IB Math AA HL — Topic 2
Polynomial Functions

Polynomial Division · Factor & Remainder Theorem · Graphs & Roots · Sum & Product of Roots

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Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:_____ Date:_____

Score:_____/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- Polynomial Division: $f(x) = d(x) \cdot q(x) + r(x)$, where $\deg r < \deg d$. Write the result as $f(x) \div d(x) = q(x) + \dfrac{r(x)}{d(x)}$.
- Remainder Theorem: The remainder when $f(x)$ is divided by $(x - a)$ equals $f(a)$.
- Factor Theorem: $(x - a)$ is a factor of $f(x)$ if and only if $f(a) = 0$.
- Vieta's Formulae for $ax^3 + bx^2 + cx + d = 0$ (roots α, β, γ): $\alpha + \beta + \gamma = -\dfrac{b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{c}{a}$, $\alpha\beta\gamma = -\dfrac{d}{a}$.
- Vieta's Formulae for $ax^4 + bx^3 + cx^2 + dx + e = 0$ (roots $\alpha, \beta, \gamma, \delta$): $\sum \alpha = -\dfrac{b}{a}$, $\sum \alpha\beta = \dfrac{c}{a}$, $\sum \alpha\beta\gamma = -\dfrac{d}{a}$, $\alpha\beta\gamma\delta = \dfrac{e}{a}$.
- Symmetric expressions without solving: $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$; $\sum \dfrac{1}{\alpha} = \dfrac{\sum \alpha\beta}{\alpha\beta\gamma}$ (for cubics).
- Complex roots in real-coefficient polynomials: non-real roots occur in conjugate pairs z and $\bar{z}$; the corresponding quadratic factor is $(x - z)(x - \bar{z}) = x^2 - 2\,\mathrm{Re}(z)\,x + |z|^2$.

GDC Tips

- Finding roots numerically: enter $y_1 = f(x)$ and use Zero (TI) or Root (Casio) to locate each x-intercept to 3 s.f.; confirm sign changes in the Table first.
- Verifying a factor: substitute the value directly using Store or the Table feature; $f(a) = 0$ confirms $(x - a)$ is a factor without long division.
- Counting real roots before algebra: graph $y = f(x)$ and count x-intercepts; use Zoom Fit to see all turning points, especially for quartics that may have 2 or 4 real roots.
- Solving polynomial models: for volume or profit problems, graph both sides of the equation and use Intersect to find solutions; always check whether solutions satisfy domain restrictions in context.
- Checking Vieta results numerically: after using Vieta formulae symbolically, verify by computing $\sum \alpha$, $\sum \alpha\beta$ etc. from the decimal roots obtained via Zero on the GDC.
- Unknown coefficients from conditions: set up the equation $f(a) = k$ symbolically, then use Solve or graph $y = f(a) - k$ as a function of the unknown coefficient to find the value.

Common Mistakes to Avoid

1. **Wrong sign in Vieta's sum.** For $ax^3 + bx^2 + cx + d$, the sum of roots is $-b/a$, not $+b/a$. Always check: the sign alternates with each Vieta formula.
2. **Wrong sign in Vieta's product.** The product of roots of a cubic is $-d/a$, not $+d/a$. The pair-sum c/a is the only Vieta formula with no leading minus sign.
3. **Solving the polynomial when "hence" forbids it.** If a Vieta question says "hence find $\sum \alpha^2$", you must use $(\sum \alpha)^2 - 2\sum \alpha\beta$; solving for the roots and squaring them earns method marks only, not full credit.
4. **Remainder vs. factor confusion.** The remainder theorem gives $f(a)$ as the remainder; for $(x - a)$ to be a factor you additionally need $f(a) = 0$. Setting $f(a) =$ remainder when asked for a factor is a common error.
5. **Repeated roots drawn as crossings.** A root of even multiplicity means the graph touches the x-axis and turns back; drawing it as a crossing loses the sketch mark. State the multiplicity explicitly alongside the factor.
6. **Forgetting the conjugate root.** If a real-coefficient polynomial has a complex root $a + bi$, then $a - bi$ is also a root. Omitting the conjugate gives the wrong degree or wrong remaining factor when dividing out.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY No Calculator [3 marks]

Let $f(x) = x^3 - 2x^2 - 5x + 6$, where $x \in \mathbb{R}$. Use the remainder theorem to find the remainder when $f(x)$ is divided by $(x - 3)$.

2. EASY Calculator [3 marks]

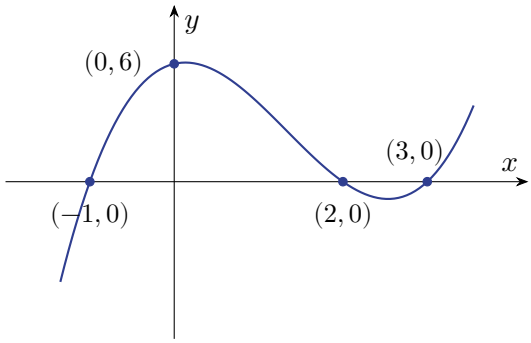
Let $p(x) = 2x^3 + 3x^2 - 11x - 6$, where $x \in \mathbb{R}$. Given that $(x - 2)$ is a factor of $p(x)$, write down the value of $p(2)$. Hence find the value of $p(-3)$.

3. EASY No Calculator [3 marks]

The polynomial $f(x) = x^3 + ax^2 - 7x + 3$, where $a \in \mathbb{R}$, has a remainder of 9 when divided by $(x - 2)$. Find the value of a.

4. **EASY** **Calculator** **[3 marks]**

Let $f(x) = x^3 - 4x^2 + x + 6$, where $x \in \mathbb{R}$. The diagram below shows the graph of $y = f(x)$.



Write down all three roots of $f(x) = 0$ and state the y-intercept of the graph.

5. **EASY** **No Calculator** **[3 marks]**

The cubic polynomial $q(x) = x^3 - 6x^2 + 11x - 6$, where $x \in \mathbb{R}$, has roots α, β, and γ. Without solving the equation, find the values of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

6. **EASY** **Calculator** **[3 marks]**

Let $g(x) = x^3 + 5x^2 - 2x - 24$, where $x \in \mathbb{R}$. Use a GDC to find all real roots of $g(x) = 0$, giving your answers to 3 significant figures where appropriate.

7. **EASY** **Calculator** **[3 marks]**

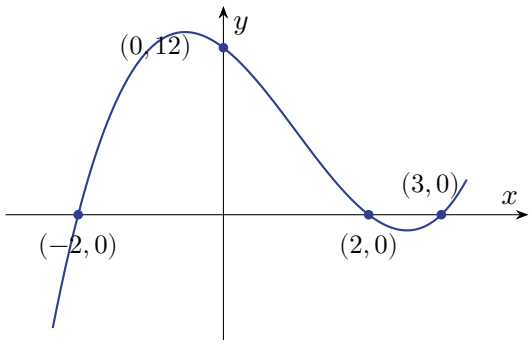
The polynomial $f(x) = x^3 + bx + 2$, where $b \in \mathbb{R}$, is exactly divisible by $(x + 1)$. Find the value of b.

8. **EASY** **No Calculator** **[3 marks]**

Divide $f(x) = 2x^3 - x^2 - 13x - 6$ by $(x - 3)$. Write down the quotient and remainder.

9. **EASY** **Calculator** **[3 marks]**

The polynomial $h(x) = x^3 - 3x^2 - 4x + 12$, where $x \in \mathbb{R}$. The diagram below shows part of the graph of $y = h(x)$.



Given that $(x - 3)$ is a factor of $h(x)$, write down the fully factorised form of $h(x)$ and verify that $x = -2$ is also a root.

10. **EASY** **Calculator** **[3 marks]**

The quartic polynomial $p(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$, where $x \in \mathbb{R}$, has four roots $\alpha, \beta, \gamma, \delta$. Without solving the equation, find the value of $\alpha\beta\gamma\delta$. Use a GDC to verify your answer by finding all four roots numerically.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator [4 marks]

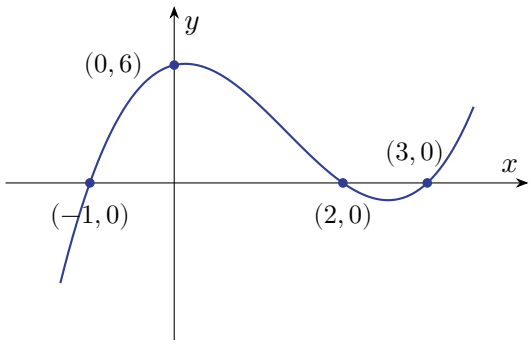
Let $f(x) = 2x^3 - 5x^2 + ax + b$, where $a, b \in \mathbb{R}$. When $f(x)$ is divided by $(x - 3)$, the remainder is 10. When $f(x)$ is divided by $(x + 1)$, the remainder is -18. Find the values of a and b.

12. MEDIUM No Calculator [4 marks]

Let $p(x) = x^3 - 4x^2 + x + 6$. It is given that $(x - 2)$ is a factor of $p(x)$.

(a) Write $p(x)$ as a product of three linear factors. [2]

(b) Hence sketch the graph of $y = p(x)$, labelling all x-intercepts and the y-intercept with their coordinates. [2]



13. MEDIUM Calculator [4 marks]

The polynomial $g(x) = x^3 + px^2 + qx - 8$, where $p, q \in \mathbb{R}$, has roots α, β, and γ. Given that $\alpha + \beta + \gamma = -3$ and $\alpha\beta + \alpha\gamma + \beta\gamma = -2$, find the values of p and q, and hence find $\alpha\beta\gamma$.

14. MEDIUM No Calculator [4 marks]

Let $h(x) = 3x^3 + 2x^2 - 7x + 2$. Use polynomial long division to divide $h(x)$ by $(x + 2)$, and hence write $h(x)$ as a product of a linear and a quadratic factor.

15. MEDIUM Calculator [4 marks]

The roots of the cubic equation $x^3 - 6x^2 + 11x - 6 = 0$ are α, β, and γ, where $\alpha, \beta, \gamma \in \mathbb{R}$.

(a) Without solving the equation, find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Without solving the equation, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. [2]

16. MEDIUM Calculator **[4 marks]**

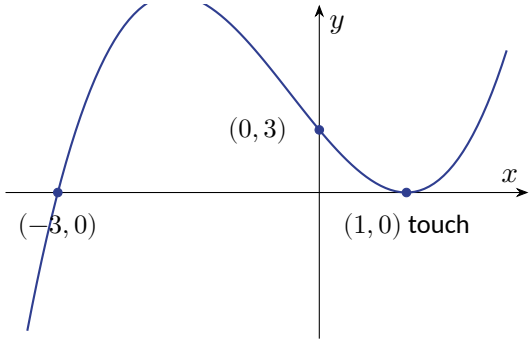
A polynomial is defined by $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$, $x \in \mathbb{R}$. Given that $x = 1 + i$ is a root of $f(x)$, where $i^2 = -1$, find all the roots of $f(x)$.

17. MEDIUM Calculator **[4 marks]**

The cubic polynomial $f(x) = x^3 + x^2 - 5x + 3$ can be written in the form $(x - a)^2(x - b)$ for some $a, b \in \mathbb{R}$.

(a) Find the values of a and b. [2]

(b) Sketch the graph of $y = f(x)$, labelling all intercepts with the axes. [2]

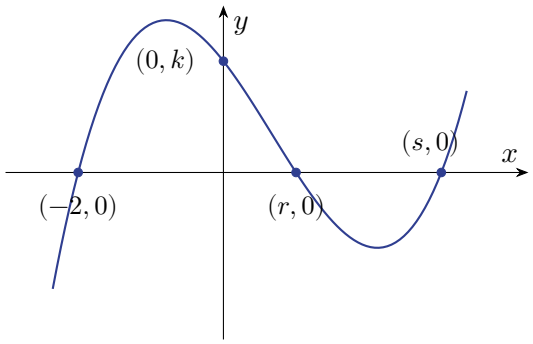


18. MEDIUM Calculator **[4 marks]**

The roots of $2x^3 - 3x^2 - 11x + 6 = 0$ are α, β, and γ. A new cubic equation has roots 2α, 2β, and 2γ. Find this new cubic equation, giving your answer in the form $x^3 + px^2 + qx + r = 0$ where $p, q, r \in \mathbb{R}$.

19. MEDIUM Calculator **[4 marks]**

Let $f(x) = x^3 - 2x^2 - 5x + k$, where $k \in \mathbb{R}$. The graph of $y = f(x)$ is shown below, and it crosses the x-axis at $x = -2$, $x = r$, and $x = s$, where r and s are positive.



Find the value of k, and the values of r and s.

20. MEDIUM Calculator **[4 marks]**

When $f(x) = x^3 + cx^2 + 5x + d$ is divided by $(x - 4)$, the remainder is 93. When $f(x)$ is divided by $(x - 1)$, the remainder is 12. Find the remainder when $f(x)$ is divided by $(x - 2)$.

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD No Calculator **[5 marks]**

Let $f(x) = 2x^3 + ax^2 + bx - 12$, where $a, b \in \mathbb{R}$. When $f(x)$ is divided by $(x - 3)$, the remainder is 42. When $f(x)$ is divided by $(x + 2)$, the remainder is -8. Find the values of a and b.

22. HARD Calculator **[5 marks]**

Let $p(x) = x^4 - 2x^3 + kx^2 + 8x + 12$, where $k \in \mathbb{R}$. It is known that $(x - 2)$ is a factor of $p(x)$.

(a) Find the value of k. [2]

(b) Hence fully factorise $p(x)$ over $\mathbb{R}$. [3]

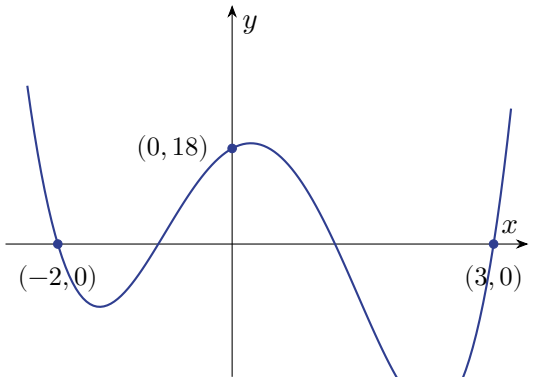
23. **HARD** **No Calculator** **[5 marks]**

The cubic polynomial $f(x) = x^3 + px^2 + qx + r$, where $p, q, r \in \mathbb{R}$, has roots α, β, and γ.

Given that $\alpha + \beta + \gamma = -3$, $\alpha\beta + \beta\gamma + \gamma\alpha = -1$, and $\alpha\beta\gamma = 5$, find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ and the value of $\alpha^2 + \beta^2 + \gamma^2$, without finding the roots of $f(x)$.

24. **HARD** **Calculator** **[5 marks]**

A polynomial is defined by $g(x) = 3x^4 - 4x^3 - 20x^2 + 9x + c$, where $c \in \mathbb{R}$. The diagram shows a sketch of part of the graph of $y = g(x)$.



The graph crosses the x-axis at $x = -2$ and $x = 3$, and has y-intercept $(0, 18)$.

(a) Write down the value of c. [1]

(b) Show that $g(x) = (x + 2)(x - 3)(3x^2 - x - 3)$. [4]

25. HARD No Calculator **[5 marks]**

The equation $2x^3 - 9x^2 + kx + 3 = 0$, where $k \in \mathbb{R}$, has roots α, β, and γ. Given that $\alpha^2 + \beta^2 + \gamma^2 = \frac{65}{4}$, find the value of k.

26. HARD Calculator **[5 marks]**

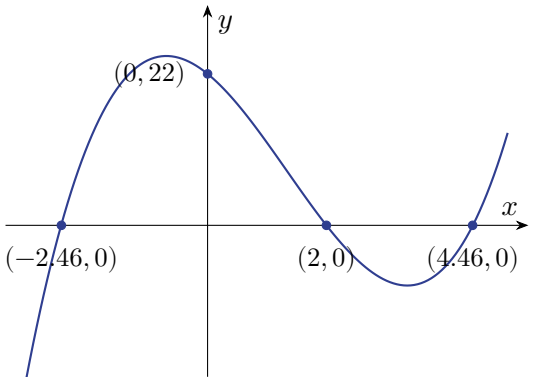
The cubic equation $x^3 - 6x^2 + 13x - 10 = 0$ has one real root and two complex roots.

(a) Show that $x = 2$ is the real root. [1]

(b) Find the two complex roots, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$. [4]

27. **HARD** **Calculator** **[5 marks]**

Let $h(x) = x^3 - 4x^2 - 7x + d$, where $d \in \mathbb{R}$. Given that $x = 1 - 2\sqrt{3}$ is a root of $h(x)$, find all roots of $h(x)$ and hence find the value of d. The diagram shows a sketch of $y = h(x)$.



28. **HARD** **Calculator** **[5 marks]**

The quartic $q(x) = x^4 + 3x^3 - 5x^2 + ax + b$, where $a, b \in \mathbb{R}$, has roots α, β, γ, δ. Given that $\alpha\beta\gamma\delta = -12$ and $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 7$, find the values of a and b.

29. HARD Calculator **[5 marks]**

The polynomial $f(x) = x^3 - 3x^2 - 13x + 15$ has three real roots. A new polynomial $g(t)$ has roots $2\alpha - 1$, $2\beta - 1$, and $2\gamma - 1$, where α, β, γ are the roots of $f(x)$.

(a) By using the substitution $x = \dfrac{t+1}{2}$, or otherwise, find $g(t)$ in the form $t^3 + pt^2 + qt + r$, where $p, q, r \in \mathbb{R}$. [4]

(b) Write down the sum of the roots of $g(t)$. [1]

30. HARD No Calculator **[5 marks]**

Show that $\left(x - \dfrac{1}{2}\right)$ is not a factor of $f(x) = 4x^3 + 6x^2 - 8x + 3$. Hence, by performing polynomial long division of $f(x)$ by $(2x - 1)$, find the quotient and the remainder, and use these to express $f(x)$ in the form $(2x - 1)Q(x) + R$, where $Q(x)$ is a quadratic and $R \in \mathbb{R}$.

Mark Schemes

Q1 — Remainder Theorem Evaluation EASY

Substitute $x = 3$ into $f(x)$:	**M1**
$f(3) = 3^3 - 2(3)^2 - 5(3) + 6 = 27 - 18 - 15 + 6$	**(A1)**
Remainder $= 0$	**A1**

Q2 — Factor Theorem and Evaluation EASY

Since $(x - 2)$ is a factor, $p(2) = 0$	**A1**
Evaluate $p(-3)$: $p(-3) = 2(-3)^3 + 3(-3)^2 - 11(-3) - 6$	**M1**
$= -54 + 27 + 33 - 6 = 0$	**A1**

Note: Award M1 for a correct substitution of $x = -3$; A1 for the value 0.

Q3 — Unknown Coefficient via Remainder Theorem EASY

Apply the remainder theorem: $f(2) = 9$	**M1**
$2^3 + a(2)^2 - 7(2) + 3 = 9 \implies 8 + 4a - 14 + 3 = 9$	**A1**
$4a - 3 = 9 \implies a = 3$	**A1**

Q4 — Reading Roots and Intercept from a Graph EASY

Roots: $x = -1$, $x = 2$, $x = 3$	**A1**

Note: Award A1 for all three roots correct.

y-intercept: $(0, 6)$	**A1**
Verify: $f(0) = 0 - 0 + 0 + 6 = 6$	**A1**

Note: Award the third A1 for correct verification or explicit statement of the y-intercept value 6.

Q5 — Vieta's Formulae for a Cubic EASY

For $q(x) = x^3 - 6x^2 + 11x - 6$, by Vieta's formulae:	**M1**
$\alpha + \beta + \gamma = -\dfrac{-6}{1} = 6$	**A1**
$\alpha\beta\gamma = -\dfrac{-6}{1} = 6$	**A1**

Note: For $ax^3 + bx^2 + cx + d$, sum of roots $= -b/a$ and product of roots $= -d/a$. Both signs must be correct.

Q6 — GDC Roots of a Cubic EASY

Using GDC, graph $y = x^3 + 5x^2 - 2x - 24$ and find zeros: **M1**

Roots found: $x = -4.00$, $x = -3.00$, $x = 2.00$ **(A1)**

$x = -4, \quad x = -3, \quad x = 2$ **A1**

Note: All three roots are integers; the GDC gives exact values. Verify: $(-4)^3 + 5(-4)^2 - 2(-4) - 24 = -64+80+8-24 = 0$; $(-3)^3+5(-3)^2-2(-3)-24 = -27+45+6-24 = 0$; $2^3+5(2)^2-2(2)-24 = 8+20-4-24 = 0$. *Award A1 for all three correct.*

Q7 — Coefficient via Factor Theorem EASY

Since $(x+1)$ is a factor, $f(-1) = 0$: **M1**

$(-1)^3 + b(-1) + 2 = 0 \implies -1 - b + 2 = 0$ **A1**

$1 - b = 0 \implies b = 1$ **A1**

Q8 — Polynomial Long Division EASY

Perform long division of $2x^3 - x^2 - 13x - 6$ by $(x-3)$: **M1**

$$2x^3 - x^2 - 13x - 6 = (x-3)(2x^2 + 5x + 2) + 0$$

Quotient $= 2x^2 + 5x + 2$ **A1**

Remainder $= 0$ **A1**

Note: Award M1 for correct first step of division giving $2x^2$. *Award A1 for the correct full quotient, A1 for remainder* 0.

Q9 — Full Factorisation from a Given Factor EASY

Since $(x-3)$ is a factor, divide: $h(x) = (x-3)(x^2-4)$ **M1**

$h(x) = (x-3)(x-2)(x+2)$ **A1**

Verify $x = -2$: $h(-2) = (-2-3)(-2-2)(-2+2) = (-5)(-4)(0) = 0$ ✓ **A1**

Note: Award M1 for obtaining the quadratic factor by inspection or division. Award A1 for fully factorised form. Award A1 for explicit verification that $h(-2) = 0$.

Q10 — Vieta Product for a Quartic EASY

For $p(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$, by Vieta's formulae: **M1**

$\alpha\beta\gamma\delta = \dfrac{-6}{1} = -6$ (product of roots of $x^4 + \ldots + d$ equals d/a) **A1**

GDC gives roots $x = -1, 1, 2, 3$; product $= (-1)(1)(2)(3) = -6$ ✓ **A1**

Note: For $x^4 + bx^3 + cx^2 + dx + e$, the product of all four roots equals e/a. Here $e = -6$, $a = 1$, so the product is -6. Award final A1 for correct GDC roots with product verified.

Q11 — Simultaneous Equations from the Remainder Theorem MEDIUM

Using the remainder theorem: $f(3) = 10$ and $f(-1) = -18$.

$f(3) = 2(27) - 5(9) + 3a + b = 54 - 45 + 3a + b = 9 + 3a + b = 10$ **M1**

So $3a + b = 1$ **A1**

$f(-1) = 2(-1) - 5(1) + a(-1) + b = -2 - 5 - a + b = -7 - a + b = -18$

So $-a + b = -11$ **A1**

Subtracting: $4a = 12$, giving $a = 3$ and $b = -8$ **A1**

Q12 — Full Factorisation and Sketch MEDIUM

(a) Since $(x - 2)$ is a factor, divide $p(x) = x^3 - 4x^2 + x + 6$ by $(x - 2)$: **M1**

$p(x) = (x - 2)(x^2 - 2x - 3) = (x - 2)(x - 3)(x + 1)$ **A1**

(b) Roots at $x = -1$, $x = 2$, $x = 3$ labelled; y-intercept $(0, 6)$ labelled; correct end behaviour (rising to $+\infty$, falling to $-\infty$). **A1 A1**

Note: Award A1 for all intercepts correctly labelled and A1 for correct end behaviour with no extra crossings.

Q13 — Vieta's Formulae for a Cubic MEDIUM

For $g(x) = x^3 + px^2 + qx - 8$, by Vieta's formulae: **M1**

$\alpha + \beta + \gamma = -p = -3 \implies p = 3$ **A1**

$\alpha\beta + \alpha\gamma + \beta\gamma = q = -2 \implies q = -2$ **A1**

$\alpha\beta\gamma = -\dfrac{-8}{1} = 8$ **A1**

Note: Signs in Vieta: for $x^3 + px^2 + qx + r$, sum $= -p$, pair-sum $= q$, product $= -r$.

Q14 — Polynomial Long Division of a Cubic MEDIUM

Dividing $3x^3 + 2x^2 - 7x + 2$ by $(x + 2)$: **M1**

$$\begin{aligned} 3x^3+2x^2-7x+2 &= (x+2)(3x^2)+(-4x^2-7x+2) \\ &= (x+2)(3x^2)+(x+2)(-4x)+(x+2) \\ &= (x+2)(3x^2-4x+1) \end{aligned}$$

M1

Quotient confirmed: $3x^2-4x+1$ with zero remainder **A1**

$h(x)=(x+2)(3x^2-4x+1)$ **A1**

Note: Full long division working must be shown line by line for both M marks.

Q15 — Symmetric Functions of Roots MEDIUM

(a) By Vieta's formulae for $x^3-6x^2+11x-6=0$: **M1**

$\alpha+\beta+\gamma=6, \quad \alpha\beta+\alpha\gamma+\beta\gamma=11$

$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=36-22=14$ **A1**

(b) $\alpha\beta\gamma=6$ (Vieta: product $=-\dfrac{-6}{1}=6$) **M1**

$\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}=\dfrac{11}{6}\ (=11/6)$ **A1**

Q16 — Complex Roots of a Real Quartic MEDIUM

Since $f(x)$ has real coefficients and $1+i$ is a root, its conjugate $1-i$ is also a root. **R1**

The quadratic factor from this pair: $(x-(1+i))(x-(1-i))=(x-1)^2+1=x^2-2x+2$ **A1**

Dividing $f(x)=x^4-7x^3+18x^2-22x+12$ by (x^2-2x+2): **M1**

$f(x)=(x^2-2x+2)(x^2-5x+6)$

Factorising the quotient: $x^2-5x+6=(x-2)(x-3)$, so the remaining roots are $x=2$ and $x=3$. **A1**

Roots: $x=1+i,\ 1-i,\ 2,\ 3$ **A1**

Note: Award R1 only if a reason is stated (real coefficients imply conjugate pair). Verify: $f(1+i)=0$ by direct substitution.

Q17 — Repeated Root Factorisation and Sketch MEDIUM

(a) Testing $x=1$: $f(1)=1+1-5+3=0$, so $(x-1)$ is a factor. **M1**

Dividing: $f(x)=(x-1)(x^2+2x-3)=(x-1)(x-1)(x+3)=(x-1)^2(x+3)$

So $a=1, b=-3$ **A1**

(b) x-intercepts: $(1,0)$ (touch, double root) and $(-3,0)$ (cross) labelled; y-intercept $(0,3)$ labelled; correct end behaviour. **A1 A1**

Note: Award A1 for intercepts correctly labelled including the touch at $x = 1$, and A1 for correct end behaviour.

Q18 — New Cubic via Root Transformation MEDIUM

For $2x^3 - 3x^2 - 11x + 6 = 0$, by Vieta's formulae: **M1**

$\alpha + \beta + \gamma = \dfrac{3}{2}, \quad \alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{-11}{2}, \quad \alpha\beta\gamma = -\dfrac{6}{2} = -3$

For roots $2\alpha, 2\beta, 2\gamma$:

Sum $= 2(\alpha + \beta + \gamma) = 3$ **A1**

Pair-sum $= 4(\alpha\beta + \alpha\gamma + \beta\gamma) = -22$

Product $= 8\alpha\beta\gamma = -24$ **A1**

New equation: $x^3 - 3x^2 - 22x + 24 = 0$ **A1**

Note: Vieta signs for $2x^3 - 3x^2 - 11x + 6$: sum $= -(-3)/2 = 3/2$, pair-sum $= -11/2$, product $= -6/2 = -3$.

Q19 — Finding k and Roots from a Graph MEDIUM

Since $x = -2$ is a root: $f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + k = -8 - 8 + 10 + k = k - 6 = 0$ **M1**

$k = 6$ **A1**

So $f(x) = x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$ **M1**

$= (x + 2)(x - 1)(x - 3)$, giving $r = 1$ and $s = 3$ **A1**

Note: GDC may be used to factorise after k is found; award M1 A1 for correct factorisation by any method.

Q20 — Remainder Theorem with Two Conditions MEDIUM

By the remainder theorem: $f(4) = 93$ and $f(1) = 12$. **M1**

$f(4) = 64 + 16c + 20 + d = 84 + 16c + d = 93 \implies 16c + d = 9$

$f(1) = 1 + c + 5 + d = 6 + c + d = 12 \implies c + d = 6$ **A1**

Subtracting: $15c = 3 \implies c = \dfrac{1}{5}, d = \dfrac{29}{5}$ **A1**

$f(2) = 8 + 4\left(\dfrac{1}{5}\right) + 10 + \dfrac{29}{5} = 18 + \dfrac{4}{5} + \dfrac{29}{5} = 18 + \dfrac{33}{5} = \dfrac{123}{5} \; (= 123/5)$ **A1**

Note: GDC may be used to solve the simultaneous equations; award A1 for each correct value of c and d, then A1 for the final remainder.

Q21 — Simultaneous Equations from the Remainder Theorem HARD

Using the remainder theorem: $f(3) = 42$ and $f(-2) = -8$.

$f(3) = 2(27) + 9a + 3b - 12 = 54 + 9a + 3b - 12 = 42 + 9a + 3b = 42$ **M1**

So $9a + 3b = 0$, giving $3a + b = 0$. **A1**

$f(-2) = 2(-8) + 4a - 2b - 12 = -16 + 4a - 2b - 12 = -28 + 4a - 2b = -8$ **M1**

So $4a - 2b = 20$, giving $2a - b = 10$.

Adding: $5a = 10 \implies a = 2$, then $b = -6$. **A1 A1**

Note: Award A1 A1 for both correct values; A1 A0 for one correct value only.

Q22 — Factor Theorem then Full Factorisation HARD

(a) $p(2) = 0$: $16 - 16 + 4k + 16 + 12 = 0$ **M1**

$4k + 28 = 0 \implies k = -7$ **A1**

(b) With $k = -7$: $p(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$.

Divide by $(x - 2)$ (GDC or long division): $p(x) = (x - 2)(x^3 - 7x - 6)$ **M1**

Testing integer factors of 6 in the cubic $x^3 - 7x - 6$: $x = -1$ gives $-1 + 7 - 6 = 0$, so $(x + 1)$ is a factor. **M1**

Dividing: $x^3 - 7x - 6 = (x + 1)(x^2 - x - 6) = (x + 1)(x - 3)(x + 2)$ **A1**

$p(x) = (x - 2)(x + 1)(x - 3)(x + 2)$ **A1**

Note: Award final A1 A1 for the complete factorisation into four linear real factors. Verify: roots $2, -1, 3, -2$ all satisfy $p(x) = 0$.

Q23 — Vieta Symmetric Expressions HARD

Using Vieta's formulae: $\alpha + \beta + \gamma = -3$, $\alpha\beta + \beta\gamma + \gamma\alpha = -1$, $\alpha\beta\gamma = 5$.

Sum of reciprocals:

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$ **M1**

$$= \frac{-1}{5} = -\frac{1}{5} \; (= -1/5)$$ **A1**

Sum of squares:

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ **M1**

$= (-3)^2 - 2(-1) = 9 + 2 = 11$ **A1 A1**

Note: Award M1 for each correct identity stated and applied. Final A1 A1: one mark per correct numerical answer. No marks if roots are found by solving the cubic.

Q24 — Full Factorisation of a Quartic with Sketch HARD

(a) $c = g(0) = 18$ **A1**

(b) Since $x = -2$ and $x = 3$ are roots, $(x+2)(x-3) = x^2 - x - 6$ is a factor. **M1**

Perform long division of $g(x) = 3x^4 - 4x^3 - 20x^2 + 9x + 18$ by $x^2 - x - 6$: **M1**

$3x^4 \div x^2 = 3x^2$; $\quad 3x^2(x^2 - x - 6) = 3x^4 - 3x^3 - 18x^2$; remainder $-x^3 - 2x^2 + 9x + 18$

$-x^3 \div x^2 = -x$; $\quad -x(x^2 - x - 6) = -x^3 + x^2 - 6x$; remainder $-3x^2 + 3x + 18$

$-3x^2 \div x^2 = -3$; $\quad -3(x^2 - x - 6) = -3x^2 + 3x + 18$; remainder 0 **A1**

Quotient: $3x^2 - x - 3$, so $g(x) = (x^2 - x - 6)(3x^2 - x - 3) = (x+2)(x-3)(3x^2 - x - 3)$ **A1 (AG)**

Note: AG — final answer is given, so the last mark is for a fully correct, consistent long division confirming zero remainder.

Q25 — Vieta with Sum of Squares HARD

For $2x^3 - 9x^2 + kx + 3 = 0$, by Vieta's formulae: **M1**

$$\alpha + \beta + \gamma = \frac{9}{2}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{k}{2}, \quad \alpha\beta\gamma = -\frac{3}{2}$$

Using the identity $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$: **M1**

$$\frac{65}{4} = \left(\frac{9}{2}\right)^2 - 2\left(\frac{k}{2}\right) = \frac{81}{4} - k$$ **A1**

$$k = \frac{81}{4} - \frac{65}{4} = \frac{16}{4} = 4$$ **A1 A1**

Note: Award M1 M1 for correct Vieta expressions and correct identity; A1 A1 for correct computation and final $k = 4$. Since k is the coefficient of x, it affects the pair-sum $\alpha\beta + \beta\gamma + \gamma\alpha = k/2$ and hence genuinely determines $\alpha^2 + \beta^2 + \gamma^2$.

Q26 — Real Cubic with Complex Roots HARD

(a) $f(2) = 8 - 24 + 26 - 10 = 0$ **A1**

So $x = 2$ is a root. **(AG)**

(b) Divide $x^3 - 6x^2 + 13x - 10$ by $(x-2)$ (GDC or synthetic division): **M1**

$x^3 - 6x^2 + 13x - 10 = (x-2)(x^2 - 4x + 5)$

Solve $x^2 - 4x + 5 = 0$ using the quadratic formula: **M1**

$$x = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$$ **A1**

$x = 2 + i$ and $x = 2 - i$ **A1**

Note: Final A1 A1 for both complex roots in the required form; award A1 A0 if only one root stated without the conjugate.

Q27 — Roots from a Given Surd Root HARD

Since the coefficients of $h(x)$ are real and $x = 1 - 2\sqrt{3}$ is a root, its conjugate surd $x = 1 + 2\sqrt{3}$ is also a root. **R1**

Sum of these two roots: $(1 - 2\sqrt{3}) + (1 + 2\sqrt{3}) = 2$

Product of these two roots: $(1 - 2\sqrt{3})(1 + 2\sqrt{3}) = 1 - 12 = -11$

So the two surd roots satisfy $x^2 - 2x - 11 = 0$. **M1**

By Vieta for $h(x) = x^3 - 4x^2 - 7x + d$: sum of all roots $= 4$.

Let the third root be ρ: $\rho + 2 = 4 \implies \rho = 2$. **A1**

So $h(x) = (x - 2)(x^2 - 2x - 11)$; by Vieta, $\alpha\beta\gamma = -d \implies 2 \times (-11) = -d \implies d = 22$. **M1**

Check: $(x - 2)(x^2 - 2x - 11) = x^3 - 2x^2 - 11x - 2x^2 + 4x + 22 = x^3 - 4x^2 - 7x + 22$ ✓ **A1**

Roots: $x = 2,\ 1 - 2\sqrt{3} \approx -2.46,\ 1 + 2\sqrt{3} \approx 4.46$; $\quad d = 22$.

Note: R1 for explicitly stating that real coefficients imply the conjugate surd is also a root.

Q28 — Vieta for a Quartic Two Unknowns HARD

For $q(x) = x^4 + 3x^3 - 5x^2 + ax + b$ with roots $\alpha, \beta, \gamma, \delta$:

By Vieta's formulae (leading coefficient $= 1$):

$\alpha\beta\gamma\delta = b$ **M1**

Given $\alpha\beta\gamma\delta = -12$, so $b = -12$. **A1**

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -a$ **M1**

Note: For $x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$, the elementary symmetric polynomial $e_3 = -c_1$ (i.e. $= -a$).

Given $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 7$, so $-a = 7 \implies a = -7$. **A1 A1**

Note: Award M1 M1 for correct identification of both Vieta relations; A1 A1 for both correct values. Deduct one A1 if sign error in either Vieta relation.

Q29 — New Polynomial via Substitution HARD

(a) Substitute $x = \dfrac{t+1}{2}$ into $f(x) = x^3 - 3x^2 - 13x + 15$: **M1**

$$g(t) = \left(\frac{t+1}{2}\right)^3 - 3\left(\frac{t+1}{2}\right)^2 - 13\left(\frac{t+1}{2}\right) + 15 = \frac{(t+1)^3}{8} - \frac{3(t+1)^2}{4} - \frac{13(t+1)}{2} + 15$$

Multiply through by 8: **M1**

$8g(t) = (t+1)^3 - 6(t+1)^2 - 52(t+1) + 120$

$(t+1)^3 = t^3 + 3t^2 + 3t + 1$; $\quad -6(t+1)^2 = -6t^2 - 12t - 6$; $\quad -52(t+1) = -52t - 52$

$8g(t) = t^3 + 3t^2 + 3t + 1 - 6t^2 - 12t - 6 - 52t - 52 + 120 = t^3 - 3t^2 - 61t + 63$ **A1**

Since $f(x)$ is monic and $x = (t+1)/2$, the monic form of $g(t)$ is $g(t) = t^3 - 3t^2 - 61t + 63$. **A1**

Note: GDC confirms roots of $f(x)$ are $\alpha = -3, \beta = 1, \gamma = 5$; roots of $g(t)$ are $t = 2\alpha - 1 = -7$, $t = 2\beta - 1 = 1$, $t = 2\gamma - 1 = 9$.

(b) Sum of roots of $g(t) = -(-3)/1 = 3$. **A1**

Note: Accept $2(\alpha+\beta+\gamma) - 3 = 2(3) - 3 = 3$ using Vieta on $f(x)$.

Q30 — Remainder Theorem then Long Division HARD

Test $x = \dfrac{1}{2}$: **M1**

$$f\left(\frac{1}{2}\right) = 4\cdot\frac{1}{8} + 6\cdot\frac{1}{4} - 8\cdot\frac{1}{2} + 3 = \frac{1}{2} + \frac{3}{2} - 4 + 3 = 1$$

Since $f\left(\dfrac{1}{2}\right) = 1 \neq 0$, $\left(x - \dfrac{1}{2}\right)$ is not a factor of $f(x)$. **R1**

Perform long division of $4x^3 + 6x^2 - 8x + 3$ by $(2x - 1)$: **M1**

$4x^3 \div 2x = 2x^2$; $\quad 2x^2(2x-1) = 4x^3 - 2x^2$; remainder $8x^2 - 8x + 3$

$8x^2 \div 2x = 4x$; $\quad 4x(2x-1) = 8x^2 - 4x$; remainder $-4x + 3$

$-4x \div 2x = -2$; $\quad -2(2x-1) = -4x + 2$; remainder 1

Quotient $Q(x) = 2x^2 + 4x - 2$, remainder $R = 1$. **A1**

$f(x) = (2x-1)(2x^2 + 4x - 2) + 1$ **A1**

Note: R1 requires an explicit statement that $f(1/2) \neq 0$ therefore $(x - 1/2)$ is not a factor. Award M1 for at least two correct steps of long division. Final A1 A1 for correct $Q(x)$ and R.