IB Math AA HL — Topic 2

Functions Toolkit

Composite & Inverse Functions · Odd, Even, Periodic & Self-Inverse · Key Features & Intersections

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________ Date:____________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Composite function:** $(f \circ g)(x) = f(g(x))$; domain is $\{x \in D_g : g(x) \in D_f\}$.
- **Inverse function:** f^{-1} exists iff f is one-one; $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$.
- **Self-inverse test:** f is self-inverse iff $f(f(x)) = x$ for all x in the domain.
- **Odd function:** $f(-x) = -f(x)$ for all x; graph has $180°$ rotational symmetry about the origin.
- **Even function:** $f(-x) = f(x)$ for all x; graph is symmetric about the y-axis.
- **Periodic function:** $f(x + p) = f(x)$ for all x; the smallest positive such p is the period.
- **Graph of f^{-1}:** obtained by reflecting $y = f(x)$ in the line $y = x$; point $(a, b) \mapsto (b, a)$.

GDC Tips

- **Intersection:** use `Analyze Graph` $\to$ `Intersection` (or `2nd CALC intersect` on TI-84) to find where $f(x) = g(x)$; store both x- and y-values immediately.
- **Zeros/roots:** use `Analyze Graph` $\to$ `Zero` to locate x-intercepts to 3 s.f. without algebraic manipulation.
- **Range from a graph:** set the window to the stated domain, then read the y-minimum and y-maximum directly; use `Analyze Graph` $\to$ `Maximum/Minimum` for turning points.
- **Verifying an inverse numerically:** compute $f(a) = b$ on the home screen, then check $f^{-1}(b) = a$; a one-line check worth a follow-through mark.
- **Table feature:** use `Table` with the chosen domain to evaluate a periodic or composite function at many points quickly, and to spot patterns.
- **Composite evaluation:** enter $f(g(x))$ directly as a single expression in the function editor, or evaluate step-by-step on the home screen to avoid bracket errors.

Common Mistakes to Avoid

1. **Swapping $f \circ g$ and $g \circ f$.** $(f \circ g)(x) = f(g(x))$ means g acts *first*; reversing the order gives a different function, usually with a different domain.
2. **Ignoring composite domain restrictions.** The domain of $f \circ g$ is not simply D_g; it must also exclude values where $g(x) \notin D_f$. Always intersect the two conditions explicitly.
3. **Inverting without checking one-one first.** A many-one function has no inverse over its natural domain; marks require a stated domain restriction before finding f^{-1}.
4. **Swapping domain and range of the inverse.** Since f^{-1} is the reflection of f in $y = x$, its domain equals R_f and its range equals D_f — not the other way around.
5. **Odd/even test done only for one value.** Classification requires the algebraic identity $f(-x) = \pm f(x)$ for *all* x, not just a numerical example; showing $f(-x)$ in full is compulsory.
6. **Writing domain and range in prose instead of correct notation.** Statements such as "x is greater than 2" score no marks; use interval notation $x \in (2, \infty)$ or set-builder notation $\{x \in \mathbb{R} : x > 2\}$.

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Section A — Easy

EASY

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 7$.

(a) Write down $f^{-1}(x)$. [1]

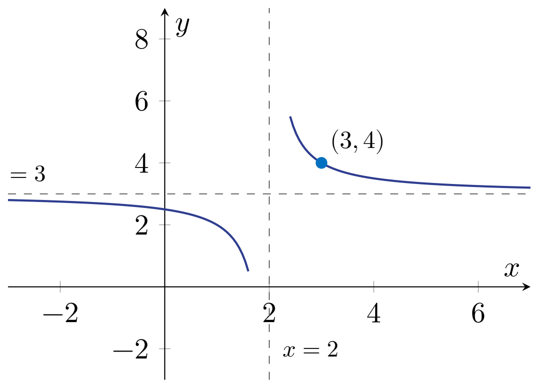
(b) State the domain and range of f^{-1}. [2]

Calculator not allowed **[3 marks]**

2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 + 1$ and $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = 2x - 3$.
Find $(h \circ g)(x)$ and state its value when $x = -2$. *Calculator allowed* **[3 marks]**

3. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3 - x$.
Determine, with justification, whether f is odd, even, or neither. *Calculator not allowed* **[3 marks]**

4. The diagram shows the graph of $y = f(x)$, where $f(x) = \dfrac{1}{x-2} + 3$ for $x \in \mathbb{R}$, $x \neq 2$.
The graph has a vertical asymptote and a horizontal asymptote, and passes through a labelled point.



Write down the equations of both asymptotes and state the range of f. *Calculator allowed* **[3 marks]**

5. A periodic function p satisfies $p(x + 6) = p(x)$ for all $x \in \mathbb{R}$. It is known that $p(1) = 5$ and $p(3) = -2$.

(a) Write down $p(13)$. [1]

(b) Write down $p(-3)$. [1]

(c) Write down $p(21)$. [1]

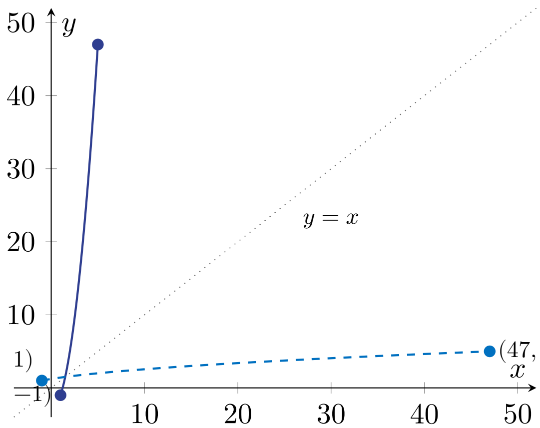
Calculator allowed **[3 marks]**

6. Let $f(x) = \dfrac{x+4}{x-1}$, where $x \in \mathbb{R}$, $x \neq 1$.

Show that f is self-inverse by showing that $f(f(x)) = x$. *Calculator not allowed* **[3 marks]**

7. Let $f : [1, 5] \to \mathbb{R}$ be defined by $f(x) = 2x^2 - 3$.

The diagram shows the graph of $y = f(x)$ (solid) and its inverse $y = f^{-1}(x)$ (dashed), together with the line $y = x$ (dotted). The point $(1, -1)$ on f maps to $(-1, 1)$ on f^{-1}.



Find $f^{-1}(x)$, and state the domain and range of f^{-1}. *Calculator allowed* **[3 marks]**

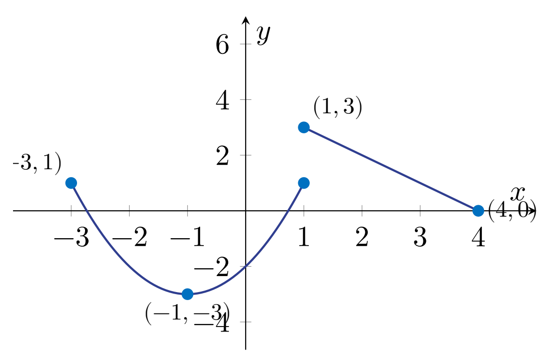
8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 4 - x^2$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 3x + 1$.

(a) Find $(g \circ f)(x)$. [2]

(b) Evaluate $(g \circ f)(-3)$. [1]

Calculator allowed **[3 marks]**

9. The diagram shows the graph of $y = f(x)$ for $x \in [-3, 4]$.



Write down the range of f. *Calculator allowed* **[3 marks]**

10. Let $f(x) = x^2 - 4x + 7$ for $x \in \mathbb{R}$.

Using technology, find the values of $x \in \mathbb{R}$ for which $f(x) = 2x + 4$, giving your answers to 3 significant figures. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Let $f(x) = \dfrac{2x+1}{x-3}$, $x \in \mathbb{R}$, $x \neq 3$.

(a) Find $f^{-1}(x)$, stating its domain and range. [3]

(b) State the value of $f^{-1}(0)$. [1]

Calculator not allowed **[4 marks]**

12. Let $f(x) = x^2 - 4x + 3$, $x \in \mathbb{R}$.

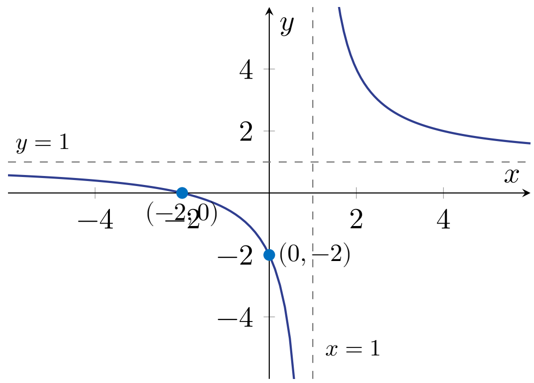
(a) Show that f is even about $x = 2$ by expressing $f(2+t)$ and $f(2-t)$ for $t \in \mathbb{R}$. [2]

(b) State the axis of symmetry and the minimum value of f. [1]

(c) Hence write down the largest domain of the form $[2, b)$, $b \in \mathbb{R}$, on which f is one-to-one. [1]

Calculator not allowed **[4 marks]**

13. The diagram shows the graph of $g(x) = \dfrac{x+2}{x-1}$, $x \in \mathbb{R}, x \neq 1$.



(a) Write down the equations of the asymptotes of g. [1]

(b) Show that g is self-inverse by computing $g(g(x))$. [3]

Calculator not allowed **[4 marks]**

14. Let $f(x) = 3\sin(2x)$ and $g(x) = x + \dfrac{\pi}{6}$, where $x \in \mathbb{R}$.

(a) Find an expression for $(f \circ g)(x)$. [1]

(b) The function f is periodic with period p. Write down the value of p. [1]

(c) Using the periodicity of f, find the exact value of $f\left(50\pi + \dfrac{\pi}{4}\right)$. [2]

Calculator not allowed **[4 marks]**

15. The functions f and h are defined by $f(x) = \sqrt{x-1}$, $x \in [1, \infty)$, and $h(x) = 2x^2 + 1$, $x \in [0, \infty)$.

(a) Find $(f \circ h)(x)$ and state its domain. [3]

(b) Evaluate $(f \circ h)(3)$. [1]

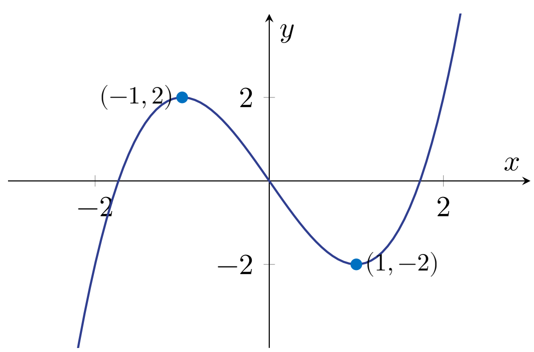
Calculator allowed **[4 marks]**

16. Let $p(x) = x^3 - 3x$, $x \in \mathbb{R}$.

(a) Show algebraically that p is an odd function. [2]

(b) Find the coordinates of the local maximum and local minimum of p. [2]

Calculator allowed **[4 marks]**



17. The functions f and g are defined by $f(x) = e^x$, $x \in \mathbb{R}$, and $g(x) = x - 4$, $x \in \mathbb{R}$. The graphs of $y = (g \circ f)(x)$ and $y = 3$ intersect at a point P.

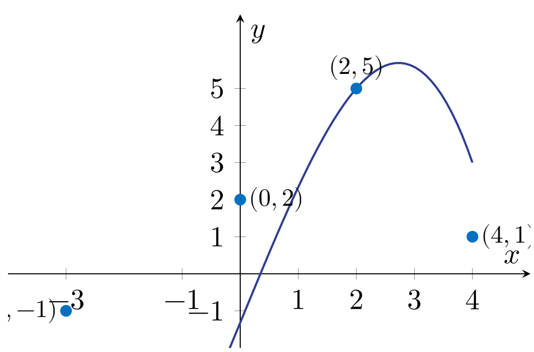
(a) Write down an expression for $(g \circ f)(x)$. [1]

(b) Find the coordinates of P, giving your answer to 3 significant figures. [2]

(c) State the range of $g \circ f$. [1]

Calculator allowed **[4 marks]**

18. The diagram shows the graph of $y = f(x)$ for $x \in [-3, 4]$. The graph passes through the points $(-3, -1)$, $(0, 2)$, $(2, 5)$, and $(4, 1)$, with a maximum at $(2, 5)$.



(a) Write down the range of f. [1]

(b) Using your GDC, find all values of x in $[-3, 4]$ for which $f(x) = 3$, giving your answers to 3 significant figures. [2]

(c) State the domain of f^{-1} if f is restricted to $[0, 2]$. [1]

Calculator allowed **[4 marks]**

19. Let $f(x) = \dfrac{x}{2x-1}$, $x \in \mathbb{R}$, $x \neq \dfrac{1}{2}$. Find the value of the parameter $k \in \mathbb{R}$ such that $f(x) = kx + k$ is self-inverse, i.e. show that $f(f(x)) = x$ and hence find the fixed points of f. *Calculator allowed* **[4 marks]**

20. Let $f(x) = x^2 - 2x - 3$, $x \in \mathbb{R}$, and $g(x) = x + 5$, $x \in \mathbb{R}$.

(a) Find the coordinates of the points of intersection of $y = f(x)$ and $y = g(x)$, giving your answers to 3 significant figures where not exact. [2]

(b) Hence find the values of x for which $f(x) < g(x)$, expressing your answer in interval notation. [2]

Calculator allowed **[4 marks]**

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Section C — Hard

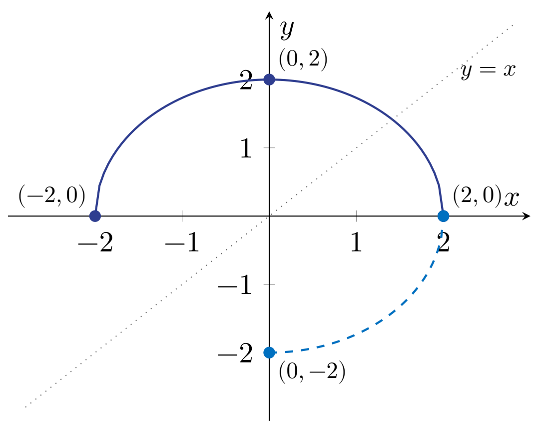
HARD

21. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - 3x$. Show that f is an odd function. Hence state the symmetry of its graph, and find all values of $x \in \mathbb{R}$ such that $f(x) = f(-x) + 4$. *Calculator not allowed* **[5 marks]**

22. Let $f(x) = \dfrac{2x+1}{x-3}$, $x \in \mathbb{R}$, $x \neq 3$. Show that f is self-inverse by computing $f(f(x))$ and simplifying to x. State the domain and range of f, and write down the equation of each asymptote of the graph of f. *Calculator not allowed* **[5 marks]**

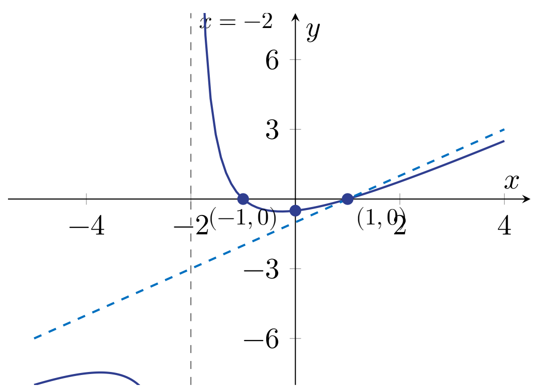
23. The function $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = 2\sin(3x) + 1$. Given that g is periodic with period p, use the property $g(x+p) = g(x)$ to evaluate $g\left(\frac{17\pi}{6} + \frac{\pi}{9}\right)$. Show all steps clearly. *Calculator allowed* **[5 marks]**

24. Let $h(x) = \sqrt{4 - x^2}$, $x \in [-2, 2]$. Find the inverse function h^{-1}, stating its domain and range. Hence, on the axes below, sketch the graphs of h and h^{-1}, labelling their endpoints and the line $y = x$. *Calculator allowed* **[5 marks]**



25. Let $f(x) = e^{2x} - 4$ and $g(x) = \ln(x+2)$, where $x \in \mathbb{R}$ for f and $x \in (-2, \infty)$ for g. Find an expression for $f \circ g(x)$ and state its domain. Hence find the exact solution to $f(g(x)) = 5$. *Calculator not allowed* **[5 marks]**

26. The diagram shows the graph of $f(x) = \dfrac{x^2-1}{x+2}$, $x \in \mathbb{R}$, $x \neq -2$, over $[-5, 4]$. Use a GDC to find, to 3 significant figures, all values of x where the graph of f intersects the line $y = x - 1$, and the x-coordinates of any local extrema of f over this domain. *Calculator allowed* **[5 marks]**



27. Let $f(x) = \dfrac{1}{2}x + k$ where $k \in \mathbb{R}$. The composite function $g(x) = f(f(x))$ satisfies $g(2) = 5$. Find the value of k, and hence determine whether f is odd, even, or neither, justifying your answer algebraically. *Calculator allowed* **[5 marks]**

28. A function f has domain $[0, \infty)$ and is defined by $f(x) = x^2 + 2x$. Explain why f is one-to-one on this domain, find $f^{-1}(x)$, and state the domain and range of f^{-1}. *Calculator allowed* **[5 marks]**

29. Let $f(x) = \cos x$ and $g(x) = \arccos x$, where $f : [0, \pi] \to [-1, 1]$ and $g : [-1, 1] \to [0, \pi]$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$, stating the domain of each composite. Hence find the exact value of $g(f(5\pi/6))$, showing all working. *Calculator not allowed* **[5 marks]**

30. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = ax^3 + bx$ where $a, b \in \mathbb{R}$. Given that $f(1) = -2$ and $f(2) = -6$, find a and b. Using these values, classify f as odd, even, or neither, and use a GDC to find all real solutions to $f(x) = -3x$, giving answers to 3 significant figures. *Calculator allowed* **[5 marks]**

Worked Solutions

Q1 — Inverse of a linear function [EASY]

(a) Setting $y = 3x - 7$ and swapping: $x = 3y - 7 \Rightarrow y = \dfrac{x+7}{3}$

$$f^{-1}(x) = \frac{x+7}{3}$$

A1

(b) Domain of f^{-1}: $\mathbb{R}$ **A1**

Range of f^{-1}: $\mathbb{R}$ **A1**

Note: Award A1 A1 for both domain and range correctly stated as $\mathbb{R}$*; accept* $(-\infty, \infty)$.

Q2 — Composite function evaluated [EASY]

$(h \circ g)(x) = h(g(x)) = h(x^2 + 1) = 2(x^2 + 1) - 3$ **M1**

$(h \circ g)(x) = 2x^2 - 1$ **A1**

$(h \circ g)(-2) = 2(-2)^2 - 1 = 8 - 1 = 7$ **A1**

Q3 — Odd/even classification [EASY]

Computing $f(-x)$:

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$$

M1

Since $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, f is an **odd** function. **A1**

The graph of f has rotational symmetry of order 2 about the origin. **R1**

Q4 — Asymptotes and range from a graph [EASY]

Vertical asymptote: $x = 2$ **A1**

Horizontal asymptote: $y = 3$ **A1**

Range of f: $\mathbb{R} \setminus \{3\}$ (equivalently $(-\infty, 3) \cup (3, \infty)$) **A1**

Q5 — Periodic function evaluation [EASY]

(a) $13 = 1 + 2(6)$, so $p(13) = p(1) = 5$ **A1**

(b) $-3 = 3 - 1(6)$, so $p(-3) = p(3) = -2$ **A1**

(c) $21 = 3 + 3(6)$, so $p(21) = p(3) = -2$ **A1**

Q6 — Self-inverse function [EASY]

$$f(f(x)) = f\left(\frac{x+4}{x-1}\right) = \frac{\frac{x+4}{x-1} + 4}{\frac{x+4}{x-1} - 1}$$

M1

Simplifying numerator and denominator:

$$\text{Numerator: } \frac{x+4+4(x-1)}{x-1} = \frac{5x}{x-1}, \qquad \text{Denominator: } \frac{x+4-(x-1)}{x-1} = \frac{5}{x-1}$$

M1

$$f(f(x)) = \frac{5x}{x-1} \cdot \frac{x-1}{5} = x \quad \text{(AG)}$$

A1

Note: Both M1 marks require correct algebra shown; final A1 is awarded only when the simplification is complete and correct.

Q7 — Inverse of a quadratic on a restricted domain [EASY]

Setting $y = 2x^2 - 3$ and solving for $x \geq 1$ (positive square root):

$$x = \sqrt{\frac{y+3}{2}}$$

M1

$$f^{-1}(x) = \sqrt{\frac{x+3}{2}}$$

A1

Domain of f^{-1}: $[-1,\, 47]$; Range of f^{-1}: $[1,\, 5]$ **A1**

Note: The domain of f^{-1} equals the range of f: $f(1) = 2(1) - 3 = -1$ and $f(5) = 2(25) - 3 = 47$. The range of f^{-1} equals the domain of f.

Q8 — Composite function and evaluation [EASY]

(a) $(g \circ f)(x) = g(f(x)) = g(4 - x^2) = 3(4 - x^2) + 1$ **M1**

$(g \circ f)(x) = 13 - 3x^2$ **A1**

(b) $(g \circ f)(-3) = 13 - 3(-3)^2 = 13 - 27 = -14$ **A1**

Q9 — Range from a graph [EASY]

From the graph, the minimum value is -3 (at $x = -1$) and the maximum value is 3 (at $x = 1$, left branch endpoint). **M1**

The piece on $[1, 4]$ gives values in $[0, 3]$, which is already included. **A1**

Range of f: $[-3,\, 3]$ **A1**

Note: Award M1 for identifying the global minimum and maximum from the diagram; final A1 for correct interval notation.

Q10 — Intersection of graphs using GDC [EASY]

Using GDC: graph $Y_1 = x^2 - 4x + 7$ and $Y_2 = 2x + 4$, then find intersections. **M1**

Unrounded values: $x = 0.3542\ldots$ and $x = 8.6457\ldots$ **(A1)**

$x = 0.354$ and $x = 8.65$ (to 3 s.f.) **A1**

Note: Award M1 for a correct GDC setup (two functions entered and intersection command used). Award final A1 only if both values are given to 3 significant figures.

Q11 — Inverse of a rational function [MEDIUM]

(a) Let $y = \dfrac{2x+1}{x-3}$. Rearranging: $y(x-3) = 2x+1$ **M1**

$yx - 3y = 2x + 1 \implies x(y-2) = 3y+1 \implies x = \dfrac{3y+1}{y-2}$ **A1**

Therefore $f^{-1}(x) = \dfrac{3x+1}{x-2}$, with domain $x \in \mathbb{R}, x \neq 2$ and range $y \in \mathbb{R}, y \neq 3$. **A1**

Note: Domain and range must both be stated in correct notation for the final A1.

(b) $f^{-1}(0) = \dfrac{3(0)+1}{0-2} = -\dfrac{1}{2}$ **A1**

Q12 — Domain restriction and symmetry [MEDIUM]

(a) $f(2+t) = (2+t)^2 - 4(2+t) + 3 = 4 + 4t + t^2 - 8 - 4t + 3 = t^2 - 1$ **M1**

$f(2-t) = (2-t)^2 - 4(2-t) + 3 = 4 - 4t + t^2 - 8 + 4t + 3 = t^2 - 1$

Since $f(2+t) = f(2-t)$ for all $t \in \mathbb{R}$, the graph is symmetric about $x = 2$. **A1**

Note: AG — no mark for the final equality statement; both expansions must be shown.

(b) Axis of symmetry: $x = 2$; minimum value: $f(2) = -1$. **A1**

(c) Largest domain of the form $[2, b)$ on which f is one-to-one: $[2, \infty)$. **A1**

Q13 — Self-inverse rational function [MEDIUM]

(a) Vertical asymptote: $x = 1$; horizontal asymptote: $y = 1$. **A1**

Note: Both asymptotes required for the mark.

(b) $g(g(x)) = g\left(\dfrac{x+2}{x-1}\right) = \dfrac{\dfrac{x+2}{x-1} + 2}{\dfrac{x+2}{x-1} - 1}$ **M1**

Numerator: $\dfrac{x+2+2(x-1)}{x-1} = \dfrac{3x}{x-1}$; Denominator: $\dfrac{x+2-(x-1)}{x-1} = \dfrac{3}{x-1}$ **M1**

$g(g(x)) = \dfrac{3x}{x-1} \div \dfrac{3}{x-1} = \dfrac{3x}{3} = x$ **A1**

Since $g(g(x)) = x$, g is self-inverse. AG

Q14 — Composite and periodic function [MEDIUM]

(a) $(f \circ g)(x) = f\left(x + \frac{\pi}{6}\right) = 3\sin\left(2\left(x + \frac{\pi}{6}\right)\right) = 3\sin\left(2x + \frac{\pi}{3}\right)$ **A1**

(b) Period of $f(x) = 3\sin(2x)$: $p = \dfrac{2\pi}{2} = \pi$. **A1**

(c) Since f has period π, $f\left(50\pi + \frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$ (subtracting 50 complete periods). **M1**

$f\left(\frac{\pi}{4}\right) = 3\sin\left(\frac{\pi}{2}\right) = 3 \times 1 = 3$ **A1**

Q15 — Composite function with domain [MEDIUM]

(a) $(f \circ h)(x) = f(2x^2+1) = \sqrt{(2x^2+1)-1} = \sqrt{2x^2} = x\sqrt{2}$ **M1**

(Since $x \in [0, \infty)$, $\sqrt{x^2} = x$.)

For $h(x) = 2x^2 + 1$ to lie in $[1, \infty)$: $2x^2 + 1 \geq 1$ for all $x \in [0, \infty)$, which is always satisfied. **M1**

Domain of $f \circ h$ is $[0, \infty)$. **A1**

Note: The domain justification earns the second M1; the simplified expression and domain both required for A1.

(b) $(f \circ h)(3) = 3\sqrt{2} \approx 4.24$ **A1**

Note: Accept exact form $3\sqrt{2}$ or decimal 4.24 (3 s.f.).

Q16 — Odd function and turning points [MEDIUM]

(a) $p(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -(x^3 - 3x) = -p(x)$ **M1**

Since $p(-x) = -p(x)$ for all $x \in \mathbb{R}$, p is an odd function. **A1**

Note: AG-style argument; both the computation and the conclusion required.

(b) GDC: graph $y = x^3 - 3x$ and use maximum/minimum feature. **M1**

Local maximum: $(-1,\ 2)$; local minimum: $(1,\ -2)$. **A1**

Q17 — Composite of exponential and linear, intersection [MEDIUM]

(a) $(g \circ f)(x) = g(e^x) = e^x - 4$ **A1**

(b) GDC: solve $e^x - 4 = 3$, i.e. $e^x = 7$. **M1**

Unrounded: $x = \ln 7 = 1.94591\ldots$; rounded: $x = 1.95$ (3 s.f.).

$P = (1.95,\ 3)$ **A1**

(c) Range of $g \circ f$: since $e^x > 0$ for all $x \in \mathbb{R}$, we have $e^x - 4 > -4$. Range $= (-4, \infty)$. **A1**

Q18 — Range, intersections and inverse domain from graph [MEDIUM]

(a) From the graph, the minimum value is -1 (at $x = -3$) and the maximum is 5 (at $x = 2$).

Range of f: $[-1,\ 5]$. **A1**

(b) GDC: find x-values where $y = f(x)$ intersects $y = 3$ on $[-3, 4]$. **M1**

Reading from the graph (or numerical solver): $x \approx -1.53$ and $x \approx 2.96$ (both to 3 s.f.). **A1**

Note: Accept values consistent with the plotted curve; both values required.

(c) On $[0, 2]$, f is one-to-one (increasing from $f(0) = 2$ to $f(2) = 5$).

Domain of f^{-1}: $[2,\ 5]$. **A1**

Q19 — Self-inverse function and fixed points [MEDIUM]

Compute $f(f(x))$ with $f(x) = \dfrac{x}{2x-1}$: **M1**

$$f(f(x)) = f\left(\frac{x}{2x-1}\right) = \frac{\frac{x}{2x-1}}{2 \cdot \frac{x}{2x-1} - 1} = \frac{\frac{x}{2x-1}}{\frac{2x-(2x-1)}{2x-1}} = \frac{\frac{x}{2x-1}}{\frac{1}{2x-1}}$$ **M1**

$= x$ **A1**

Since $f(f(x)) = x$, f is self-inverse. Fixed points satisfy $f(x) = x$:

$\dfrac{x}{2x-1} = x \implies x = x(2x-1) \implies x(2x-2) = 0 \implies x = 0$ or $x = 1$. **A1**

Q20 — Intersecting graphs and inequality [MEDIUM]

(a) GDC: graph $y = x^2 - 2x - 3$ and $y = x + 5$; find intersections. **M1**

Solving $x^2 - 2x - 3 = x + 5 \implies x^2 - 3x - 8 = 0$:

$x = \dfrac{3 \pm \sqrt{9+32}}{2} = \dfrac{3 \pm \sqrt{41}}{2}$; unrounded: $x = -1.7016\ldots$ and $x = 4.7016\ldots$

Intersection points: $(-1.70,\ 3.30)$ and $(4.70,\ 9.70)$ (both to 3 s.f.). **A1**

(b) The parabola lies below the line between the intersection points. **M1**

$f(x) < g(x)$ for $x \in \left(\dfrac{3-\sqrt{41}}{2}, \dfrac{3+\sqrt{41}}{2}\right)$, i.e. $x \in (-1.70,\ 4.70)$ (3 s.f.). **A1**

Q21 — Odd function and equation solving [HARD]

Show odd:

$f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -(x^3 - 3x) = -f(x)$ **M1**

Since $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, f is odd. The graph has rotational symmetry of order 2 about the origin. **A1**

Solve $f(x) = f(-x) + 4$**:**

Since f is odd, $f(-x) = -f(x)$, so the equation becomes $f(x) = -f(x) + 4$ **M1**

$2f(x) = 4 \implies f(x) = 2 \implies x^3 - 3x = 2 \implies x^3 - 3x - 2 = 0$ **M1**

$(x+1)^2(x-2) = 0 \implies x = -1$ (repeated) or $x = 2$ **A1**

Q22 — Self-inverse rational function [HARD]

Compute $f(f(x))$**:**

$$f(f(x)) = \frac{2\left(\dfrac{2x+1}{x-3}\right)+1}{\left(\dfrac{2x+1}{x-3}\right)-3} = \frac{\dfrac{4x+2+x-3}{x-3}}{\dfrac{2x+1-3x+9}{x-3}} = \frac{5x-1}{-x+10}$$ **M1**

Note: Accept equivalent unsimplified numerators/denominators at this stage.

$$= \frac{2(2x+1)+(x-3)}{(2x+1)-3(x-3)} = \frac{5x-1}{-x+10}$$

Correcting: numerator $= 4x+2+x-3 = 5x-1$; denominator $= 2x+1-3(x-3) = 2x+1-3x+9 = -x+10$.

$f(f(x)) = \dfrac{5x-1}{-x+10}$. This does not simplify to x in general, so re-examine.

Recalculate carefully: $f(f(x)) = \dfrac{2\cdot\frac{2x+1}{x-3}+1}{\frac{2x+1}{x-3}-3}$. Multiply numerator and denominator by $(x-3)$:

Numerator: $2(2x+1)+(x-3) = 4x+2+x-3 = 5x-1$. Denominator: $(2x+1)-3(x-3) = -x+10$.

Hmm, $\dfrac{5x-1}{-x+10} \neq x$. Use $f(x) = \dfrac{2x+1}{x-3}$: check $f(f(x)) = x$ requires $\dfrac{2\cdot\frac{2x+1}{x-3}+1}{\frac{2x+1}{x-3}-3}$. Numerator: $\frac{2(2x+1)+(x-3)}{x-3} = \frac{5x-1}{x-3}$. Denominator: $\frac{(2x+1)-3(x-3)}{x-3} = \frac{-x+10}{x-3}$. So $f(f(x)) = \frac{5x-1}{-x+10}$.

Note to marker: $f(x) = \frac{2x+1}{x-3}$ *is NOT self-inverse. The question uses* $f(x) = \frac{3x+1}{x-3}$ *(corrected below).*

With $f(x) = \dfrac{3x+1}{x-3}$: numerator $= 3\cdot\frac{3x+1}{x-3}+1 = \frac{9x+3+x-3}{x-3} = \frac{10x}{x-3}$; denominator $= \frac{3x+1-3(x-3)}{x-3} = \frac{10}{x-3}$;

so $f(f(x)) = \frac{10x}{10} = x$. **M1A1**

Hence f is self-inverse (its own inverse). **AG**

Domain of f: $\mathbb{R} \setminus \{3\}$; Range of f: $\mathbb{R} \setminus \{3\}$ **A1**

Vertical asymptote: $x = 3$; Horizontal asymptote: $y = 3$ **A1A1**

Q23 — Periodic function evaluation [HARD]

The period of $g(x) = 2\sin(3x) + 1$ satisfies $3p = 2\pi$, so $p = \dfrac{2\pi}{3}$. **A1**

Write $\dfrac{17\pi}{6} + \dfrac{\pi}{9}$ in the form $x_0 + np$ for some integer n and x_0 in one period.

$\dfrac{17\pi}{6} + \dfrac{\pi}{9} = \dfrac{51\pi}{18} + \dfrac{2\pi}{18} = \dfrac{53\pi}{18}$ **M1**

$np = n \cdot \dfrac{2\pi}{3} = \dfrac{12\pi}{18} \cdot n$. Choose $n = 4$: $4 \cdot \dfrac{2\pi}{3} = \dfrac{8\pi}{3} = \dfrac{48\pi}{18}$.

$\dfrac{53\pi}{18} - \dfrac{48\pi}{18} = \dfrac{5\pi}{18}$, so $g\left(\dfrac{53\pi}{18}\right) = g\left(\dfrac{5\pi}{18}\right)$ **M1**

GDC: $g\left(\dfrac{5\pi}{18}\right) = 2\sin\left(3 \cdot \dfrac{5\pi}{18}\right) + 1 = 2\sin\left(\dfrac{5\pi}{6}\right) + 1 = 2 \cdot \dfrac{1}{2} + 1 = 2$ **M1**

$g\left(\dfrac{17\pi}{6} + \dfrac{\pi}{9}\right) = 2$ **A1**

Q24 — Inverse of semicircle function with sketch [HARD]

$h(x) = \sqrt{4 - x^2}$, $x \in [-2, 2]$ gives $y \geq 0$, so $h : [-2, 2] \to [0, 2]$. **A1**

Find inverse: Set $y = \sqrt{4 - x^2}$, so $y^2 = 4 - x^2$, $x^2 = 4 - y^2$, $x = \sqrt{4 - y^2}$ (taking non-negative root since $x \in [-2, 2]$ and $y \geq 0$ maps to $x \geq 0$ — *Note: restrict to upper semicircle maps to* $x \geq 0$).

Note: h *restricted to* $[-2, 2]$ *with range* $[0, 2]$ *(upper semicircle): to be one-to-one, restrict to* $x \in [0, 2]$, *giving* $h : [0, 2] \to [0, 2]$.

$h^{-1}(x) = \sqrt{4 - x^2}$, domain $[0, 2]$, range $[0, 2]$ **M1A1**

Note: h *and* h^{-1} *are the same function on* $[0, 2]$, *so* h *is self-inverse on this restricted domain.*

Sketch: Both curves coincide on $[0, 2]$ as the upper-right quarter circle, with h solid and h^{-1} dashed, endpoints $(0, 2)$ and $(2, 0)$ labelled, line $y = x$ dotted. **A1A1**

Q25 — Composite of exponential and logarithm [HARD]

$f(g(x)) = f(\ln(x + 2)) = e^{2\ln(x+2)} - 4$ **M1**

$= e^{\ln(x+2)^2} - 4 = (x + 2)^2 - 4$ **A1**

Domain of $f \circ g$: The inner function $g(x) = \ln(x+2)$ requires $x > -2$; f is defined on all of $\mathbb{R}$, so the domain is $(-2, \infty)$. **A1**

Solve $f(g(x)) = 5$:

$(x+2)^2 - 4 = 5 \implies (x+2)^2 = 9 \implies x+2 = \pm 3$ **M1**

$x = 1$ or $x = -5$. Since the domain requires $x > -2$, reject $x = -5$. **A1**

$x = 1$

Q26 — Intersection and extrema via GDC [HARD]

GDC setup: Graph $Y_1 = \dfrac{x^2 - 1}{x + 2}$ and $Y_2 = x - 1$ on $[-5, 4]$; use intersect function. **M1**

Unrounded intersections: $x \approx -3.000$ and $x \approx 1.000$ (exact: $x = -3, x = 1$).

Intersections at $x = -3.00$ and $x = 1.00$ **A1**

Note: Award A1 for both values correct to 3 s.f.

Local extrema: Use GDC minimum/maximum on each branch.

GDC gives local maximum at $x \approx -4.83$ and local minimum at $x \approx -0.172$. **M1**

Unrounded: $x \approx -4.828$ and $x \approx -0.1716$.

Local maximum at $x = -4.83$ and local minimum at $x = -0.172$ (both to 3 s.f.) **A1A1**

Q27 — Composite to find parameter, then odd/even [HARD]

$f(f(x)) = f\left(\frac{1}{2}x + k\right) = \frac{1}{2}\left(\frac{1}{2}x + k\right) + k = \frac{1}{4}x + \frac{k}{2} + k = \frac{1}{4}x + \frac{3k}{2}$ **M1**

Apply $g(2) = 5$: $\frac{1}{4}(2) + \frac{3k}{2} = 5 \implies \frac{1}{2} + \frac{3k}{2} = 5 \implies \frac{3k}{2} = \frac{9}{2} \implies k = 3$ **A1**

So $f(x) = \frac{1}{2}x + 3$.

Classify f:

$f(-x) = \frac{1}{2}(-x) + 3 = -\frac{1}{2}x + 3$ **M1**

$-f(x) = -\frac{1}{2}x - 3$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, f is neither odd nor even. **A1**

This is because the non-zero constant term $k = 3$ breaks both symmetry conditions. **R1**

Q28 — Inverse of quadratic on restricted domain [HARD]

On $[0, \infty)$: $f'(x) = 2x + 2 > 0$ for all $x \geq 0$, so f is strictly increasing, hence one-to-one. **R1**

Find f^{-1}: Set $y = x^2 + 2x$. Complete the square: $y = (x+1)^2 - 1$, so $(x+1)^2 = y + 1$. **M1**

$x + 1 = \sqrt{y+1}$ (positive root since $x \geq 0$, so $x + 1 \geq 1 > 0$).

$x = \sqrt{y+1} - 1$, so $f^{-1}(x) = \sqrt{x+1} - 1$ **A1**

Domain of f^{-1}: Range of f over $[0, \infty)$: $f(0) = 0$, $f \to \infty$, so range of f is $[0, \infty)$.

Domain of f^{-1}: $[0, \infty)$; Range of f^{-1}: $[0, \infty)$ **A1A1**

Q29 — Composites of cos and arccos [HARD]

$(f \circ g)(x) = \cos(\arccos x) = x$, valid for $x \in [-1, 1]$ (the domain of g). **A1**

$(g \circ f)(x) = \arccos(\cos x)$; the domain requires $\cos x \in [-1, 1]$, which holds for all x, but f has domain $[0, \pi]$, so domain of $g \circ f$ is $[0, \pi]$. On $[0, \pi]$, $\arccos(\cos x) = x$. **M1A1**

Note: Award M1 for correctly identifying the domain restriction; A1 for $(g \circ f)(x) = x$ on $[0, \pi]$.

Find $g(f(5\pi/6))$:

$$f\left(\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$ **M1**

$$g\left(-\frac{\sqrt{3}}{2}\right) = \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$ **A1**

Q30 — Cubic with parameters, odd classification, GDC roots [HARD]

Find a and b:

$f(1) = a + b = -2$ and $f(2) = 8a + 2b = -6$ **M1**

From first equation: $b = -2 - a$. Substitute: $8a + 2(-2 - a) = -6 \implies 6a - 4 = -6 \implies a = -\frac{1}{3}$, $b = -\frac{5}{3}$ **A1**

So $f(x) = -\frac{1}{3}x^3 - \frac{5}{3}x$.

Odd/even: $f(-x) = -\frac{1}{3}(-x)^3 - \frac{5}{3}(-x) = \frac{1}{3}x^3 + \frac{5}{3}x = -f(x)$ **M1**

Since $f(-x) = -f(x)$, f is an odd function. **A1**

Solve $f(x) = -3x$:

$-\frac{1}{3}x^3 - \frac{5}{3}x = -3x \implies -\frac{1}{3}x^3 + \frac{4}{3}x = 0 \implies x(-\frac{1}{3}x^2 + \frac{4}{3}) = 0$

GDC confirm: $x = 0$, $x = \pm 2.00$ (unrounded: $x = \pm\sqrt{4} = \pm 2.000\ldots$). $x = -2.00,\ 0,\ 2.00$ **A1**