IB Math AI HL — Topic 3 [Set 1]

Modelling with Vectors

Kinematics with Vectors · Constant & Variable Velocity · Position, Speed & Distance

Difficulty	EASY	MEDIUM	HARD	
Questions	10	10	10	**30 total**
Marks each	3	4	5	
Section marks	30	40	50	**120 total**

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Name:____________________________ Date:______________

Score:__________/120

Instructions: Answer all questions. Show all working unless "write down" is stated. Give answers correct to 3 significant figures unless otherwise stated. A GDC is permitted throughout.

Key Formulae — Vectors & Kinematics

- **Position (const. velocity):** $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ **Position (variable):** $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt + \mathbf{c}$
- **Velocity:** $\mathbf{v}(t) = \mathbf{r}'(t)$ **Acceleration:** $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$
- **Speed:** $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$ **Distance travelled** (t_1 to t_2): $\int_{t_1}^{t_2} |\mathbf{v}(t)|\,dt$
- **Displacement:** $\mathbf{r}(t_2) - \mathbf{r}(t_1)$ **Closest approach:** minimise $|\mathbf{r}_1(t) - \mathbf{r}_2(t)|$
- **Perpendicular:** $\mathbf{u} \cdot \mathbf{v} = 0$ **Parallel:** $\mathbf{u} = k\mathbf{v}$ **Angle:** $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

GDC Tips

- Use **nDeriv** / **d/dx** to differentiate vector components; **fnInt** / $\int$ to integrate.
- **Closest approach:** store $|\mathbf{r}_1(t) - \mathbf{r}_2(t)|^2$ as Y1, use **minimum** in the graph menu.
- **Interception:** set component equations equal; solve the system using **solve( )**.
- **Graph** $|\mathbf{v}(t)|$ to locate when the particle is at rest (speed $= 0$).
- Store $\sqrt{(x_1(t))^2 + (y_1(t))^2}$ as a function of t to find distance from origin.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY GDC **[3 marks]**

A surveillance drone starts from the point with position vector $\mathbf{r}_0 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ m and moves with constant velocity $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ m s^{-1}.

(a) Write down the position vector $\mathbf{r}(t)$ of the drone at time t seconds. [1]

(b) Find the speed of the drone. [1]

(c) Find the position of the drone after 4 seconds. [1]

2. EASY GDC **[3 marks]**

A particle moves from point $P(1, -2, 3)$ to point $Q(5, -3, 5)$. Distances are in metres.

(a) Write down the displacement vector $\overrightarrow{PQ}$. [1]

(b) Find the distance PQ. [1]

(c) Write down a unit vector in the direction of $\overrightarrow{PQ}$. [1]

3. EASY GDC **[3 marks]**

A cargo ship is at position $\begin{pmatrix}-3\\7\end{pmatrix}$ km at time $t = 0$ hours and at position $\begin{pmatrix}5\\3\end{pmatrix}$ km at time $t = 4$ hours. The ship travels with constant velocity.

(a) Find the velocity vector of the ship in km h^{-1}. [2]

(b) Find the time at which the x-coordinate of the ship equals 9 km. [1]

4. EASY GDC **[3 marks]**

A remote-controlled car moves so that its position vector at time t seconds is given by $\mathbf{r}(t) = \begin{pmatrix}3t^2 - 2\\4t\end{pmatrix}$ metres.

(a) Write down the velocity vector $\mathbf{v}(t)$. [1]

(b) Find the speed of the car when $t = 1$. [1]

(c) Find the value of t when the horizontal component of velocity equals 6 m s^{-1}. [1]

5. EASY GDC **[3 marks]**

A boat has velocity vector $\mathbf{v}(t) = \begin{pmatrix}2\\3t\end{pmatrix}$ m s^{-1} at time t seconds. At $t = 0$, the boat is at position $\begin{pmatrix}1\\-2\end{pmatrix}$ m.

(a) Find the position vector $\mathbf{r}(t)$. [2]

(b) Find the position of the boat when $t = 3$. [1]

6. EASY GDC **[3 marks]**

Drone A travels with velocity $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ m s^{-1} and Drone B travels with velocity $\mathbf{v}_2 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ m s^{-1}.

(a) Show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. [1]

(b) State what this result tells you about the directions of motion of the two drones. [1]

(c) Find the speed of Drone A. [1]

7. EASY GDC **[3 marks]**

A particle moves with constant acceleration $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ m s^{-2}. The initial velocity of the particle is $\mathbf{v}(0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ m s^{-1}.

(a) Write down the velocity vector $\mathbf{v}(t)$. [1]

(b) Find the value of t when the y-component of velocity is zero. [1]

(c) Find the speed of the particle at this time. [1]

8. EASY GDC **[3 marks]**

A particle moves so that its position vector at time t seconds is $\mathbf{r}(t) = \begin{pmatrix} 5\cos t \\ 5\sin t \end{pmatrix}$ metres, for $t \geq 0$.

(a) Write down the velocity vector $\mathbf{v}(t)$. [1]

(b) Show that the speed of the particle is constant and equal to 5 m s^{-1}. [1]

(c) Write down the shape of the path traced by the particle. [1]

9. EASY GDC **[3 marks]**

A particle moves along a straight track. Its velocity at time t seconds is $v(t) = 6t - t^2$ m s^{-1}, for $0 \leq t \leq 6$. The displacement from the origin at $t = 0$ is zero.

(a) Find the displacement $s(t)$ from the origin. [1]

(b) Find the displacement when $t = 6$. [1]

(c) Find the maximum speed of the particle in the interval $0 \leq t \leq 6$. [1]

10. EASY GDC **[3 marks]**

The table gives the position of a kayak at selected times.

t (s)	x (m)	y (m)
0	2	-1
2	8	5
4	14	11
6	20	17

(a) Write down the position vector of the kayak at $t = 0$. [1]

(b) Find the average velocity vector over the interval $0 \leq t \leq 6$ s. [1]

(c) Write down the constant velocity vector of the kayak. [1]

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM GDC **[4 marks]**

A hiking path has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$, where distances are in km. A ranger station is located at the point $A(7, 0)$ km.

(a) Find the value of t at the point on the path closest to the ranger station. [2]

(b) Hence find the shortest distance from the ranger station to the path. [2]

12. MEDIUM GDC **[4 marks]**

A particle P moves with velocity $\mathbf{v}(t) = \begin{pmatrix} 2t-1 \\ e^t \end{pmatrix}$ m s^{-1}. At $t = 0$, the position of P is $\mathbf{r}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ m.

(a) Find the position vector $\mathbf{r}(t)$. [2]

(b) Find the distance of P from the origin when $t = 2$. [2]

13. MEDIUM GDC **[4 marks]**

Two particles move along straight paths. Their position vectors are

$$\mathbf{r}_1(t) = t\begin{pmatrix}3\\2\end{pmatrix} \quad \text{and} \quad \mathbf{r}_2(t) = \begin{pmatrix}4\\6\end{pmatrix} + t\begin{pmatrix}1\\-1\end{pmatrix},$$

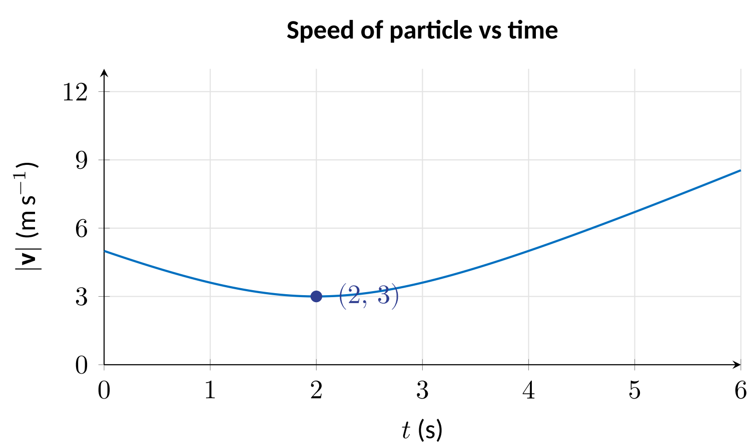
where t is in seconds and distances in metres.

(a) Find the value of t at which the two particles are at the same position. [3]

(b) Write down the position vector of this point. [1]

14. MEDIUM GDC **[4 marks]**

A particle moves so that its position vector at time t seconds is $\mathbf{r}(t) = \begin{pmatrix}t^2 - 4t\\3t\end{pmatrix}$ metres. The graph shows the speed $|\mathbf{v}|$ of the particle against time.



(a) Find the velocity vector $\mathbf{v}(t)$. [1]

(b) Show that $|\mathbf{v}|^2 = 4t^2 - 16t + 25$. [1]

(c) Find the minimum speed of the particle and the value of t at which it occurs. [2]

15. MEDIUM GDC **[4 marks]**

A ball rolls so that its velocity at time t seconds is $\mathbf{v}(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ m s^{-1}. The ball is at the origin at $t = 0$.

(a) Find the position vector $\mathbf{r}(t)$. [2]

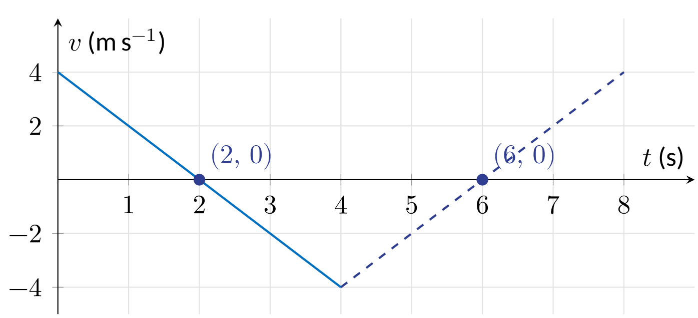
(b) Find the displacement of the ball from the origin when $t = \pi$. [1]

(c) Show that the speed of the ball is constant and equal to 1 m s^{-1}. Hence find the total distance travelled from $t = 0$ to $t = \pi$. [1]

16. MEDIUM GDC **[4 marks]**

The graph shows the velocity v (m s^{-1}) of a particle moving along a straight line for $0 \leq t \leq 8$ s.

Velocity of particle vs time



(a) Write down the velocity function as a piecewise expression. [1]

(b) Find the total displacement of the particle from $t = 0$ to $t = 8$. [2]

(c) Find the total distance travelled from $t = 0$ to $t = 8$. [1]

17. MEDIUM GDC **[4 marks]**

A particle P has constant acceleration $\mathbf{a} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ m s^{-2}. At $t = 0$, its velocity is $\mathbf{v}(0) = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ m s^{-1} and its position is $\mathbf{r}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ m.

(a) Find the velocity vector $\mathbf{v}(t)$. [1]

(b) Find the position vector $\mathbf{r}(t)$. [1]

(c) Find the values of t when the particle passes through the y-axis ($x = 0$). [2]

18. MEDIUM GDC **[4 marks]**

A beam of light travels along the line with vector equation $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + t\begin{pmatrix}2\\1\\2\end{pmatrix}$, where distances are in metres. A sensor is located at the point $B(4,\,1,\,3)$.

(a) Show that the point on the beam closest to the sensor corresponds to $t = 1$. [2]

(b) Find the shortest distance from the sensor to the beam of light. [2]

19. MEDIUM GDC **[4 marks]**

At $t = 0$ hours, ship P is at position $\begin{pmatrix}0\\0\end{pmatrix}$ km with velocity $\begin{pmatrix}3\\1\end{pmatrix}$ km h^{-1}, and ship Q is at position $\begin{pmatrix}10\\4\end{pmatrix}$ km with velocity $\begin{pmatrix}1\\3\end{pmatrix}$ km h^{-1}.

(a) Write down the position vectors of ships P and Q at time t. [1]

(b) Find an expression for the vector from P to Q at time t. [1]

(c) Find the minimum distance between the two ships, to the nearest 0.1 km. [2]

20. MEDIUM GDC **[4 marks]**

Two particles have velocity vectors $\mathbf{v}_1 = \begin{pmatrix} a \\ 3 \end{pmatrix}$ m s^{-1} and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ b \end{pmatrix}$ m s^{-1}, where a and b are constants. The particles move in perpendicular directions.

(a) Write down an equation relating a and b. [1]

(b) Given that $a = 6$, find the value of b. [1]

(c) Particle 1 starts at position $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ m. Find its position vector when $t = 2$ s. [2]

Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)

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21. **HARD** **GDC** **[5 marks]**

Two aircraft are in the same horizontal plane. At $t = 0$ minutes, aircraft A is at $(0, 0)$ km with velocity $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ km min^{-1}, and aircraft B is at $(12, -2)$ km with velocity $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ km min^{-1}. Aviation regulations require a minimum separation of 5 km.

(a) Write down the position vectors of aircraft A and B at time t. [1]

(b) Show that $|\overrightarrow{AB}|^2 = 17t^2 - 100t + 148$. [2]

(c) Find the time at which the aircraft are closest. [1]

(d) Determine whether the aviation regulation is satisfied at this time. Justify your answer. [1]

22. HARD GDC **[5 marks]**

A particle moves so that its position vector at time t seconds is $\mathbf{r}(t) = \begin{pmatrix} 4\cos 2t \\ 4\sin 2t \end{pmatrix}$ metres.

(a) Find the velocity vector $\mathbf{v}(t)$. [1]

(b) Find the acceleration vector $\mathbf{a}(t)$. [1]

(c) Show that the acceleration is always directed towards the origin and find its magnitude. [2]

(d) Find the distance travelled by the particle from $t = 0$ to $t = \pi/2$. [1]

23. HARD GDC **[5 marks]**

Drone X has position vector $\mathbf{r}_X(t) = \begin{pmatrix} k \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and drone Y has position vector $\mathbf{r}_Y(t) = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, where k is a constant, t is in seconds, and distances in metres. The drones intercept (occupy the same position at the same time).

(a) Find the value of k and the time T at which the drones intercept. [3]

(b) Write down the position vector of the interception point. [1]

(c) Find the angle between the velocity vectors of the two drones. [1]

24. HARD GDC **[5 marks]**

Particle A has position vector $\mathbf{r}_A(t) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and particle B has position vector $\mathbf{r}_B(t) = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ p \end{pmatrix}$, where p is a constant. The particles are closest to each other when $t = 1$.

(a) Find an expression for $|\mathbf{r}_A(t) - \mathbf{r}_B(t)|^2$ in terms of t and p. [2]

(b) Hence find the value of p. [2]

(c) Find the minimum distance between the particles. [1]

25. HARD GDC **[5 marks]**

A particle moves so that its position vector at time t seconds ($t \geq 0$) is $\mathbf{r}(t) = \begin{pmatrix} t^2 \\ t^3/3 \end{pmatrix}$ metres.

(a) Find the velocity vector $\mathbf{v}(t)$. [1]

(b) Show that the speed of the particle is $t\sqrt{4+t^2}$ m s^{-1}. [1]

(c) Find the exact distance travelled from $t = 0$ to $t = 3$. [2]

(d) State one reason why the distance travelled is greater than the magnitude of the displacement. [1]

26. HARD GDC **[5 marks]**

A particle moves so that its position at time $t \geq 0$ seconds is $\mathbf{r}(t) = \begin{pmatrix} t^3/3 - t \\ t^2/2 \end{pmatrix}$ metres.

(a) Find the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ of the particle. [2]

(b) Show that $\mathbf{a}(t) \cdot \mathbf{v}(t) = t(2t^2 - 1)$. [1]

(c) Find the positive value of t at which the acceleration is perpendicular to the velocity. [1]

(d) Find the speed of the particle at this time. Give your answer in exact form. [1]

27. HARD GDC **[5 marks]**

A ball is launched from the point $(1, -1)$ m with velocity $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ m s^{-1}, so its position at time $t \geq 0$ s is $\mathbf{r}(t) = \begin{pmatrix} t+1 \\ 2t-1 \end{pmatrix}$ m. A circular boundary has equation $x^2 + y^2 = 25$.

(a) Show that $|\mathbf{r}(t)|^2 = 5t^2 - 2t + 2$. [1]

(b) Find the time at which the ball crosses the boundary of the circular region. [2]

(c) Determine whether the ball is inside or outside the boundary when $t = 3$. Justify your answer. [1]

(d) Write down the coordinates of the point where the ball crosses the boundary. [1]

28. HARD GDC **[5 marks]**

Particle P has position vector $\mathbf{r}_P(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} p \\ 1 \end{pmatrix}$ and particle Q has position vector $\mathbf{r}_Q(t) = \begin{pmatrix} 7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ p \end{pmatrix}$, where p is a positive constant. At $t = 1$, both particles are the same distance from the origin.

(a) Write expressions for $|\mathbf{r}_P(1)|^2$ and $|\mathbf{r}_Q(1)|^2$ in terms of p. [2]

(b) Hence find the value of p. [2]

(c) Find the distance of each particle from the origin when $t = 1$. [1]

29. HARD GDC **[5 marks]**

A particle moves so that its position vector at time t seconds ($t \geq 0$) is $\mathbf{r}(t) = \begin{pmatrix} t^2 \\ 2t^3/3 \end{pmatrix}$ metres.

(a) Find the velocity vector $\mathbf{v}(t)$. [1]

(b) Show that the speed is $2t\sqrt{1+t^2}$ m s^{-1}. [1]

(c) Using the substitution $u = 1 + t^2$, find the exact distance travelled from $t = 0$ to $t = 2$. Express your answer in the form $k(5\sqrt{5} - 1)$ for rational k. [2]

(d) Write down the numerical value of this distance, correct to 3 significant figures. [1]

30. HARD GDC **[5 marks]**

Particle A has position vector $\mathbf{r}_A(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} a \\ 3 \end{pmatrix}$ and particle B has position vector $\mathbf{r}_B(t) = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + t\begin{pmatrix} -1 \\ b \end{pmatrix}$, where a and b are positive constants. The particles intercept at time $t = T$. It is also given that $b = 2a$.

(a) By setting the position vectors equal and using $b = 2a$, show that $a = 4$ and find the value of T. [3]

(b) Find the position vector of the point of interception. [1]

(c) Find the angle between the velocity vectors of the two particles. [1]

Worked Solutions

M = Method   A = Accuracy   R = Reasoning   AG = Answer Given (no mark)

Section A — Easy

Q1 — Constant Velocity: Position & Speed EASY

(a) $\mathbf{r}(t) = \begin{pmatrix} 2+3t \\ 5-t \end{pmatrix}$ **A1**

(b) Speed $= |\mathbf{v}| = \sqrt{3^2 + (-1)^2} = \sqrt{10} \approx 3.16$ m s^{-1} **A1**

(c) $\mathbf{r}(4) = \begin{pmatrix} 2+12 \\ 5-4 \end{pmatrix} = \begin{pmatrix} 14 \\ 1 \end{pmatrix}$ m **A1**

Q2 — Displacement Vector & Unit Vector EASY

(a) $\overrightarrow{PQ} = Q - P = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ **A1**

(b) $PQ = \sqrt{16+1+4} = \sqrt{21} \approx 4.58$ m **M1 A1**

(c) Unit vector $= \frac{1}{\sqrt{21}} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ **A1**

Q3 — Velocity from Two Positions EASY

(a) $\mathbf{v} = \dfrac{\begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 7 \end{pmatrix}}{4} = \dfrac{\begin{pmatrix} 8 \\ -4 \end{pmatrix}}{4} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ km h^{-1} **M1 A1**

(b) $x(t) = -3 + 2t = 9 \Rightarrow 2t = 12 \Rightarrow t = 6$ hours **A1**

Q4 — Velocity by Differentiation EASY

(a) $\mathbf{v}(t) = \begin{pmatrix} 6t \\ 4 \end{pmatrix}$ **A1**

(b) Speed at $t = 1$: $|\mathbf{v}(1)| = |(6, 4)| = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13} \approx 7.21$ m s^{-1} **M1 A1**

(c) $6t = 6 \Rightarrow t = 1$ s **A1**

Q5 — Position by Integration EASY

(a) Integrate: $\mathbf{r}(t) = \begin{pmatrix} 2t + c_1 \\ 3t^2/2 + c_2 \end{pmatrix}$. Apply $\mathbf{r}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$: $c_1 = 1, c_2 = -2$ **M1**

$\mathbf{r}(t) = \begin{pmatrix} 2t + 1 \\ 3t^2/2 - 2 \end{pmatrix}$ **A1**

(b) $\mathbf{r}(3) = \begin{pmatrix} 7 \\ 27/2 - 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 11.5 \end{pmatrix}$ m **A1**

Q6 — Perpendicular Velocities EASY

(a) $\mathbf{v}_1 \cdot \mathbf{v}_2 = (3)(-4) + (4)(3) = -12 + 12 = 0$ **A1** (AG)

(b) The two drones are moving in perpendicular directions. **R1**

(c) Speed of Drone A $= |(3, 4)| = \sqrt{9 + 16} = 5$ m s^{-1} **A1**

Q7 — Constant Acceleration EASY

(a) $\mathbf{v}(t) = \begin{pmatrix} 5 - t \\ -2 + 3t \end{pmatrix}$ **A1**

(b) $-2 + 3t = 0 \Rightarrow t = \frac{2}{3}$ s **A1**

(c) $\mathbf{v}(\frac{2}{3}) = \begin{pmatrix} 5 - 2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 13/3 \\ 0 \end{pmatrix}$; speed $= \frac{13}{3} \approx 4.33$ m s^{-1} **M1 A1**

Q8 — Circular Motion & Constant Speed EASY

(a) $\mathbf{v}(t) = \begin{pmatrix} -5 \sin t \\ 5 \cos t \end{pmatrix}$ **A1**

(b) $|\mathbf{v}|^2 = 25 \sin^2 t + 25 \cos^2 t = 25(\sin^2 t + \cos^2 t) = 25$, so $|\mathbf{v}| = 5$ m s^{-1} (constant). **M1 A1** (AG)

(c) The particle traces a **circle** of radius 5 m centred at the origin. **A1**

Q9 — 1D Kinematics: Displacement & Max Speed EASY

(a) $s(t) = \int (6t - t^2)\, dt = 3t^2 - \frac{t^3}{3}$ ($s(0) = 0$, so no constant) **M1 A1**

(b) $s(6) = 3(36) - 72 = 108 - 72 = 36$ m **A1**

(c) $v'(t) = 6 - 2t = 0 \Rightarrow t = 3$; $v(3) = 18 - 9 = 9$ m s^{-1} (max speed) **M1 A1**

Q10 — Position from Table EASY

(a) $\mathbf{r}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ m **A1**

(b) Average velocity $= \dfrac{\mathbf{r}(6) - \mathbf{r}(0)}{6} = \dfrac{\begin{pmatrix} 18 \\ 18 \end{pmatrix}}{6} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ m s^{-1} **M1 A1**

(c) Constant velocity $= \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ m s^{-1} (confirmed by uniform spacing in table) **A1**

Section B — Medium

Q11 — Shortest Distance to a Line MEDIUM

(a) Point on path: $\mathbf{r}(t) = (1 + 3t,\ 2 + t)$. Vector to A: $(6 - 3t,\ -2 - t)$.

Perpendicularity condition: $(6 - 3t,\ -2 - t) \cdot (3, 1) = 0$ **M1**

$3(6 - 3t) + 1(-2 - t) = 18 - 9t - 2 - t = 16 - 10t = 0 \Rightarrow t = 1.6$ **A1**

(b) Foot of perpendicular: $\mathbf{r}(1.6) = (5.8,\ 3.6)$.

Distance $= |(7 - 5.8,\ 0 - 3.6)| = |(1.2,\ -3.6)| = \sqrt{1.44 + 12.96} = \sqrt{14.4} \approx 3.79$ km **M1 A1**

Q12 — Integration to Find Position MEDIUM

(a) $\mathbf{r}(t) = \begin{pmatrix} t^2 - t + c_1 \\ e^t + c_2 \end{pmatrix}$. Apply $\mathbf{r}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$: $c_1 = 3$, $e^0 + c_2 = 1 \Rightarrow c_2 = 0$ **M1**

$\mathbf{r}(t) = \begin{pmatrix} t^2 - t + 3 \\ e^t \end{pmatrix}$ **A1 A1**

(b) $\mathbf{r}(2) = \begin{pmatrix} 4 - 2 + 3 \\ e^2 \end{pmatrix} = \begin{pmatrix} 5 \\ e^2 \end{pmatrix}$; $|\mathbf{r}(2)| = \sqrt{25 + e^4} \approx \sqrt{25 + 54.60} \approx 8.92$ m **M1 A1**

Q13 — Particle Interception MEDIUM

(a) Set $\mathbf{r}_1(t) = \mathbf{r}_2(t)$:

x: $3t = 4 + t \Rightarrow 2t = 4 \Rightarrow t = 2$ **M1**

Check y: $\mathbf{r}_{1y}(2) = 2(2) = 4$; $\mathbf{r}_{2y}(2) = 6 - 2 = 4$? **M1 A1**

$t = 2$ s

(b) $\mathbf{r}_1(2) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ m **A1**

Q14 — Speed Function & Minimum MEDIUM

(a) $\mathbf{v}(t) = \begin{pmatrix} 2t-4 \\ 3 \end{pmatrix}$ **A1**

(b) $|\mathbf{v}|^2 = (2t-4)^2 + 9 = 4t^2 - 16t + 16 + 9 = 4t^2 - 16t + 25$ **M1 A1** (AG)

(c) Minimise: $\frac{d}{dt}(4t^2 - 16t + 25) = 8t - 16 = 0 \Rightarrow t = 2$ s **M1**

Min speed $= \sqrt{4(4) - 32 + 25} = \sqrt{9} = 3$ m s^{-1} **A1**

Q15 — Trig Velocity: Distance vs Displacement MEDIUM

(a) $\mathbf{r}(t) = \begin{pmatrix} -\cos t + c_1 \\ \sin t + c_2 \end{pmatrix}$. Apply $\mathbf{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $c_1 = 1, c_2 = 0$ **M1**

$\mathbf{r}(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}$ **A1 A1**

(b) $\mathbf{r}(\pi) = \begin{pmatrix} 1-(-1) \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$; $\quad$ |displacement| $= 2$ m **A1**

(c) $|\mathbf{v}|^2 = \sin^2 t + \cos^2 t = 1$, so speed $= 1$ m s^{-1} (constant). Distance $= 1 \times \pi = \pi \approx 3.14$ m **M1 A1**

Q16 — Piecewise Velocity: Displacement & Distance MEDIUM

(a) $v(t) = \begin{cases} 4-2t & 0 \le t \le 4 \\ 2t-12 & 4 < t \le 8 \end{cases}$ **A1**

(b) Displacement $= \int_0^4 (4-2t)\,dt + \int_4^8 (2t-12)\,dt = [4t - t^2]_0^4 + [t^2 - 12t]_4^8$ **M1**

$= (16 - 16 - 0) + (64 - 96 - 16 + 48) = 0 + 0 = \mathbf{0}$ m **M1 A1**

(c) $v = 0$ at $t = 2$ and $t = 6$. Distance $= |[4t-t^2]_0^2| + |[4t-t^2]_2^4| + |[t^2-12t]_4^6| + |[t^2-12t]_6^8|$ **M1**

$= 4 + 4 + 4 + 4 = \mathbf{16}$ m **A1**

Q17 — Constant Acceleration in 2D MEDIUM

(a) $\mathbf{v}(t) = \mathbf{v}(0) + t\mathbf{a} = \begin{pmatrix} 1+6t \\ 5-2t \end{pmatrix}$ **A1**

(b) $\mathbf{r}(t) = \mathbf{r}(0) + \int \mathbf{v}\,dt = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} t+3t^2 \\ 5t-t^2 \end{pmatrix} = \begin{pmatrix} t+3t^2 \\ 2+5t-t^2 \end{pmatrix}$ **M1 A1**

(c) $x = 0$: $t + 3t^2 = 0 \Rightarrow t(1+3t) = 0 \Rightarrow t = 0$ or $t = -\frac{1}{3}$ **M1**

Only $t = 0$ satisfies $t \ge 0$. The particle starts on the y-axis and does not return to it. **A1 R1**

Q18 — Shortest Distance in 3D MEDIUM

(a) At $t = 1$: $\mathbf{r}(1) = (3, 1, 4)$. Vector to B: $(4-3,\ 1-1,\ 3-4) = (1,\ 0,\ -1)$ **M1**

Check: $(1, 0, -1) \cdot (2, 1, 2) = 2 + 0 - 2 = 0$ ⍰, so perpendicular holds at $t = 1$. **M1 A1** (AG)

(b) Distance $= |(1, 0, -1)| = \sqrt{1 + 0 + 1} = \sqrt{2} \approx 1.41$ m **M1 A1**

Q19 — Minimum Distance Between Moving Ships MEDIUM

(a) $\mathbf{r}_P(t) = \begin{pmatrix} 3t \\ t \end{pmatrix}$; $\quad \mathbf{r}_Q(t) = \begin{pmatrix} 10 + t \\ 4 + 3t \end{pmatrix}$ **A1**

(b) $\overrightarrow{PQ}(t) = \mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix} 10 - 2t \\ 4 + 2t \end{pmatrix}$ **M1 A1**

(c) $|PQ|^2 = (10 - 2t)^2 + (4 + 2t)^2 = 8t^2 - 24t + 116$. Setting $\frac{d}{dt} = 0$: $16t - 24 = 0 \Rightarrow t = \frac{3}{2}$ **M1**

Min distance $= \sqrt{8(9/4) - 36 + 116} = \sqrt{98} = 7\sqrt{2} \approx 9.9$ km **A1**

Q20 — Perpendicular Velocities & Position MEDIUM

(a) $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$: $2a + 3b = 0$ **M1 A1**

(b) $2(6) + 3b = 0 \Rightarrow b = -4$ **A1**

(c) $\mathbf{r}_1(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 6 \end{pmatrix}$ m **M1 A1**

Section C — Hard

Q21 — Aircraft Closest Approach & Safety Decision HARD

(a) $\mathbf{r}_A(t) = \begin{pmatrix} 5t \\ 3t \end{pmatrix}$; $\quad \mathbf{r}_B(t) = \begin{pmatrix} 12 + t \\ -2 + 4t \end{pmatrix}$ **A1**

(b) $\overrightarrow{AB}(t) = \mathbf{r}_A - \mathbf{r}_B = \begin{pmatrix} 4t - 12 \\ -t + 2 \end{pmatrix}$ **M1**

$|\overrightarrow{AB}|^2 = (4t - 12)^2 + (-t + 2)^2 = 16t^2 - 96t + 144 + t^2 - 4t + 4 = 17t^2 - 100t + 148$ **M1 A1** (AG)

(c) $\frac{d}{dt}(17t^2 - 100t + 148) = 34t - 100 = 0 \Rightarrow t = \frac{50}{17} \approx 2.94$ min **A1**

(d) $|\overrightarrow{AB}|^2 = 17 \cdot \frac{2500}{289} - 100 \cdot \frac{50}{17} + 148 = \frac{42500 - 85000 + 42772}{289} = \frac{272}{289}$ **M1**

$|\overrightarrow{AB}| = \sqrt{272/289} = \frac{4\sqrt{17}}{17} \approx 0.970$ km < 5 km **A1**

The regulation is **violated**: the separation is only ≈ 0.970 km, which is less than 5 km. **R1**

Q22 — Circular Motion: Acceleration Toward Origin HARD

(a) $\mathbf{v}(t) = \begin{pmatrix} -8\sin 2t \\ 8\cos 2t \end{pmatrix}$ **A1**

(b) $\mathbf{a}(t) = \begin{pmatrix} -16\cos 2t \\ -16\sin 2t \end{pmatrix}$ **A1**

(c) $\mathbf{a}(t) = -16(\cos 2t, \sin 2t) = -4 \cdot 4(\cos 2t, \sin 2t) = -4\,\mathbf{r}(t)$ **M1**

Since $\mathbf{a} = -4\mathbf{r}$, the acceleration points **opposite** to $\mathbf{r}$, i.e. towards the origin. **A1** (AG)

Magnitude: $|\mathbf{a}| = 4|\mathbf{r}| = 4 \times 4 = 16$ m s^{-2} **A1**

(d) Speed $= |\mathbf{v}| = \sqrt{64\sin^2 2t + 64\cos^2 2t} = 8$ m s^{-1}. Distance $= 8 \times \frac{\pi}{2} = 4\pi \approx 12.6$ m **M1 A1**

Q23 — Interception with Unknown Parameter HARD

(a) Set $\mathbf{r}_X(T) = \mathbf{r}_Y(T)$:

x: $k + 3T = 2T \Rightarrow k = -T$ ...(i) **M1**

y: $2 + T = 5 - T \Rightarrow 2T = 3 \Rightarrow T = 1.5$ s **M1 A1**

From (i): $k = -1.5$ m **A1**

(b) $\mathbf{r}_X(1.5) = \begin{pmatrix} -1.5 + 4.5 \\ 2 + 1.5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3.5 \end{pmatrix}$ m **A1**

(c) $\mathbf{v}_X = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_Y = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. $\cos\theta = \dfrac{(3)(2) + (1)(-1)}{\sqrt{10}\cdot\sqrt{5}} = \dfrac{5}{\sqrt{50}} = \dfrac{1}{\sqrt{2}}$ **M1**

$\theta = 45°$ **A1**

Q24 — Unknown Parameter from Closest-Approach Condition HARD

(a) $\mathbf{d}(t) = \mathbf{r}_A - \mathbf{r}_B = (-3, -3, 2) + t(1, 2, -1-p)$ **M1**

$|\mathbf{d}|^2 = (-3+t)^2 + (-3+2t)^2 + (2 - (1+p)t)^2$

$= (5 + (1+p)^2)t^2 - (18 + 4(1+p))t + 22$ **M1 A1 A1**

(b) Minimise at $t = 1$: $\frac{d}{dt}|\mathbf{d}|^2\big|_{t=1} = 0$ **M1**

$2(5 + (1+p)^2) - (18 + 4(1+p)) = 0$. Let $q = 1 + p$:

$2q^2 - 4q - 8 = 0 \Rightarrow q^2 - 2q - 4 = 0 \Rightarrow q = 1 \pm \sqrt{5}$ **M1**

$p = q - 1 = \sqrt{5} \approx 2.24$ **A1**

(c) $\mathbf{d}(1) = (-2, -1, 1 - \sqrt{5})$. Min distance $= \sqrt{4 + 1 + (1-\sqrt{5})^2} = \sqrt{11 - 2\sqrt{5}} \approx 2.55$ m **M1 A1**

Q25 — Arc Length by Substitution HARD

(a) $\mathbf{v}(t) = \begin{pmatrix} 2t \\ t^2 \end{pmatrix}$ **A1**

(b) $|\mathbf{v}|^2 = 4t^2 + t^4 = t^2(4 + t^2)$, so $|\mathbf{v}| = t\sqrt{4 + t^2}$ (for $t \geq 0$) **M1 A1** (AG)

(c) Distance $= \int_0^3 t\sqrt{4+t^2}\,dt$. Let $u = 4+t^2$, $du = 2t\,dt$:

$$= \int_4^{13} \frac{\sqrt{u}}{2}\,du = \left[\frac{u^{3/2}}{3}\right]_4^{13} = \frac{13\sqrt{13}-8}{3}\text{ m}$$

M1 M1 A1

(d) The particle travels a **curved** path, so the arc length exceeds the straight-line distance between start and end points. **R1**

Q26 — Acceleration Perpendicular to Velocity HARD

(a) $\mathbf{v}(t) = \begin{pmatrix} t^2-1 \\ t \end{pmatrix}$; $\mathbf{a}(t) = \begin{pmatrix} 2t \\ 1 \end{pmatrix}$ **A1 A1**

(b) $\mathbf{a}\cdot\mathbf{v} = (2t)(t^2-1) + (1)(t) = 2t^3 - 2t + t = 2t^3 - t = t(2t^2-1)$ **M1 A1** (AG)

(c) $t(2t^2-1) = 0, t > 0$: $2t^2 = 1 \Rightarrow t = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ s **A1**

(d) $\mathbf{v}\left(\frac{1}{\sqrt{2}}\right) = \begin{pmatrix} 1/2-1 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/\sqrt{2} \end{pmatrix}$; speed $= \sqrt{1/4+1/2} = \sqrt{3/4} = \frac{\sqrt{3}}{2}$ m s^{-1} (exact) **M1 A1**

Q27 — Particle Crossing a Circular Boundary HARD

(a) $|\mathbf{r}(t)|^2 = (t+1)^2 + (2t-1)^2 = t^2 + 2t + 1 + 4t^2 - 4t + 1 = 5t^2 - 2t + 2$ **M1 A1** (AG)

(b) $5t^2 - 2t + 2 = 25 \Rightarrow 5t^2 - 2t - 23 = 0$ **M1**

$t = \frac{2+\sqrt{4+460}}{10} = \frac{2+\sqrt{464}}{10} \approx \frac{2+21.54}{10} \approx 2.35$ s **M1 A1**

(c) $|\mathbf{r}(3)|^2 = 45 - 6 + 2 = 41 > 25$, so the ball is **outside** the boundary at $t = 3$. **M1 R1**

(d) At $t \approx 2.354$: coordinates $\approx (3.35,\ 3.71)$ m **A1**

Q28 — Equal Distance from Origin HARD

(a) $\mathbf{r}_P(1) = \begin{pmatrix} 1+p \\ 3 \end{pmatrix}$: $|\mathbf{r}_P(1)|^2 = (1+p)^2 + 9$ **M1 A1**

$\mathbf{r}_Q(1) = \begin{pmatrix} 8 \\ p \end{pmatrix}$: $|\mathbf{r}_Q(1)|^2 = 64 + p^2$ **A1**

(b) $(1+p)^2 + 9 = 64 + p^2 \Rightarrow 1 + 2p + p^2 + 9 = 64 + p^2 \Rightarrow 2p = 54 \Rightarrow p = 27$ **M1 A1**

(c) $|\mathbf{r}_P(1)|^2 = (28)^2 + 9 = 793$; distance $= \sqrt{793} \approx 28.2$ m **M1 A1**

Q29 — Exact Arc Length via Substitution HARD

(a) $\mathbf{v}(t) = \begin{pmatrix} 2t \\ 2t^2 \end{pmatrix}$ **A1**

(b) $|\mathbf{v}|^2 = 4t^2 + 4t^4 = 4t^2(1+t^2)$, so $|\mathbf{v}| = 2t\sqrt{1+t^2}$ (for $t \geq 0$) **M1 A1** (AG)

(c) Distance $= \displaystyle\int_0^2 2t\sqrt{1+t^2}\,dt$. Let $u = 1+t^2$, $du = 2t\,dt$:

$$= \int_1^5 \sqrt{u}\,du = \left[\frac{2u^{3/2}}{3}\right]_1^5 = \frac{2}{3}(5^{3/2}-1) = \frac{2}{3}(5\sqrt{5}-1)$$

So $k = \dfrac{2}{3}$. **M1 M1 A1**

(d) $\dfrac{2}{3}(5 \times 2.236 - 1) \approx \dfrac{2}{3}(10.18) \approx 6.79$ m **A1**

Q30 — Interception with Two Unknowns HARD

(a) Set $\mathbf{r}_A(T) = \mathbf{r}_B(T)$:

x: $2 + aT = 5 - T \Rightarrow T(a+1) = 3$ ...(i) **M1**

y: $1 + 3T = -2 + bT \Rightarrow T(b-3) = 3$ ...(ii) **M1**

From (i) & (ii): $a + 1 = b - 3 \Rightarrow b = a + 4$. With $b = 2a$: $2a = a + 4 \Rightarrow a = 4$ **A1**

Substitute in (i): $T(5) = 3 \Rightarrow T = 0.6$ s **A1** (AG shown)

(b) $\mathbf{r}_A(0.6) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0.6\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4.4 \\ 2.8 \end{pmatrix}$ m **M1 A1**

(c) $\mathbf{v}_A = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{v}_B = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$. $|\mathbf{v}_A| = 5$, $|\mathbf{v}_B| = \sqrt{65}$ **M1**

$\cos\theta = \dfrac{(4)(-1)+(3)(8)}{5\sqrt{65}} = \dfrac{20}{5\sqrt{65}} = \dfrac{4}{\sqrt{65}}$; $\theta = \arccos\left(\frac{4}{\sqrt{65}}\right) \approx 60.3°$ **A1**

Common Mistakes to Avoid

1. **Distance vs displacement.** Distance $= \int |\mathbf{v}|\,dt$ (always positive). Displacement $= \mathbf{r}(t_2) - \mathbf{r}(t_1)$ (can be zero or negative). Never substitute displacement for distance.
2. **Constant of integration.** When integrating $\mathbf{v}(t)$ to find $\mathbf{r}(t)$, always apply the initial condition $\mathbf{r}(0)$ to find **both** constants c_1 and c_2 separately.
3. **Speed $\neq$ velocity component.** Speed is $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$. Setting $v_x = 0$ alone does not make speed zero — both components must be zero for a 2D particle to be at rest.
4. **Closest approach.** Minimise $|\mathbf{r}_1 - \mathbf{r}_2|^2$ (not $|\mathbf{r}_1 - \mathbf{r}_2|$) for easier algebra. Check the second derivative or

plot to confirm it is a minimum, not a maximum.

5. **"Show that" questions.** Write every algebraic step explicitly. The final printed answer carries **no mark** (AG) — the mark is on the preceding justified line.
6. **Domain restriction.** When solving for time, discard negative values of t unless the question explicitly allows $t < 0$.

End of Worksheet — Total: 120 marks

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