if lines intersect, use two *different* parameter names (λ and μ). Setting both equal loses one degree of freedom and leads to wrong conclusions.

3. **Forgetting to verify the third equation.** Two lines in 3D intersect only if all three component equations are consistent simultaneously. Solving two equations and not checking the third is the most common intersection error.

4. **Direction vector vs. position vector.** The vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ uses $\mathbf{a}$ (position) and $\mathbf{d}$ (direction). Swapping them gives a completely different line.

5. **Foot of perpendicular formula.** To find the foot F, set $\overrightarrow{FP} \cdot \mathbf{d} = 0$ (not $\overrightarrow{AP} \cdot \mathbf{d} = 0$). The vector must be from the *foot* (unknown) to P, not from a fixed point.

6. **Skew distance formula sign.** The formula $d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$ uses any connecting vector between the two lines. Compute $\mathbf{d}_1 \times \mathbf{d}_2$ carefully — sign errors in the cross product are very common.

Worked Solutions — Section A (Easy)

Q1 — Vector & Parametric Equations EASY

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$ **A1**

Note: Accept any equivalent form; the direction vector may be any non-zero scalar multiple.

(b) $x = 1 + 2\lambda, \quad y = -3 + 4\lambda, \quad z = 5 - \lambda$ **A1 A1**

Note: Award A1 for any two correct equations, A1 for the third.

Q2 — Reading a Vector Equation EASY

(a) $(3, -1, 0)$ **A1**

Note: Accept any point of the form $(3 - 2\lambda, -1 + 5\lambda, 3\lambda)$ *with* λ *stated.*

(b) $\begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}$ (or any scalar multiple) **A1**

(c) $\mathbf{r}(\lambda = 2) = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 6 \end{pmatrix}$ **A1**

Coordinates: $(-1, 9, 6)$

Q3 — Vector Equation from Two Points EASY

(a) $\overrightarrow{PQ} = Q - P = \begin{pmatrix} 10 - 4 \\ 7 - 1 \\ 1 - (-2) \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}$ **A1**

(b) $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$ **A1**

Note: Accept $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} 10 \\ 7 \\ 1 \end{pmatrix}$ *as jump-on/direction respectively. A1 awarded for correct structure.*

(c) $\mathbf{r}(\tfrac{1}{2}) = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -\tfrac{1}{2} \end{pmatrix}$ **A1**

Coordinates: $(7,\ 4,\ -\tfrac{1}{2})$

Q4 — Does a Point Lie on a Line? EASY

If $R(14, -1, 11)$ lies on ℓ, then $\begin{pmatrix}14\\-1\\11\end{pmatrix} = \begin{pmatrix}5\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\4\end{pmatrix}$ for some λ. **M1**

From x: $14 = 5 + 3\lambda \Rightarrow \lambda = 3$. **A1**

Check y: $2 + 3(-1) = -1$ ✓; Check z: $-1 + 3(4) = 11$ ✓. **A1**

All three equations are satisfied with $\lambda = 3$, so R **does** lie on the power line.

Note: Award M1 for setting up the equation. A1 for $\lambda = 3$. A1 for checking all three components and stating conclusion.

Q5 — Angle Between Two Lines (2D) EASY

Direction vectors: $\mathbf{d}_1 = \begin{pmatrix}3\\-2\\0\end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix}1\\5\\0\end{pmatrix}$. **A1**

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 3(1) + (-2)(5) + 0 = 3 - 10 = -7$ **M1**

$|\mathbf{d}_1| = \sqrt{9+4} = \sqrt{13}$; $|\mathbf{d}_2| = \sqrt{1+25} = \sqrt{26}$

$\cos\theta = \dfrac{|-7|}{\sqrt{13}\cdot\sqrt{26}} = \dfrac{7}{\sqrt{338}}$ **A1**

$\theta = \arccos\left(\dfrac{7}{\sqrt{338}}\right) = 67.6^\circ$ (3 s.f.)

Q6 — Intersection with Coordinate Plane EASY

(a) $\begin{pmatrix}1\\3\\-2\end{pmatrix}$ **A1**

(b) $z = 0 \Rightarrow 5 - 2t = 0 \Rightarrow t = \frac{5}{2}$ **A1**

(c) $x = 2 + \frac{5}{2} = \frac{9}{2}$; $y = -1 + 3\cdot\frac{5}{2} = \frac{13}{2}$ **A1**

Point: $(\frac{9}{2}, \frac{13}{2}, 0)$ or $(4.5, 6.5, 0)$.

Q7 — Angle Between Direction Vectors EASY

$\mathbf{d}_1 \cdot \mathbf{d}_2 = (3)(0) + (4)(5) + (0)(12) = 20$ **M1**

$|\mathbf{d}_1| = \sqrt{9+16} = 5$; $|\mathbf{d}_2| = \sqrt{0+25+144} = 13$

$\cos\theta = \dfrac{|20|}{5 \times 13} = \dfrac{20}{65} = \dfrac{4}{13}$ **A1**

$\theta = \arccos\left(\dfrac{4}{13}\right) = 72.1^\circ$ (1 d.p.) **A1**

Q8 — Finding a Point at Given Height EASY

(a) $z = 2 \Rightarrow 8 - 3\lambda = 2 \Rightarrow \lambda = 2$ **A1**

(b) $x = -2 + 4(2) = 6; \quad y = 2(2) = 4$ **A1 A1**

Point: $(6, 4, 2)$.

Q9 — Coordinates on a Line EASY

(a) y-coordinate of starting point $= 6$ **A1**

(b) $x = -1 + 2\lambda = 7 \Rightarrow 2\lambda = 8 \Rightarrow \lambda = 4$ **A1**

(c) Displacement from start $= \lambda\mathbf{d} = 4\begin{pmatrix}2\\-3\\1\end{pmatrix}$; distance $= 4\sqrt{4+9+1} = 4\sqrt{14} \approx 14.97\ldots \approx 15.0$ m

A1

Note: Award A1 for $4|\mathbf{d}| = 4\sqrt{14}$; accept $14.97\ldots$ or 15.0.

Q10 — Foot of Perpendicular Parameter EASY

(a) General point on ℓ: $\mathbf{r}(\lambda) = \begin{pmatrix}2+\lambda\\1+2\lambda\\3+2\lambda\end{pmatrix}$.

Vector from $\mathbf{r}(\lambda)$ to S: $\begin{pmatrix}5-(2+\lambda)\\3-(1+2\lambda)\\1-(3+2\lambda)\end{pmatrix} = \begin{pmatrix}3-\lambda\\2-2\lambda\\-2-2\lambda\end{pmatrix}$ **AG**

(b) For perpendicularity: $\begin{pmatrix}3-\lambda\\2-2\lambda\\-2-2\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} = 0$ **M1**

$(3-\lambda) + 2(2-2\lambda) + 2(-2-2\lambda) = 0$

$3 - \lambda + 4 - 4\lambda - 4 - 4\lambda = 0$

$3 - 9\lambda = 0 \Rightarrow \lambda = \dfrac{1}{3}$ **A1**

Worked Solutions — Section B (Medium)

Q11 — Foot of Perpendicular & Distance MEDIUM

(a) General point on ℓ: $F = \begin{pmatrix} 2\lambda \\ 3-\lambda \\ 1+2\lambda \end{pmatrix}$.

$\overrightarrow{FH} = \begin{pmatrix} 5-2\lambda \\ 2-(3-\lambda) \\ 7-(1+2\lambda) \end{pmatrix} = \begin{pmatrix} 5-2\lambda \\ -1+\lambda \\ 6-2\lambda \end{pmatrix}$ **M1**

Perpendicularity: $\overrightarrow{FH} \cdot \mathbf{d} = 0$; $\mathbf{d} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$:

$2(5-2\lambda) + (-1)(-1+\lambda) + 2(6-2\lambda) = 10 - 4\lambda + 1 - \lambda + 12 - 4\lambda = 23 - 9\lambda = 0$

$\lambda = \dfrac{23}{9}$ **A1**

Foot: $F = \begin{pmatrix} 2(23/9) \\ 3-(23/9) \\ 1+2(23/9) \end{pmatrix} = \begin{pmatrix} 46/9 \\ 4/9 \\ 55/9 \end{pmatrix}$

i.e. $F = \left(\frac{46}{9}, \frac{4}{9}, \frac{55}{9}\right)$.

(b) $\overrightarrow{FH} = \begin{pmatrix} 5-46/9 \\ 2-4/9 \\ 7-55/9 \end{pmatrix} = \begin{pmatrix} -1/9 \\ 14/9 \\ 8/9 \end{pmatrix}$ **M1**

$|FH| = \dfrac{1}{9}\sqrt{1+196+64} = \dfrac{\sqrt{261}}{9} = \dfrac{3\sqrt{29}}{9} = \dfrac{\sqrt{29}}{3} \approx 1.80$ km **A1**

Note: Accept $\sqrt{29}/3$ or 1.80 (3 s.f.).

Q12 — Angle & Intersection of Lines MEDIUM

(a) $\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$

$\mathbf{d}_1 \cdot \mathbf{d}_2 = -2 + 3 - 4 = -3; \quad |\mathbf{d}_1| = 3; \quad |\mathbf{d}_2| = \sqrt{1+9+4} = \sqrt{14}$

$\cos\theta = \dfrac{3}{3\sqrt{14}} = \dfrac{1}{\sqrt{14}}; \quad \theta = \arccos\left(\dfrac{1}{\sqrt{14}}\right) = 74.5^\circ$ (1 d.p.) **M1 A1**

(b) Equating: $\begin{pmatrix} 1+2\lambda \\ 2\lambda \\ -3+2\lambda \end{pmatrix} = \begin{pmatrix} 4-\mu \\ 1+3\mu \\ 3-2\mu \end{pmatrix}$

From y: $2\lambda = 1 + 3\mu$. From x: $1 + 2\lambda = 4 - \mu \Rightarrow 2\lambda + \mu = 3$. **M1**

Substituting $2\lambda = 1 + 3\mu$: $1 + 3\mu + \mu = 3 \Rightarrow \mu = \frac{1}{2}; \quad \lambda = \frac{5}{4}$.

Check z: ℓ_1: $-3 + 2(\frac{5}{4}) = -3 + \frac{5}{2} = -\frac{1}{2}$; $\quad \ell_2$: $3 - 2(\frac{1}{2}) = 2$. $-\frac{1}{2} \neq 2$, contradiction.
The lines are **skew** (no intersection). **A1**

Q13 — Shortest Distance: Point to Line MEDIUM

$$\overrightarrow{AP} = P - A = \begin{pmatrix} 6-3 \\ 5-0 \\ 1-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}; \quad \mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$ **M1**

$$\overrightarrow{AP} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 5 & -1 \\ 1 & 2 & -1 \end{vmatrix} = \mathbf{i}(5 \cdot (-1) - (-1) \cdot 2) - \mathbf{j}(3 \cdot (-1) - (-1) \cdot 1) + \mathbf{k}(3 \cdot 2 - 5 \cdot 1)$$

$$= \mathbf{i}(-5+2) - \mathbf{j}(-3+1) + \mathbf{k}(6-5) = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$ **A1**

$$|\overrightarrow{AP} \times \mathbf{d}| = \sqrt{9+4+1} = \sqrt{14}; \quad |\mathbf{d}| = \sqrt{1+4+1} = \sqrt{6}$$

$$d = \frac{\sqrt{14}}{\sqrt{6}} = \sqrt{\frac{14}{6}} = \sqrt{\frac{7}{3}} = 1.527\ldots \approx 1.53 \text{ m}$$ **M1 A1**

Note: Accept $\sqrt{7/3}$ in exact form.

Q14 — Intersection of Two Lines MEDIUM

Equating ℓ_1 and ℓ_2: $\begin{pmatrix} 1+2\lambda \\ 2-\lambda \\ 3+\lambda \end{pmatrix} = \begin{pmatrix} 3+\mu \\ 0+\mu \\ 4-\mu \end{pmatrix}$ **M1**

x: $1 + 2\lambda = 3 + \mu$ ... (i)
y: $2 - \lambda = \mu$ ... (ii)
z: $3 + \lambda = 4 - \mu$ ... (iii)

From (ii): $\mu = 2 - \lambda$. Sub into (i): $1 + 2\lambda = 3 + (2 - \lambda) = 5 - \lambda \Rightarrow 3\lambda = 4 \Rightarrow \lambda = \frac{4}{3}, \mu = \frac{2}{3}$. **A1**

Check (iii): LHS $= 3 + \frac{4}{3} = \frac{13}{3}$; RHS $= 4 - \frac{2}{3} = \frac{10}{3}$. $\frac{13}{3} \neq \frac{10}{3}$ — inconsistent. **M1**

The lines **do not intersect** (they are skew). **A1**

Note: Award M1A1 for solving two equations, M1 for checking third, A1 for correct conclusion.

Q15 — Closest Approach of Two Moving Objects MEDIUM

(a) $\mathbf{r}_A - \mathbf{r}_B = \begin{pmatrix} 4+3t-(14+t) \\ 1+2t-(-3+4t) \\ 0 \end{pmatrix} = \begin{pmatrix} -10+2t \\ 4-2t \\ 0 \end{pmatrix}$ **A1**

(b) $|\mathbf{r}_A - \mathbf{r}_B|^2 = (-10+2t)^2 + (4-2t)^2 = 100 - 40t + 4t^2 + 16 - 16t + 4t^2 = 8t^2 - 56t + 116$ **M1**

Minimise: $\dfrac{d}{dt}(8t^2 - 56t + 116) = 16t - 56 = 0 \Rightarrow t = 3.5$ s **A1**

Min distance$^2 = 8(3.5)^2 - 56(3.5) + 116 = 98 - 196 + 116 = 18$

Min distance $= \sqrt{18} = 3\sqrt{2} \approx 4.24$ cm **A1**

Note: Accept $3\sqrt{2}$ or 4.24 (3 s.f.). Award FT from correct expression in (a).

Q16 — Parallel Lines & Distance Between Them MEDIUM

(a) $\mathbf{d}_2 = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = -\mathbf{d}_1$, so $\mathbf{d}_2$ is a scalar multiple of $\mathbf{d}_1$. **A1**

Hence $\ell_1 \parallel \ell_2$. AG

(b) Take $A(0,1,5)$ on ℓ_1 and $B(9,5,-1)$ on ℓ_2. $\overrightarrow{AB} = \begin{pmatrix} 9 \\ 4 \\ -6 \end{pmatrix}$ **M1**

$\mathbf{d}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$; $\quad |\mathbf{d}_1| = \sqrt{14}$

$$\overrightarrow{AB} \times \mathbf{d}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & 4 & -6 \\ 3 & 2 & -1 \end{vmatrix} = \mathbf{i}(4(-1) - (-6)(2)) - \mathbf{j}(9(-1) - (-6)(3)) + \mathbf{k}(9 \cdot 2 - 4 \cdot 3)$$

$$= \mathbf{i}(-4+12) - \mathbf{j}(-9+18) + \mathbf{k}(18-12) = \begin{pmatrix} 8 \\ -9 \\ 6 \end{pmatrix}$$ **A1**

$|\overrightarrow{AB} \times \mathbf{d}_1| = \sqrt{64+81+36} = \sqrt{181}$

$d = \dfrac{\sqrt{181}}{\sqrt{14}} = \sqrt{\dfrac{181}{14}} = 3.593\ldots \approx 3.59$ m **M1 A1**

Q17 — Unknown Parameter for Perpendicularity MEDIUM

Direction vectors: $\mathbf{d}_\ell = \begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix}$, $\mathbf{d}_m = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ **M1**

Perpendicularity requires $\mathbf{d}_\ell \cdot \mathbf{d}_m = 0$:

$2(1) + k(1) + 1(-3) = 0 \Rightarrow 2 + k - 3 = 0 \Rightarrow k - 1 = 0 \Rightarrow k = 1$ **A1**

Verification: $\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=2+1-3=0$ ✓ **A1**

Hence $k=1$ makes ℓ perpendicular to m.

Note: Award M1 for setting dot product to zero; A1 for $k=1$; A1 for verification or justification.

Q18 — Angle Between Road and Path MEDIUM

(a) $\overrightarrow{AB}=B-A=\begin{pmatrix}3\\-3\\3\end{pmatrix}$; $\mathbf{r}_L=\begin{pmatrix}2\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-3\\3\end{pmatrix}$ (or equivalent) **A1**

(b) $\overrightarrow{BC}=C-B=\begin{pmatrix}3\\-2\\-1\end{pmatrix}$ **A1**

(c) $\mathbf{d}_L=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ (simplified), $\mathbf{d}_{BC}=\begin{pmatrix}3\\-2\\-1\end{pmatrix}$

$\mathbf{d}_L\cdot\mathbf{d}_{BC}=3+2-1=4;\quad |\mathbf{d}_L|=\sqrt{3};\quad |\mathbf{d}_{BC}|=\sqrt{9+4+1}=\sqrt{14}$

$\cos\theta=\dfrac{|4|}{\sqrt{3}\cdot\sqrt{14}}=\dfrac{4}{\sqrt{42}}$ **M1**

$\theta=\arccos\left(\dfrac{4}{\sqrt{42}}\right)=51.9^\circ$ (3 s.f.) **A1**

Q19 — Shortest Distance Between Skew Lines MEDIUM

$\mathbf{d}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\mathbf{d}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$

$\mathbf{n}=\mathbf{d}_1\times\mathbf{d}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\0&1&1\end{vmatrix}=\mathbf{i}(0-1)-\mathbf{j}(1-0)+\mathbf{k}(1-0)=\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ **M1**

$|\mathbf{n}|=\sqrt{3}$

Connecting vector: $\mathbf{a}_2-\mathbf{a}_1=\begin{pmatrix}2\\3\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}2\\3\\0\end{pmatrix}$ **A1**

$$d = \frac{\left|\begin{pmatrix}2\\3\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right|}{\sqrt{3}} = \frac{|-2-3+0|}{\sqrt{3}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3} \approx 2.89 \text{ m}$$ **M1 A1**

Note: Accept $\frac{5}{\sqrt{3}}$, $\frac{5\sqrt{3}}{3}$, *or* 2.89 *(3 s.f.).*

Q20 — Closest Distance from Origin to Line MEDIUM

(a) General point on path: $\mathbf{r}(\lambda) = \begin{pmatrix}10-2\lambda\\3\lambda\\4+\lambda\end{pmatrix}$

Vector from $\mathbf{r}(\lambda)$ to origin: $\mathbf{v} = -\mathbf{r}(\lambda) = \begin{pmatrix}-10+2\lambda\\-3\lambda\\-4-\lambda\end{pmatrix}$

For foot of perpendicular: $\mathbf{v}\cdot\mathbf{d} = 0$, $\mathbf{d} = \begin{pmatrix}-2\\3\\1\end{pmatrix}$: **M1**

$(-10+2\lambda)(-2) + (-3\lambda)(3) + (-4-\lambda)(1) = 0$

$20 - 4\lambda - 9\lambda - 4 - \lambda = 0 \Rightarrow 16 - 14\lambda = 0 \Rightarrow \lambda = \frac{8}{7}$

Foot: $F = \begin{pmatrix}10-16/7\\24/7\\4+8/7\end{pmatrix} = \begin{pmatrix}54/7\\24/7\\36/7\end{pmatrix}$ **A1**

(b) $|OF| = \frac{1}{7}\sqrt{54^2+24^2+36^2} = \frac{1}{7}\sqrt{2916+576+1296} = \frac{1}{7}\sqrt{4788}$

$= \frac{\sqrt{4788}}{7} = \frac{6\sqrt{133}}{7} \approx 9.87$ km **M1 A1**

Note: Full precision: $\sqrt{4788}/7 = 9.869\ldots \approx 9.87$ *km.*

Worked Solutions — Section C (Hard)

Q21 — Unknown Parameter: Perpendicularity & Angle HARD

(a) $\mathbf{d}_1 = \begin{pmatrix} k \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ k \\ -1 \end{pmatrix}$

Perpendicularity: $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ **M1**

$k(1) + 2(k) + 3(-1) = 0 \Rightarrow k + 2k - 3 = 0 \Rightarrow 3k = 3 \Rightarrow k = 1$ **A1**

Verify: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1 + 2 - 3 = 0$ ✓ **A1**

(b) With $k = 1$: $\mathbf{d}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{d}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\cos\theta = \dfrac{|1+2+0|}{\sqrt{14}\cdot\sqrt{2}} = \dfrac{3}{\sqrt{28}} = \dfrac{3}{2\sqrt{7}}$ **M1**

$\theta = \arccos\left(\dfrac{3}{2\sqrt{7}}\right) = 55.4°$ (3 s.f.) **A1**

Q22 — Skew Lines: Show Cross Product & Find Distance HARD

(a) $\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$

$= \mathbf{i}(1\cdot 2 - 0\cdot 0) - \mathbf{j}(2\cdot 2 - 0\cdot 1) + \mathbf{k}(2\cdot 0 - 1\cdot 1)$

$= \mathbf{i}(2) - \mathbf{j}(4) + \mathbf{k}(-1) = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}$ **M1 AG**

Note: Both M1 lines of the determinant expansion must be shown.

(b) $|\mathbf{d}_1 \times \mathbf{d}_2| = \sqrt{4+16+1} = \sqrt{21}$

$\mathbf{a}_2 - \mathbf{a}_1 = \begin{pmatrix} 0-1 \\ 4-0 \\ 1-3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}$ **M1**

$(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{d}_1 \times \mathbf{d}_2) = (-1)(2) + (4)(-4) + (-2)(-1) = -2 - 16 + 2 = -16$

$d = \dfrac{|-16|}{\sqrt{21}} = \dfrac{16}{\sqrt{21}} = \dfrac{16\sqrt{21}}{21}$ m **A1 A1**

Note: Award A1 for $|-16|/\sqrt{21}$*; A1 for exact rationalised form* $\frac{16\sqrt{21}}{21}$*. Accept* 3.49 *m (3 s.f.) for final A1.*

Q23 — Three Lines: Determine Which Two Intersect HARD

Test $\ell_1 \cap \ell_2$**:**

$$\begin{pmatrix} 1+2\lambda \\ 3+\lambda \\ 2-\lambda \end{pmatrix} = \begin{pmatrix} 3+\mu \\ 4-\mu \\ 1 \end{pmatrix}$$

From z: $2-\lambda = 1 \Rightarrow \lambda = 1$. From y: $3+1 = 4-\mu \Rightarrow \mu = 0$. **M1**

Check x: $1+2(1) = 3; \quad 3+0 = 3$ ✓. **A1**

ℓ_1 and ℓ_2 **intersect**. Point: $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, i.e. $(3, 4, 1)$. **A1**

Test $\ell_1 \cap \ell_3$ (for completeness):

$$\begin{pmatrix} 1+2\lambda \\ 3+\lambda \\ 2-\lambda \end{pmatrix} = \begin{pmatrix} 5-\nu \\ 2+2\nu \\ 0+\nu \end{pmatrix}$$

From z: $2-\lambda = \nu$. From x: $1+2\lambda = 5-\nu \Rightarrow 2\lambda+\nu = 4$. Sub: $2\lambda + (2-\lambda) = 4 \Rightarrow \lambda = 2, \nu = 0$. **M1**

Check y: $3+2 = 5; \quad 2+0 = 2$. $5 \neq 2$: **inconsistent**, so ℓ_1 and ℓ_3 do not intersect.

The two fences that intersect are ℓ_1 and ℓ_2, at the point $(3, 4, 1)$. **A1**

Note: Award M1A1 for each consistent/inconsistent conclusion with justification. Award final A1 for correct identification and point.

Q24 — Foot of Perpendicular & Area of Triangle HARD

(a) $\ell : \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. General point $F = \begin{pmatrix} \lambda \\ 2+2\lambda \\ 1+2\lambda \end{pmatrix}$.

$\overrightarrow{FW} = W - F = \begin{pmatrix} 4-\lambda \\ 3-(2+2\lambda) \\ -1-(1+2\lambda) \end{pmatrix} = \begin{pmatrix} 4-\lambda \\ 1-2\lambda \\ -2-2\lambda \end{pmatrix}$ **M1**

$\overrightarrow{FW} \cdot \mathbf{d} = 0$: $(4-\lambda)(1) + (1-2\lambda)(2) + (-2-2\lambda)(2) = 0$

$4-\lambda+2-4\lambda-4-4\lambda = 0 \Rightarrow 2-9\lambda = 0 \Rightarrow \lambda = \dfrac{2}{9}$ **A1**

$$F = \begin{pmatrix} 2/9 \\ 2+4/9 \\ 1+4/9 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 22/9 \\ 13/9 \end{pmatrix}$$ **A1**

(b) $\overrightarrow{FW} = \begin{pmatrix} 4-2/9 \\ 3-22/9 \\ -1-13/9 \end{pmatrix} = \begin{pmatrix} 34/9 \\ 5/9 \\ -22/9 \end{pmatrix}$; $\overrightarrow{FV} = V - F = \begin{pmatrix} 7-2/9 \\ 1-22/9 \\ 5-13/9 \end{pmatrix} = \begin{pmatrix} 61/9 \\ -13/9 \\ 32/9 \end{pmatrix}$ **M1**

$$\overrightarrow{FW} \times \overrightarrow{FV} = \frac{1}{81}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 34 & 5 & -22 \\ 61 & -13 & 32 \end{vmatrix}$$

$$= \frac{1}{81}[\mathbf{i}(5\cdot 32 - (-22)(-13)) - \mathbf{j}(34\cdot 32 - (-22)\cdot 61) + \mathbf{k}(34(-13) - 5\cdot 61)]$$

$$= \frac{1}{81}[\mathbf{i}(160-286) - \mathbf{j}(1088+1342) + \mathbf{k}(-442-305)]$$

$$= \frac{1}{81}\begin{pmatrix} -126 \\ -2430 \\ -747 \end{pmatrix}$$

$$|\overrightarrow{FW} \times \overrightarrow{FV}| = \frac{1}{81}\sqrt{126^2 + 2430^2 + 747^2} = \frac{1}{81}\sqrt{15876 + 5904900 + 558009} = \frac{\sqrt{6478785}}{81}$$

$$= \frac{2545.35\ldots}{81} \approx 31.42\ldots$$

Area $= \frac{1}{2}|\overrightarrow{FW} \times \overrightarrow{FV}| \approx \frac{1}{2}(31.42) \approx 15.7$ m^2 **A1**

Note: Award M1 for setting up the cross product; A1 for ≈ 15.7 m^2 (3 s.f.).

Q25 — Ship Closest Approach & Safety Check HARD

(a) $\mathbf{r}_P - \mathbf{r}_Q = \begin{pmatrix} 3+2t \\ -1+4t \\ 0 \end{pmatrix} - \begin{pmatrix} 7-t \\ 5+t \\ 0 \end{pmatrix} = \begin{pmatrix} -4+3t \\ -6+3t \\ 0 \end{pmatrix}$ **M1**

$|\mathbf{r}_P - \mathbf{r}_Q|^2 = (-4+3t)^2 + (-6+3t)^2 = 16 - 24t + 9t^2 + 36 - 36t + 9t^2 = 18t^2 - 60t + 52$ **A1**

(b) $\frac{d}{dt}(18t^2 - 60t + 52) = 36t - 60 = 0 \Rightarrow t = \frac{5}{3}$ h (after noon) **M1**

Min $|\mathbf{r}_P - \mathbf{r}_Q|^2 = 18\left(\frac{5}{3}\right)^2 - 60\left(\frac{5}{3}\right) + 52 = 18\cdot\frac{25}{9} - 100 + 52 = 50 - 100 + 52 = 2$

Min distance $= \sqrt{2} \approx 1.41$ km at $t = \frac{5}{3}$ h ≈ 1 h 40 min after noon **A1**

(c) Min distance $= \sqrt{2} \approx 1.41$ km < 3 km. The safety rule **is violated**. **R1**

Q26 — Unknown m for Intersection & Angle HARD

(a) For intersection: $\begin{pmatrix} 2+\lambda \\ -1+2\lambda \\ 4-\lambda \end{pmatrix} = \begin{pmatrix} m+3\mu \\ 1 \\ 2+\mu \end{pmatrix}$

From y: $-1+2\lambda = 1 \Rightarrow \lambda = 1$. **M1**

From z: $4-1 = 2+\mu \Rightarrow \mu = 1$. **A1**

From x: $2+1 = m+3(1) \Rightarrow 3 = m+3 \Rightarrow m = 0$ **A1**

(b) $\mathbf{d}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 3+0-1 = 2$; $\quad |\mathbf{d}_1| = \sqrt{6}$; $\quad |\mathbf{d}_2| = \sqrt{10}$

$\cos\theta = \dfrac{2}{\sqrt{6}\cdot\sqrt{10}} = \dfrac{2}{\sqrt{60}} = \dfrac{1}{\sqrt{15}}$; $\quad \theta = \arccos\left(\dfrac{1}{\sqrt{15}}\right) = 75.0^\circ$ (3 s.f.) **M1 A1**

Q27 — Foot of Perpendicular & Triangle Area HARD

(a) General point: $F = \begin{pmatrix} 1+2\lambda \\ 3\lambda \\ 4-\lambda \end{pmatrix}$.

$\overrightarrow{FB} = B - F = \begin{pmatrix} 3-(1+2\lambda) \\ -1-3\lambda \\ 5-(4-\lambda) \end{pmatrix} = \begin{pmatrix} 2-2\lambda \\ -1-3\lambda \\ 1+\lambda \end{pmatrix}$ **M1**

$\overrightarrow{FB}\cdot\mathbf{d} = 0$, $\mathbf{d} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$:

$2(2-2\lambda) + 3(-1-3\lambda) + (-1)(1+\lambda) = 4-4\lambda-3-9\lambda-1-\lambda = 0$

$0 - 14\lambda = 0 \Rightarrow \lambda = 0$ **A1**

$F = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$ (the starting point of the line). **A1**

(b) $\overrightarrow{FB} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $\quad \overrightarrow{FC} = C - F = \begin{pmatrix} 4 \\ 9 \\ -2 \end{pmatrix}$

$\overrightarrow{FB}\times\overrightarrow{FC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 4 & 9 & -2 \end{vmatrix} = \mathbf{i}(2-9) - \mathbf{j}(-4-4) + \mathbf{k}(18+4) = \begin{pmatrix} -7 \\ 8 \\ 22 \end{pmatrix}$ **M1**

Area $= \frac{1}{2}|\overrightarrow{FB}\times\overrightarrow{FC}| = \frac{1}{2}\sqrt{49+64+484} = \frac{1}{2}\sqrt{597} \approx \frac{1}{2}(24.43) \approx 12.2$ m^2 **A1**

Note: Accept $\frac{\sqrt{597}}{2}$ (exact) or 12.2 m^2 (3 s.f.).

Q28 — Closest Points on Skew Lines **HARD**

(a) Let $F_1 = \begin{pmatrix} \lambda \\ 1+\lambda \\ 2 \end{pmatrix}$ on m_1 and $F_2 = \begin{pmatrix} 3 \\ 0+\mu \\ 5+\mu \end{pmatrix}$ on m_2.

$\overrightarrow{F_1F_2} = \begin{pmatrix} 3-\lambda \\ \mu-1-\lambda \\ 3+\mu \end{pmatrix}$

For shortest segment: $\overrightarrow{F_1F_2} \perp \mathbf{d}_1$ and $\overrightarrow{F_1F_2} \perp \mathbf{d}_2$. **M1**

$\mathbf{d}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$: $(3-\lambda)+(\mu-1-\lambda)+0=0 \Rightarrow 2-2\lambda+\mu=0$ ...(i)

$\mathbf{d}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$: $0+(\mu-1-\lambda)+(3+\mu)=0 \Rightarrow 2\mu-\lambda+2=0$ ...(ii) **A1**

From (i): $\mu = 2\lambda-2$. Sub into (ii): $2(2\lambda-2)-\lambda+2=0 \Rightarrow 3\lambda-2=0 \Rightarrow \lambda=\frac{2}{3}$, $\mu=-\frac{2}{3}$.

$F_1 = \begin{pmatrix} 2/3 \\ 5/3 \\ 2 \end{pmatrix}$; $F_2 = \begin{pmatrix} 3 \\ -2/3 \\ 13/3 \end{pmatrix}$ **A1**

(b) $\overrightarrow{F_1F_2} = \begin{pmatrix} 3-2/3 \\ -2/3-5/3 \\ 13/3-2 \end{pmatrix} = \begin{pmatrix} 7/3 \\ -7/3 \\ 7/3 \end{pmatrix}$

$|F_1F_2| = \frac{7}{3}\sqrt{3} = \frac{7\sqrt{3}}{3} \approx 4.04$ m **M1 A1**

Note: Accept $7\sqrt{3}/3$ (exact) or 4.04 m (3 s.f.).

Q29 — Show Point Not on Line; Cross Product Distance **HARD**

(a) If $T(6,-3,9)$ lies on ℓ: $\begin{pmatrix} 6 \\ -3 \\ 9 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ 5-2\lambda \\ 1+2\lambda \end{pmatrix}$

From x: $\lambda=4$. Check y: $5-8=-3$ ✓. Check z: $1+8=9$ ✓. **M1**

All three equations are satisfied with $\lambda=4$, so T **does lie on** ℓ. **A1**

Note: Since T lies on ℓ, the shortest distance from T to ℓ is 0. This is the solution to part (b).

(b) $\overrightarrow{AT} = T - A = \begin{pmatrix} 4 \\ -8 \\ 8 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$\overrightarrow{AT} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -8 & 8 \\ 1 & -2 & 2 \end{vmatrix} = \mathbf{i}((-8)(2) - (8)(-2)) - \mathbf{j}((4)(2) - (8)(1)) + \mathbf{k}((4)(-2) - (-8)(1))$$

$$= \mathbf{i}(-16 + 16) - \mathbf{j}(8 - 8) + \mathbf{k}(-8 + 8) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ **M1**

$|\overrightarrow{AT} \times \mathbf{d}| = 0 \Rightarrow d = 0$ **A1 A1**

The shortest distance is 0 m, confirming that T lies on ℓ.

Note: Award M1 for attempting the cross product correctly; A1 for **0** *vector; A1 for $d = 0$ with conclusion. Part (a) must clearly demonstrate that T is on ℓ to earn both marks.*

Q30 — Values of k for Perpendicularity & Shortest Distance HARD

(a) $\mathbf{d}_1 = \begin{pmatrix} 3 \\ k \\ -2 \end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix} k \\ -1 \\ 3 \end{pmatrix}$

Perpendicularity: $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ **M1**

$3k + k(-1) + (-2)(3) = 0 \Rightarrow 3k - k - 6 = 0 \Rightarrow 2k = 6 \Rightarrow k = 3$ **A1**

Note: There is only one value since $3k - k - 6 = 2k - 6 = 0$ gives a unique solution. **A1**

(b) With $k = 3$: $\ell_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$. Point $P(4, 2, 0)$.

$$\overrightarrow{AP} = P - A_1 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \mathbf{d}_1 = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$$

$$\overrightarrow{AP} \times \mathbf{d}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -2 \\ 3 & 3 & -2 \end{vmatrix} = \mathbf{i}(2(-2) - (-2)(3)) - \mathbf{j}(3(-2) - (-2)(3)) + \mathbf{k}(3 \cdot 3 - 2 \cdot 3)$$

$$= \mathbf{i}(-4 + 6) - \mathbf{j}(-6 + 6) + \mathbf{k}(9 - 6) = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$ **M1**

$|\overrightarrow{AP} \times \mathbf{d}_1| = \sqrt{4 + 0 + 9} = \sqrt{13}; \quad |\mathbf{d}_1| = \sqrt{9 + 9 + 4} = \sqrt{22}$

$$d = \frac{\sqrt{13}}{\sqrt{22}} = \sqrt{\frac{13}{22}} = 0.7689\ldots \approx 0.769$$ **A1**

Note: Accept $\sqrt{13/22}$ (exact) or 0.769 (3 s.f.). Units are those of the coordinate system.

End of Worksheet

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