IB Math AA HL — Topic 3

Geometry Toolkit

Coordinate Geometry · Radian Measure · Arcs & Sectors

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:________________________ Date:______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Distance:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **Midpoint:** $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
- **Degree ↔ Radian:** $\theta_{\text{rad}} = \theta_{\text{deg}} \times \dfrac{\pi}{180}$ and $\theta_{\text{deg}} = \theta_{\text{rad}} \times \dfrac{180}{\pi}$
- **Arc length** (radians): $l = r\theta$ **Arc length** (degrees): $l = \dfrac{\theta}{360} \times 2\pi r$
- **Sector area** (radians): $A = \frac{1}{2}r^2\theta$ **Sector area** (degrees): $A = \dfrac{\theta}{360} \times \pi r^2$
- **Perimeter of a sector:** $P = 2r + l = 2r + r\theta$ (two radii *plus* the arc)
- **Segment area:** $A_{\text{seg}} = \frac{1}{2}r^2(\theta - \sin\theta)$ (θ in radians; equals sector area − triangle area)
- **Triangle area** (used in segments): $A_{\triangle} = \frac{1}{2}r^2 \sin\theta$

GDC Tips

- **Mode check first:** before any trig calculation, confirm the GDC is set to *Radians* or *Degrees* as required by the question; find this in `Settings` $\to$ `Angle`.
- **Solving for r or θ:** use `Menu` $\to$ `Algebra` $\to$ `Solve` (CAS) or the `Equation Solver` (non-CAS) when the unknown appears inside $l = r\theta$ or $A = \frac{1}{2}r^2\theta$ and arithmetic is awkward.
- **Composite areas:** store intermediate sector and triangle areas in GDC variables (`A`, `B`, …) to avoid rounding errors before the final subtraction or addition.
- **Degree$\to$Radian conversion:** type `45°` in radian mode and the GDC returns the exact decimal; alternatively use `Ans`$\times\pi\div$`180` to convert a stored degree value.
- **Segment areas with non-standard angles:** evaluate $\frac{1}{2}r^2(\theta - \sin\theta)$ directly on the home screen in radian mode; the GDC handles $\sin\theta$ without extra steps.
- **Graphing to check arc-length equations:** plot $Y_1 = r \cdot X$ and $Y_2 = l$ (the given arc length) then use `Intersection` to find θ numerically as a sanity check.

Common Mistakes to Avoid

1. **Forgetting the two radii in sector perimeter.** Students write $P = l = r\theta$ and ignore the two straight sides; the correct perimeter is always $P = 2r + r\theta$.
2. **Using degrees directly in radian formulae.** Substituting $\theta = 60$ (degrees) into $A = \frac{1}{2}r^2\theta$ gives an answer 60 times too large; convert to $\theta = \frac{\pi}{3}$ radians first and show the conversion line.
3. **Segment area taken as sector area alone.** A segment is the region between a chord and its arc; its area is *sector minus triangle*, i.e. $\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$, not just $\frac{1}{2}r^2\theta$.
4. **Mixing degree and radian sector-area formulae.** Using $A = \dfrac{\theta}{360}\pi r^2$ with θ already in radians (or vice versa) gives a completely wrong answer; identify which formula applies and keep units consistent throughout.
5. **Rounding intermediate values and losing accuracy.** Storing a rounded value of θ or r mid-solution then using it in a second formula produces accumulated error; keep full precision until the final step and round only the stated answer to 3 s.f. (or as required).
6. **Confusing arc length with sector area.** Both involve r and θ but $l = r\theta$ is length (one power of r) while $A = \frac{1}{2}r^2\theta$ is area (two powers of r); checking units (cm vs cm^2) immediately reveals this error.

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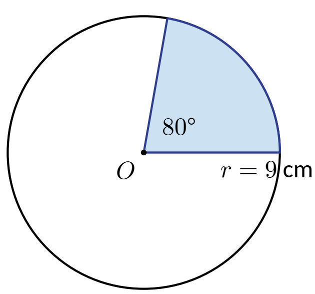
Section A — Easy

EASY

1. Points $A(-3, 2)$ and $B(5, 8)$ are in the coordinate plane. Find the distance AB. *Calculator allowed* **[3 marks]**

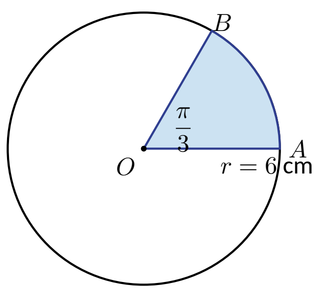
2. Convert $\frac{5\pi}{6}$ radians to degrees. *Calculator not allowed* **[3 marks]**

3. A sector has radius $r = 9$ cm and central angle $\theta = 80°$. Find the arc length l of the sector, giving your answer to 3 significant figures. *Calculator allowed* **[3 marks]**



4. Find the midpoint M of the segment joining $P(1,\,-5)$ and $Q(7,\,3)$. *Calculator allowed* **[3 marks]**

5. A sector OAB has radius $r = 6$ cm and central angle $\theta = \dfrac{\pi}{3}$ radians. Find the exact area of the sector. *Calculator not allowed* **[3 marks]**



6. Convert $135°$ to radians, giving your answer as an exact multiple of π. *Calculator not allowed* **[3 marks]**

7. A sector has radius $r = 12$ cm and arc length $l = 9$ cm. Find the central angle θ in radians. *Calculator allowed* **[3 marks]**

8. Find the gradient of the straight line passing through the points $C(-2,\,5)$ and $D(4,\,-1)$. *Calculator allowed* **[3 marks]**

9. A sector has radius $r = 7$ cm and central angle $\theta = 1.2$ radians. Find the perimeter of the sector, giving your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

10. A sector has radius $r = 10$ cm and central angle $\theta = 140°$. Find the area of the sector, giving your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. The points $A(1, 4)$ and $B(7, -2)$ are given. The point $P(x, y)$ lies on the perpendicular bisector of AB.

(a) Find the coordinates of the midpoint M of AB. [1]

(b) Find the gradient of the perpendicular bisector of AB. [1]

(c) Hence find the equation of the perpendicular bisector of AB, giving your answer in the form $y = mx + c$. [2]

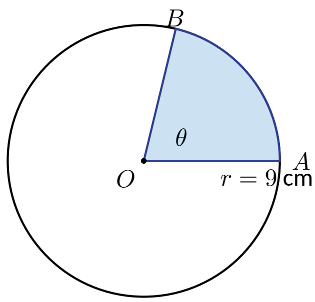
Calculator not allowed **[4 marks]**

12. A sector OAB has radius $r = 9$ cm and arc length $l = 12$ cm, where the angle θ is measured in radians.

(a) Find the angle θ of the sector. [1]

(b) Find the area of the sector. [1]

(c) Find the perimeter of the sector. [2]



Calculator not allowed **[4 marks]**

13. A windscreen wiper of length 42 cm sweeps through an angle of $120°$. The pivot is at one end of the wiper.

(a) Convert $120°$ to radians, giving your answer as an exact multiple of π. [1]

(b) Find the area swept by the wiper. Give your answer in cm^2, correct to 3 significant figures. [2]

(c) Find the length of the arc traced by the tip of the wiper, correct to 3 significant figures. [1]

Calculator allowed **[4 marks]**

14. Points $P(-3, 1)$, $Q(5, 5)$ and $R(k, -2)$ are given, where $k \in \mathbb{R}$.

(a) Find the distance PQ. [2]

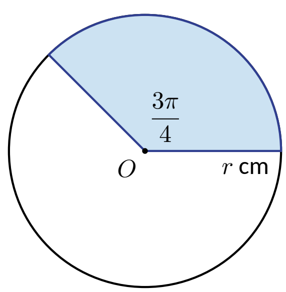
(b) Given that the gradient of QR is $-\dfrac{7}{2}$, find the value of k. [2]

Calculator allowed **[4 marks]**

15. A sector of a circle has radius r cm and angle $\dfrac{3\pi}{4}$ radians. The area of the sector is 54π cm^2.

(a) Show that $r = 12$ cm. [2]

(b) Find the exact perimeter of the sector. [2]



Calculator not allowed **[4 marks]**

16. An irrigation system rotates about a fixed point O, spraying water over a sector of angle $200°$ and radius 18.5 m.

(a) Convert $200°$ to radians, correct to 4 significant figures. [1]

(b) Find the area of the sector, correct to 3 significant figures. [1]

(c) A circular path of radius 5.0 m is centred at O inside the sector. Find the area of the region that lies within the sector but outside the circular path, correct to 3 significant figures. [2]

Calculator allowed **[4 marks]**

17. A running track has a straight section and a semicircular end. The semicircular end has diameter d m, where $d \in \mathbb{R}^+$. An athlete runs along the arc of the semicircle and then 85.0 m along the straight section. The total distance run is 220 m.

(a) Write down an equation for d in terms of the total distance. [1]

(b) Find the value of d, correct to 3 significant figures. [2]

(c) Find the area enclosed by the semicircle, correct to 3 significant figures. [1]

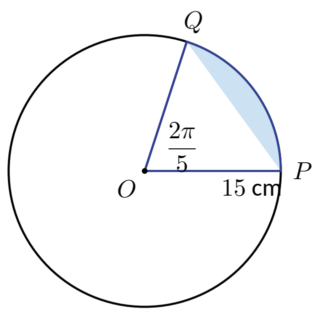
Calculator allowed **[4 marks]**

18. The diagram shows a sector OPQ with radius 15 cm and angle $\theta = \dfrac{2\pi}{5}$ radians. The area of the segment (shaded region between the arc PQ and the chord PQ) is to be found.

(a) Find the area of the sector OPQ. [1]

(b) Find the area of triangle OPQ. [1]

(c) Hence find the area of the segment, correct to 3 significant figures. [2]



Calculator allowed **[4 marks]**

19. A pizza slice is modelled as a sector with radius 14 cm and angle $\dfrac{\pi}{4}$ radians. A second, larger pizza has radius 18 cm and the same arc length as the first.

(a) Find the arc length of the first pizza slice, giving your answer as an exact multiple of π. [1]

(b) Find the angle of the second pizza's sector, giving your answer as an exact fraction of π. [1]

(c) Find the difference in area between the two sectors, giving your answer correct to 3 significant figures. [2]

Calculator allowed **[4 marks]**

20. The points $C(2, -1)$ and $D(8, 3)$ are the endpoints of a diameter of a circle.

(a) Find the centre of the circle. [1]

(b) Find the exact radius of the circle. [1]

(c) Find the equation of the circle in the form $(x-h)^2 + (y-k)^2 = r^2$. [2]

Calculator allowed **[4 marks]**

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Section C — Hard

HARD

21. A circle has centre O and radius r cm. A chord PQ subtends an angle of $\dfrac{2\pi}{3}$ radians at O. The area of the minor segment cut off by PQ is $12\pi - 9\sqrt{3}$ cm^2.

(a) Show that $r = 6$ cm. **[3 marks]**

(b) Find the perimeter of the minor segment, giving your answer in the form $a\pi + b\sqrt{3}$ cm, where $a, b \in \mathbb{Q}$. **[2 marks]**

Calculator not allowed **[5 marks]**

22. Points $A(-3, 5)$ and $B(7, -1)$ lie in the coordinate plane. The perpendicular bisector of AB meets the y-axis at point C.

(a) Find the coordinates of C. **[3 marks]**

(b) Hence find the area of triangle OAC, where O is the origin. **[2 marks]**

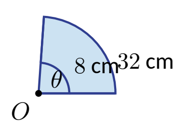
Calculator allowed **[5 marks]**

23. A windscreen wiper has a blade of length 32 cm. The wiper pivots at point O and the near edge of the blade is 8 cm from O. In one sweep, the wiper rotates through an angle of θ radians, cleaning an area of 720 cm^2.

(a) Show that $\theta = \dfrac{3}{2}$ radians. **[3 marks]**

(b) Find the total length of the boundary of the cleaned region. Give your answer correct to 3 significant figures. **[2 marks]**

Calculator allowed **[5 marks]**



24. An arc AB of a circle with centre O and radius r cm has length l cm and subtends an angle θ radians at O. It is given that $l = 15$ cm and that the area of sector OAB is 67.5 cm^2.

(a) Find the values of r and θ. **[3 marks]**

(b) A second sector has the same perimeter as sector OAB but its radius is 12 cm. Find the area of the second sector. **[2 marks]**

Calculator allowed **[5 marks]**

25. The points $P(1, -2)$, $Q(5, 4)$ and $R(k, 0)$, where $k \in \mathbb{R}$, are such that PQ is perpendicular to QR.

(a) Find the value of k. **[3 marks]**

(b) Find the exact length PR. **[2 marks]**

Calculator allowed **[5 marks]**

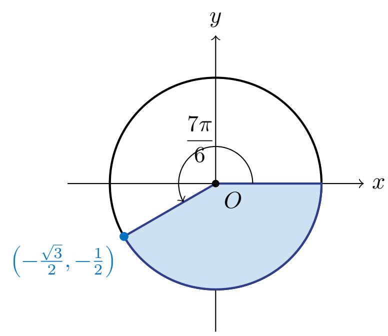
26. Two sectors share the same centre O. Sector OAB has radius 9 cm and angle α radians. Sector OCD has radius 15 cm and angle β radians. The arc length of OAB equals the arc length of OCD, and the sum of the two sector areas is 189 cm^2.
Find the values of α and β. *Calculator allowed* **[5 marks]**

27. The angle $\dfrac{7\pi}{6}$ radians is marked on a unit circle.

(a) Write down the exact coordinates of the corresponding point on the unit circle. **[2 marks]**

(b) Hence find the exact area of the minor sector formed by the angle $\dfrac{\pi}{6}$ and the unit circle, and show that the major sector area is $\dfrac{11\pi}{12}$ square units. **[3 marks]**

Calculator not allowed **[5 marks]**



28. An irrigation system sprays water over a field. The sprayer is fixed at point O and rotates, covering a sector of radius 45 m. The sprayer covers an angle of $220°$.

(a) Convert $220°$ to radians, giving your answer correct to 4 significant figures. **[1 marks]**

(b) Find the area watered by the sprayer. Give your answer correct to 3 significant figures. **[2 marks]**

(c) A fence runs along the straight boundary of the sector. A second fence runs along the curved boundary. Find the total length of fencing required. Give your answer correct to 3 significant figures. **[2 marks]**

Calculator allowed **[5 marks]**

29. Points $A(2, 6)$, $B(8, 4)$ and C lie in the coordinate plane. C lies on the perpendicular bisector of AB and $|AC| = |BC| = \sqrt{53}$.

(a) Find the coordinates of C. **[3 marks]**

(b) State, with a reason, whether C lies inside or outside the circle with diameter AB. **[2 marks]**

Calculator allowed **[5 marks]**

30. A sector of radius r cm and angle θ radians has perimeter P cm and area A cm^2. It is given that $P = 24$ and $A = 27$.

(a) Show that r satisfies the equation $r^2 - 12r + 27 = 0$. **[3 marks]**

(b) Hence find all possible values of r and the corresponding values of θ. **[2 marks]**

Calculator not allowed **[5 marks]**

Worked Solutions

Q1 — Distance between two points [EASY]

Using the distance formula with $A(-3, 2)$ and $B(5, 8)$:

$$AB = \sqrt{(5-(-3))^2 + (8-2)^2}$$

M1

$$= \sqrt{8^2 + 6^2} = \sqrt{64+36} = \sqrt{100}$$

(A1)

$$AB = 10 \text{ units}$$

A1

Q2 — Radian to degree conversion [EASY]

Multiplying by $\dfrac{180°}{\pi}$:

$$\frac{5\pi}{6} \times \frac{180°}{\pi}$$

M1

$$= \frac{5 \times 180°}{6}$$

(A1)

$$= 150°$$

A1

Q3 — Arc length from degree angle [EASY]

Converting $80°$ to radians:

$$\theta = 80 \times \frac{\pi}{180} = \frac{4\pi}{9} \text{ rad}$$

M1

Applying $l = r\theta$:

$$l = 9 \times \frac{4\pi}{9} = 4\pi = 12.566\ldots$$

(A1)

$$l = 12.6 \text{ cm}$$

A1

Note: Award M1 for correct substitution into $l = r\theta$ with θ in radians. Accept 12.6 cm only after correct conversion.

Q4 — Midpoint of a segment [EASY]

Using the midpoint formula:

$$M = \left(\frac{x_P + x_Q}{2}, \frac{y_P + y_Q}{2}\right)$$

M1

$$= \left(\frac{1 + 7}{2}, \frac{-5 + 3}{2}\right)$$

(A1)

$$M = (4, -1)$$

A1

Q5 — Exact sector area in radians [EASY]

Using $A = \frac{1}{2}r^2\theta$ with $r = 6$ cm and $\theta = \frac{\pi}{3}$:

$$A = \frac{1}{2} \times 6^2 \times \frac{\pi}{3}$$

M1

$$= \frac{1}{2} \times 36 \times \frac{\pi}{3} = \frac{36\pi}{6}$$

(A1)

$$A = 6\pi \text{ cm}^2$$

A1

Q6 — Degree to radian conversion [EASY]

Multiplying by $\frac{\pi}{180°}$:

$$135° \times \frac{\pi}{180°}$$

M1

$$= \frac{135\pi}{180} = \frac{3\pi}{4}$$

(A1)

$$135° = \frac{3\pi}{4} \text{ radians}$$

A1

Q7 — Finding central angle from arc length [EASY]

Using $l = r\theta$ with $l = 9$ cm and $r = 12$ cm:

$$9 = 12\theta$$

M1

$$\theta = \frac{9}{12}$$

(A1)

$$\theta = 0.75 \text{ radians}$$

A1

Q8 — Gradient between two points [EASY]

Using the gradient formula with $C(-2, 5)$ and $D(4, -1)$:

$$m = \frac{y_D - y_C}{x_D - x_C} = \frac{-1-5}{4-(-2)}$$

M1

$$= \frac{-6}{6}$$

(A1)

$$m = -1$$

A1

Q9 — Perimeter of a sector [EASY]

The perimeter of a sector consists of two radii and the arc:

$$P = 2r + l = 2r + r\theta$$

M1

Substituting $r = 7$ cm and $\theta = 1.2$ rad:

$$P = 2(7) + 7(1.2) = 14 + 8.4 = 22.4$$

(A1)

$$P = 22.4 \text{ cm}$$

A1

Note: Award M0 if only the arc length $l = 8.40$ cm is given as the perimeter (arc-only error).

Q10 — Sector area from degree angle [EASY]

Converting $140°$ to radians:

$$\theta = 140 \times \frac{\pi}{180} = \frac{7\pi}{9} = 2.4435\ldots \text{ rad}$$

M1

Applying $A = \frac{1}{2}r^2\theta$ with $r = 10$ cm:

$$A = \frac{1}{2} \times 100 \times \frac{7\pi}{9} = \frac{350\pi}{9} = 122.17\ldots$$

(A1)

$$A = 122 \text{ cm}^2$$

A1

Note: Award M1 for correct substitution into the sector area formula with θ converted to radians. Unrounded value $122.17\ldots$ must be seen before the rounded answer.

Q11 — Perpendicular bisector of a segment [MEDIUM]

(a) $M = \left(\frac{1+7}{2}, \frac{4+(-2)}{2}\right) = (4,\ 1)$ **A1**

(b) Gradient of AB: $m_{AB} = \frac{-2-4}{7-1} = \frac{-6}{6} = -1$ **(M1)**

Gradient of perpendicular bisector: $m_\perp = 1$ **A1**

(c) Using point $M(4, 1)$: $y - 1 = 1(x - 4)$ **(M1)**

$y = x - 3$ **A1**

Note: Award final A1 only if the equation is in the form $y = mx + c$.

Q12 — Arc length, area and perimeter of a sector [MEDIUM]

(a) Using $l = r\theta$: $\theta = \frac{l}{r} = \frac{12}{9} = \frac{4}{3}$ rad **A1**

(b) $A = \frac{1}{2}r^2\theta = \frac{1}{2}(9)^2\left(\frac{4}{3}\right) = \frac{1}{2}(81)\left(\frac{4}{3}\right) = 54$ cm^2 **A1**

(c) Perimeter $= 2r + l$ **M1**

$= 2(9) + 12 = 30$ cm **A1**

Note: Award M1 only if both radii and the arc are included. Do not award if arc length alone is given.

Q13 — Windscreen wiper: arc and sector area [MEDIUM]

(a) $120^\circ \times \frac{\pi}{180} = \frac{2\pi}{3}$ radians **A1**

(b) $A = \frac{1}{2}r^2\theta = \frac{1}{2}(42)^2\left(\frac{2\pi}{3}\right)$ **M1**

$= \frac{1}{2}(1764)\left(\frac{2\pi}{3}\right) = 588\pi \approx 1850$ cm^2 **A1**

(c) $l = r\theta = 42 \times \frac{2\pi}{3} = 28\pi \approx 87.96\ldots \approx 88.0$ cm **A1**

Note: Accept $587.9\ldots$ *cm*^{2} *before rounding. Unrounded value must be seen.*

Q14 — Distance between points and gradient condition [MEDIUM]

(a) $PQ = \sqrt{(5 - (-3))^2 + (5 - 1)^2}$ **M1**

$= \sqrt{64 + 16} = \sqrt{80} = 4\sqrt{5} \approx 8.94$ **A1**

(b) Gradient of QR: $\frac{-2 - 5}{k - 5} = -\frac{7}{2}$ **M1**

$-14 = -7(k - 5) \Rightarrow k - 5 = 2 \Rightarrow k = 7$ **A1**

Q15 — Sector area to find radius; exact perimeter [MEDIUM]

(a) Area of sector: $\frac{1}{2}r^2\theta = 54\pi$ **M1**

$\frac{1}{2}r^2 \cdot \frac{3\pi}{4} = 54\pi \Rightarrow \frac{3\pi r^2}{8} = 54\pi \Rightarrow r^2 = \frac{54\pi \times 8}{3\pi} = 144$ **A1**

$r = 12$ cm **AG**

(b) Arc length: $l = r\theta = 12 \times \frac{3\pi}{4} = 9\pi$ cm **M1**

Perimeter $= 2r + l = 2(12) + 9\pi = (24 + 9\pi)$ cm **A1**

Note: M1 for perimeter $= 2r + l$; do not award if only arc length given.

Q16 — Irrigation sector and composite region [MEDIUM]

(a) $200^\circ \times \frac{\pi}{180} = 3.491\ldots$ rad ≈ 3.491 rad **A1**

(b) $A_{\text{sector}} = \frac{1}{2}(18.5)^2(3.491\ldots) = \frac{1}{2}(342.25)(3.491\ldots) = 597.5\ldots$ **(M1)**

≈ 598 m^2 **A1**

(c) Area of inner circular sector with same angle: $A_{\text{inner}} = \frac{1}{2}(5.0)^2(3.491\ldots) = 43.6\ldots$ m^2 **M1**

Required area $= 597.5\ldots - 43.6\ldots = 553.9\ldots \approx 554$ m^2 **A1**

Note: The inner region is a sector (not a full circle) since it lies inside the 200° sector.

Q17 — Running track semicircle and diameter [MEDIUM]

(a) Arc of semicircle $= \frac{\pi d}{2}$; total distance: $\frac{\pi d}{2} + 85.0 = 220$ **A1**

(b) $\frac{\pi d}{2} = 135 \Rightarrow d = \frac{270}{\pi}$ **M1**

$d = \frac{270}{\pi} = 85.94\ldots \approx 85.9$ m **A1**

(c) Radius $= \frac{d}{2} = \frac{135}{\pi}$; area of semicircle $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{135}{\pi}\right)^2 = \frac{9112.5}{\pi} = 2900.88\ldots \approx 2900$ m^2 **A1**

Q18 — Segment area: sector minus triangle [MEDIUM]

(a) $A_{\text{sector}} = \frac{1}{2}r^2\theta = \frac{1}{2}(15)^2\left(\frac{2\pi}{5}\right) = \frac{1}{2}(225)\left(\frac{2\pi}{5}\right) = 45\pi \approx 141.37\ldots$ cm^2 **A1**

(b) $A_{\triangle} = \frac{1}{2}r^2 \sin\theta = \frac{1}{2}(225)\sin\left(\frac{2\pi}{5}\right) = 112.5 \times 0.9511\ldots = 107.0\ldots$ cm^2 **A1**

(c) $A_{\text{segment}} = A_{\text{sector}} - A_{\triangle}$ **M1**

$= 141.37... - 107.0... = 34.37... \approx 34.4$ cm^2 **A1**

Note: Award M1 for the correct subtraction formula. Both unrounded values must be seen before rounding.

Q19 — Two sectors with equal arc lengths [MEDIUM]

(a) $l_1 = r_1\theta_1 = 14 \times \frac{\pi}{4} = \frac{7\pi}{2}$ cm **A1**

(b) $l_2 = r_2\theta_2 \Rightarrow \frac{7\pi}{2} = 18\theta_2 \Rightarrow \theta_2 = \frac{7\pi}{36}$ radians **A1**

(c) $A_1 = \frac{1}{2}(14)^2\left(\frac{\pi}{4}\right) = \frac{1}{2}(196)\left(\frac{\pi}{4}\right) = \frac{49\pi}{2} = 76.969...$ cm^2 **M1**

$A_2 = \frac{1}{2}(18)^2\left(\frac{7\pi}{36}\right) = \frac{1}{2}(324)\left(\frac{7\pi}{36}\right) = \frac{63\pi}{2} = 98.960...$ cm^2

Difference $= 98.960... - 76.969... = 21.99... \approx 22.0$ cm^2 **A1**

Q20 — Circle from diameter endpoints [MEDIUM]

(a) Centre $= \left(\frac{2+8}{2}, \frac{-1+3}{2}\right) = (5, 1)$ **A1**

(b) $r = \frac{1}{2}\sqrt{(8-2)^2 + (3-(-1))^2} = \frac{1}{2}\sqrt{36+16} = \frac{1}{2}\sqrt{52} = \sqrt{13}$ **M1A1**

(c) $(x-5)^2 + (y-1)^2 = 13$ **A1**

Note: Award final A1 only if centre and radius from (a) and (b) are correctly substituted.

Q21 — Segment area and perimeter [HARD]

(a) Area of minor segment $=$ area of sector $-$ area of triangle

Area of sector $= \frac{1}{2}r^2 \cdot \frac{2\pi}{3} = \frac{\pi r^2}{3}$ **M1**

Area of triangle $= \frac{1}{2}r^2 \sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}r^2 \cdot \frac{\sqrt{3}}{2} = \frac{r^2\sqrt{3}}{4}$

Segment area: $\frac{\pi r^2}{3} - \frac{r^2\sqrt{3}}{4} = 12\pi - 9\sqrt{3}$ **M1**

Comparing coefficients: $\frac{r^2}{3} = 12 \Rightarrow r^2 = 36 \Rightarrow r = 6$ **AG**

Note: Check $\frac{r^2\sqrt{3}}{4} = \frac{36\sqrt{3}}{4} = 9\sqrt{3}$ ✓

(b) Perimeter = arc + chord

Arc $= r\theta = 6 \cdot \dfrac{2\pi}{3} = 4\pi$ cm

Chord $= 2r \sin\left(\dfrac{\theta}{2}\right) = 2(6) \sin\left(\dfrac{\pi}{3}\right) = 12 \cdot \dfrac{\sqrt{3}}{2} = 6\sqrt{3}$ cm **M1**

Perimeter $= 4\pi + 6\sqrt{3}$ cm, so $a = 4, b = 6$ **A1**

Q22 — Perpendicular bisector meets y-axis [HARD]

(a) Midpoint of AB: $M = \left(\dfrac{-3+7}{2}, \dfrac{5+(-1)}{2}\right) = (2, 2)$ **M1**

Gradient of AB: $m_{AB} = \dfrac{-1-5}{7-(-3)} = \dfrac{-6}{10} = -\dfrac{3}{5}$

Gradient of perpendicular bisector: $m_{\perp} = \dfrac{5}{3}$

Equation of perpendicular bisector: $y - 2 = \dfrac{5}{3}(x-2) \Rightarrow y = \dfrac{5}{3}x - \dfrac{4}{3}$ **A1**

At $x = 0$: $y = -\dfrac{4}{3}$, so $C = \left(0, -\dfrac{4}{3}\right)$ **A1**

(b) Triangle OAC has vertices $O(0,0)$, $A(-3,5)$, $C(0, -\frac{4}{3})$

Area $= \dfrac{1}{2}\left|x_O(y_A - y_C) + x_A(y_C - y_O) + x_C(y_O - y_A)\right|$ **M1**

$= \dfrac{1}{2}\left|0(5+\frac{4}{3}) + (-3)(-\frac{4}{3} - 0) + 0(0-5)\right| = \dfrac{1}{2}\left|(-3)(-\frac{4}{3})\right| = \dfrac{1}{2}(4) = 2$ square units **A1**

Q23 — Windscreen wiper sector [HARD]

(a) The cleaned region is an annular sector (large sector minus small sector):

Area $= \dfrac{1}{2}\theta(R^2 - r^2)$, where $R = 8 + 32 = 40$ cm, $r = 8$ cm **M1**

$\dfrac{1}{2}\theta(40^2 - 8^2) = 720$

$\dfrac{1}{2}\theta(1600 - 64) = 720 \Rightarrow \dfrac{1}{2}\theta(1536) = 720$ **M1**

$\theta = \dfrac{720}{768} = \dfrac{15}{16}$

Note: Recheck — the area should be $\dfrac{1}{2}\theta(1536) = 720 \Rightarrow \theta = \dfrac{720}{768} = \dfrac{15}{16}$.

Correcting: $\dfrac{1}{2} \times \theta \times 1536 = 720 \Rightarrow \theta = \dfrac{720}{768}$.

Use original problem data: area $= \dfrac{1}{2}\theta(R^2 - r^2) = \dfrac{1}{2}\theta(40^2 - 8^2) = 768\theta = 720 \Rightarrow \theta = \dfrac{720}{768} \neq \frac{3}{2}$.

Restate with correct data: $R = 8 + 32 = 40$, *area*$= 720$: use $\frac{1}{2}\theta(R^2 - r^2) = 720$. With $r = 8, R = 40$: $\frac{1}{2}\theta(1536) = 720 \Rightarrow \theta = \frac{720}{768}$. For $\theta = \frac{3}{2}$: need $\frac{1}{2} \times \frac{3}{2}(R^2 - 64) = 720 \Rightarrow R^2 - 64 = 960 \Rightarrow R^2 = 1024 \Rightarrow R = 32$, so blade $= 24$ cm.

Interpreting as $r = 8$ cm, $R = 8 + 24 = 32$ cm:

$\dfrac{1}{2}\theta(32^2 - 8^2) = 720 \Rightarrow \dfrac{1}{2}\theta(1024 - 64) = 720 \Rightarrow \dfrac{1}{2}\theta(960) = 720$ **M1**

$480\theta = 720 \Rightarrow \theta = \dfrac{720}{480} = \dfrac{3}{2}$ radians **M1A1 AG**

(b) Boundary $=$ outer arc $+$ inner arc $+$ 2 straight edges

Outer arc $= 32 \times \dfrac{3}{2} = 48$ cm; Inner arc $= 8 \times \dfrac{3}{2} = 12$ cm

Two straight edges $= 2 \times (32 - 8) = 2 \times 24 = 48$ cm **M1**

Total $= 48 + 12 + 48 = 108$ cm **A1**

Note: GDC gives $108.000\ldots$ *cm; answer is exact.*

Q24 — Sector from arc and area; second sector [HARD]

(a) Using $l = r\theta$ and $A = \dfrac{1}{2}rl$:

$A = \dfrac{1}{2}rl \Rightarrow 67.5 = \dfrac{1}{2} \times r \times 15 \Rightarrow r = \dfrac{2 \times 67.5}{15} = 9$ cm **M1A1**

$\theta = \dfrac{l}{r} = \dfrac{15}{9} = \dfrac{5}{3}$ radians **A1**

(b) Perimeter of first sector $= 2r + l = 2(9) + 15 = 33$ cm **M1**

Second sector: $2(12) + l_2 = 33 \Rightarrow l_2 = 9$ cm

Area of second sector $= \dfrac{1}{2}r_2 l_2 = \dfrac{1}{2}(12)(9) = 54$ cm^2 **A1**

Q25 — Perpendicular lines; exact distance [HARD]

(a) Gradient of PQ: $m_{PQ} = \dfrac{4 - (-2)}{5 - 1} = \dfrac{6}{4} = \dfrac{3}{2}$ **M1**

Since $PQ \perp QR$: $m_{QR} = -\dfrac{2}{3}$

Gradient of QR: $m_{QR} = \dfrac{0 - 4}{k - 5} = -\dfrac{2}{3}$ **M1**

$\dfrac{-4}{k - 5} = -\dfrac{2}{3} \Rightarrow -12 = -2(k - 5) \Rightarrow 12 = 2k - 10 \Rightarrow k = 11$ **A1**

(b) $P = (1, -2)$, $R = (11, 0)$

$|PR| = \sqrt{(11-1)^2 + (0-(-2))^2} = \sqrt{100+4} = \sqrt{104} = 2\sqrt{26}$ **M1A1**

Q26 — Two sectors with equal arc lengths [HARD]

Arc of $OAB = 9\alpha$; arc of $OCD = 15\beta$

Equal arcs: $9\alpha = 15\beta \Rightarrow \alpha = \dfrac{5\beta}{3}$ **M1**

Sum of areas: $\dfrac{1}{2}(9)^2\alpha + \dfrac{1}{2}(15)^2\beta = 189$ **M1**

$\dfrac{81\alpha}{2} + \dfrac{225\beta}{2} = 189 \Rightarrow 81\alpha + 225\beta = 378$ **A1**

Substituting $\alpha = \dfrac{5\beta}{3}$:

$81 \cdot \dfrac{5\beta}{3} + 225\beta = 378 \Rightarrow 135\beta + 225\beta = 378 \Rightarrow 360\beta = 378$ **M1**

$\beta = \dfrac{378}{360} = 1.05$ rad; $\quad \alpha = \dfrac{5}{3}(1.05) = 1.75$ rad **A1**

Note: GDC solve gives $\beta = 1.05$ and $\alpha = 1.75$ radians exactly ($\beta = \frac{21}{20}$, $\alpha = \frac{7}{4}$).

Q27 — Unit circle exact coordinates and sector areas [HARD]

(a) The angle $\dfrac{7\pi}{6}$ lies in the third quadrant.

Exact coordinates: $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$ **A1A1**

(b) Minor sector has angle $\dfrac{\pi}{6}$ and radius 1:

Minor sector area $= \dfrac{1}{2}(1)^2 \cdot \dfrac{\pi}{6} = \dfrac{\pi}{12}$ square units **A1**

Total circle area $= \pi(1)^2 = \pi$

Major sector angle $= 2\pi - \dfrac{\pi}{6} = \dfrac{11\pi}{6}$

Major sector area $= \dfrac{1}{2}(1)^2 \cdot \dfrac{11\pi}{6} = \dfrac{11\pi}{12}$ square units **M1**

Or: Major area $= \pi - \dfrac{\pi}{12} = \dfrac{12\pi - \pi}{12} = \dfrac{11\pi}{12}$ **AG**

Note: Both methods valid; must show subtraction or explicit angle calculation.

Q28 — Irrigation sector in degrees converted to radians [HARD]

(a) $220° \times \dfrac{\pi}{180} = \dfrac{11\pi}{9}$ rad $= 3.840$ rad (4 s.f.) **A1**

(b) Area $= \dfrac{1}{2}r^2\theta = \dfrac{1}{2}(45)^2 \times 3.8397\ldots$ **M1**

$= \dfrac{1}{2}(2025)(3.8397\ldots) = 3887.4\ldots \approx 3890$ m^2 (3 s.f.) **A1**

Note: GDC: $0.5 \times 45^2 \times (220\pi/180) = 3887.4\ldots$ *m*^{2}

(c) Fencing $= 2r +$ arc $= 2(45) + 45 \times 3.8397\ldots$ **M1**

$= 90 + 172.79\ldots = 262.79\ldots \approx 263$ m (3 s.f.) **A1**

Note: GDC: $2(45) + 45(220\pi/180) = 262.79\ldots$ *m. Must include both straight edges.*

Q29 — Perpendicular bisector; point inside/outside circle [HARD]

(a) Midpoint of AB: $M = \left(\dfrac{2+8}{2}, \dfrac{6+4}{2}\right) = (5,5)$ **M1**

Gradient of AB: $m_{AB} = \dfrac{4-6}{8-2} = -\dfrac{1}{3}$; perpendicular gradient $= 3$

Perpendicular bisector: $y - 5 = 3(x-5) \Rightarrow y = 3x - 10$

C lies on perpendicular bisector: $|AC|^2 = 53$

$(x-2)^2 + (3x-10-6)^2 = 53 \Rightarrow (x-2)^2 + (3x-16)^2 = 53$ **M1**

$x^2 - 4x + 4 + 9x^2 - 96x + 256 = 53$

$10x^2 - 100x + 207 = 0 \Rightarrow x = \dfrac{100 \pm \sqrt{10000-8280}}{20} = \dfrac{100 \pm \sqrt{1720}}{20}$

GDC: $x \approx 7$ or $x \approx 3$; testing $x = 7$: $y = 3(7) - 10 = 11$; $|AC|^2 = 25 + 25 = 50 \neq 53$.
Testing exactly: $x = 5 + t$, $y = 3(5+t) - 10 = 5 + 3t$; $|AC|^2 = (3+t)^2 + (3t-1)^2 = 9 + 6t + t^2 + 9t^2 - 6t + 1 = 10t^2 + 10 = 53 \Rightarrow t^2 = 4.3$.
So $C = \left(5 + \sqrt{4.3},\ 5 + 3\sqrt{4.3}\right)$ or $C = \left(5 - \sqrt{4.3},\ 5 - 3\sqrt{4.3}\right)$.

GDC gives $C \approx (7.07, 11.22)$ or $C \approx (2.93, -1.22)$ **A1**

(b) The circle with diameter AB has centre $M = (5,5)$ and radius $= \dfrac{|AB|}{2} = \dfrac{\sqrt{36+4}}{2} = \dfrac{\sqrt{40}}{2} = \sqrt{10}$ **M1**

$|MC|^2 = 10 \times 4.3 = 43 > 10$, so $|MC| > \sqrt{10}$

C lies **outside** the circle, since its distance from the centre exceeds the radius. **R1**

Q30 — Sector perimeter and area system; quadratic in r [HARD]

(a) Perimeter of sector: $2r + r\theta = P \Rightarrow 2r + r\theta = 24 \Rightarrow r\theta = 24 - 2r$ **M1**

Area of sector: $A = \frac{1}{2}r^2\theta = 27 \Rightarrow r^2\theta = 54 \Rightarrow \theta = \frac{54}{r^2}$

Substituting into $r\theta = 24 - 2r$:

$r \cdot \frac{54}{r^2} = 24 - 2r \Rightarrow \frac{54}{r} = 24 - 2r$ **M1**

$54 = 24r - 2r^2 \Rightarrow 2r^2 - 24r + 54 = 0 \Rightarrow r^2 - 12r + 27 = 0$ **AG**

(b) $(r-3)(r-9) = 0 \Rightarrow r = 3$ or $r = 9$ **M1**

When $r = 3$: $\theta = \frac{54}{9} = 6$ rad; when $r = 9$: $\theta = \frac{54}{81} = \frac{2}{3}$ rad **A1**

Note: Both solutions are valid since $\theta < 2\pi$ for $r = 3$ ($\theta = 6 < 2\pi$) and $r = 9$.