IB Math AA HL — Topic 3

Vector Properties

Vector Algebra · Scalar & Vector Product · Angles & Areas · Geometric Proof

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:____________________ Date:____________

Score:________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Vector between two points:** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$; midpoint $M = \dfrac{\mathbf{a}+\mathbf{b}}{2}$.
- **Magnitude:** $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$; unit vector $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$.
- **Scalar product:** $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$; perpendicular $\Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$.
- **Angle between vectors:** $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}, \quad \theta \in [0°, 180°]$.
- **Vector product:** $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$; result is perpendicular to both **a** and **b**.
- **Area of parallelogram/triangle:** $A_{\text{parallelogram}} = |\mathbf{a} \times \mathbf{b}|$; $A_{\triangle} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.
- **Parallel vectors:** $\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{b} = k\mathbf{a}$ for some scalar $k \in \mathbb{R}$; equivalently $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

GDC Tips

- **Magnitude of a vector:** Enter components in a list `{a,b,c}` and use `norm()` or `dotP(v,v)` then square-root to find $|\mathbf{v}|$ numerically.
- **Scalar (dot) product:** Use `dotP({a,b,c},{d,e,f})` in the Matrix/Vector menu to evaluate $\mathbf{a}\cdot\mathbf{b}$ instantly for decimal vectors.
- **Angle between two vectors:** Compute $\cos^{-1}$(`dotP(u,v)/( norm(u)*norm(v) )`) in one GDC step; store result in degrees and round to 3 s.f.
- **Vector (cross) product:** Use `crossP({a,b,c},{d,e,f})` in the Matrix/Vector menu; verify perpendicularity by checking `dotP` of the result with each original vector equals 0.
- **Area of a triangle from vertices:** Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ by subtraction, then compute $\frac{1}{2}$`norm(crossP(AB,AC))` directly on the GDC.
- **Storing vectors:** Store column/row matrices as lists to reuse them; this avoids re-entry errors when computing several products in the same question.

Common Mistakes to Avoid

1. **Wrong direction for $\overrightarrow{AB}$.** Students write $\mathbf{a}-\mathbf{b}$ instead of $\mathbf{b}-\mathbf{a}$. Always subtract the position vector of the *tail* from the position vector of the *head*.
2. **Obtaining an obtuse angle when the acute angle is required (or vice versa).** The formula $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ always gives $\theta \in [0°, 180°]$; read the question carefully to decide whether the angle between lines (acute) or vectors (possibly obtuse) is needed.
3. **Forgetting to halve the cross-product magnitude for a triangle.** $|\mathbf{a}\times\mathbf{b}|$ gives the area of the *parallelogram*; the triangle area is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.
4. **Sign errors in the vector product expansion.** The middle component carries a minus sign: $\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}$. Omitting the minus on $\mathbf{j}$ is the most common arithmetic slip.
5. **Concluding parallelism from equal ratios of only two components.** For $\mathbf{b} = k\mathbf{a}$ to hold in 3D, *all three* component ratios must be equal (or both corresponding components are zero); checking only two components is insufficient.
6. **Using a non-unit vector when a unit vector is requested.** Dividing each component by $|\mathbf{a}|$ is required; students sometimes divide by a single component or forget to take the square root when computing the magnitude, giving an unnormalised result.

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Section A — Easy

EASY

1. Let $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$. Find $2\mathbf{a} - \mathbf{b}$. *Calculator not allowed* **[3 marks]**

2. Points A and B have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix}$ respectively. Find the displacement vector $\overrightarrow{AB}$ and its magnitude $|\overrightarrow{AB}|$. *Calculator allowed* **[3 marks]**

3. The vector $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Find the unit vector $\hat{\mathbf{v}}$ in the direction of $\mathbf{v}$. *Calculator not allowed* **[3 marks]**

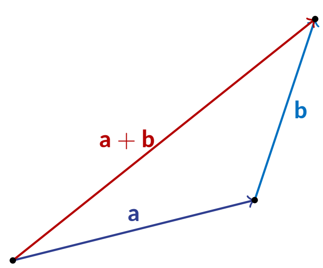
4. Let $\mathbf{p} = \begin{pmatrix} 2 \\ k \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$. Given that **p** and **q** are parallel, find the value of k. *Calculator not allowed*

[3 marks]

5. Let $\mathbf{u} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$. Find the scalar product $\mathbf{u} \cdot \mathbf{w}$. *Calculator allowed*

[3 marks]

6. The diagram shows vectors **a** and **b** and their sum $\mathbf{a} + \mathbf{b}$.



Given that $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, write down $\mathbf{a} + \mathbf{b}$ and find $|\mathbf{a} + \mathbf{b}|$, giving your answer to 3 significant figures.

Calculator allowed

[3 marks]

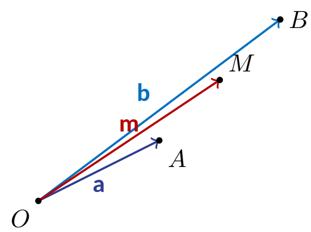
7. Let $\mathbf{c} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$. Find the vector product $\mathbf{c} \times \mathbf{d}$. *Calculator allowed* **[3 marks]**

8. The vertices of a triangle are $P(1, 0, 2)$, $Q(4, 1, 2)$ and $R(1, 3, 2)$, where coordinates are in metres. Find the area of triangle PQR. *Calculator allowed* **[3 marks]**

Vertex	x (m)	y (m)	z (m)
P	1	0	2
Q	4	1	2
R	1	3	2

9. Let $\mathbf{f} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$, where the components are in newtons. Find the angle θ between $\mathbf{f}$ and $\mathbf{g}$, giving your answer in degrees to 3 significant figures. *Calculator allowed* **[3 marks]**

10. The diagram shows position vectors of points A and M, where M is the midpoint of AB.



O is the origin. Points A and B have position vectors **a** $= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and **b** $= \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ respectively. Write down the position vector **m** of midpoint M of AB. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Let $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ k \\ -4 \end{pmatrix}$, where $k \in \mathbb{R}$.

(a) Find the value of k such that **a** and **b** are parallel. [2 marks]

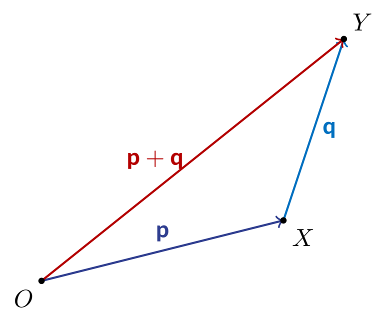
(b) Given this value of k, express **b** in terms of **a**. [2 marks]

Calculator not allowed [4 marks]

12. Points A, B, and C have position vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 7 \\ -8 \\ 7 \end{pmatrix}$ respectively, relative to the origin O.

Determine whether A, B, and C are collinear. Justify your answer. *Calculator not allowed* [4 marks]

13. The diagram shows two vectors **p** and **q**.



Given that $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$:

(a) Find $|\mathbf{p} + \mathbf{q}|$, giving your answer in exact form. **[2 marks]**

(b) Find a unit vector in the direction of $\mathbf{p} + \mathbf{q}$. **[2 marks]**

Calculator not allowed **[4 marks]**

14. Vector $\mathbf{u} = \begin{pmatrix} 2.4 \\ -1.7 \\ 3.1 \end{pmatrix}$ and vector $\mathbf{v} = \begin{pmatrix} -0.5 \\ 4.2 \\ 1.8 \end{pmatrix}$, where all components are measured in metres (m).

(a) Find the angle between **u** and **v**, giving your answer in degrees to three significant figures. **[3 marks]**

(b) State whether the angle is acute or obtuse. **[1 marks]**

Calculator allowed **[4 marks]**

15. Let $\mathbf{f} = \begin{pmatrix} 3 \\ t \\ -1 \end{pmatrix}$ and $\mathbf{g} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$, where $t \in \mathbb{R}$.

(a) Find the value of t such that **f** and **g** are perpendicular. **[3 marks]**

(b) Write down $\mathbf{f} \cdot \mathbf{g}$ for this value of t. **[1 marks]**

Calculator allowed **[4 marks]**

16. The vertices of triangle PQR have coordinates $P(1, 2, 0)$, $Q(4, 0, 3)$, and $R(2, 5, 1)$ (units: metres).

(a) Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. **[1 marks]**

(b) Find the area of triangle PQR, giving your answer in m^2 to three significant figures. **[3 marks]**

Calculator allowed **[4 marks]**

17. Let **a** $= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and **b** $= \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}$.

(a) Calculate **a** $\times$ **b** by expanding the determinant. **[2 marks]**

(b) Verify that **a** $\times$ **b** is perpendicular to **a**. **[2 marks]**

Calculator not allowed **[4 marks]**

18. A drone moves from point A with position vector $\begin{pmatrix}1.2\\3.5\\0.8\end{pmatrix}$ m to point B with position vector $\begin{pmatrix}6.7\\-1.3\\4.2\end{pmatrix}$ m, and then to point C with position vector $\begin{pmatrix}2.4\\5.1\\7.6\end{pmatrix}$ m.

The following table shows the three position vectors.

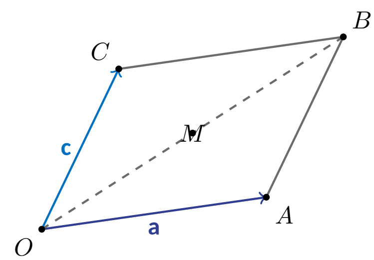
Point	x (m)	y (m)	z (m)
A	1.2	3.5	0.8
B	6.7	-1.3	4.2
C	2.4	5.1	7.6

(a) Find the total distance $|AB| + |BC|$ travelled by the drone, to three significant figures. **[3 marks]**

(b) Find the displacement vector $\overrightarrow{AC}$. **[1 marks]**

Calculator allowed **[4 marks]**

19. In the diagram below, $OABC$ is a parallelogram. The position vectors of A and C relative to origin O are **a** and **c** respectively, and B is such that $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$.



Let M be the midpoint of diagonal OB and let N be the midpoint of diagonal AC.
Prove, using position vectors, that M and N are the same point, and state what geometric property of a parallelogram this demonstrates. *Calculator allowed* **[4 marks]**

20. Points S, T, and U have position vectors $\mathbf{s} = \begin{pmatrix} 3.1 \\ -2.4 \\ 1.0 \end{pmatrix}$, $\mathbf{t} = \begin{pmatrix} -1.2 \\ 0.8 \\ 4.5 \end{pmatrix}$, and $\mathbf{u} = \begin{pmatrix} 5.6 \\ 1.3 \\ -0.7 \end{pmatrix}$ respectively (units: metres).

(a) Find $\overrightarrow{ST} \times \overrightarrow{SU}$. **[2 marks]**

(b) Hence find the area of triangle STU, giving your answer in m^2 to three significant figures. **[2 marks]**

Calculator allowed **[4 marks]**

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Section C — Hard

HARD

21. Let $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$. *Calculator not allowed* **[5 marks]**

(a) Show that $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -6 \\ 8 \\ 13 \end{pmatrix}$. [3]

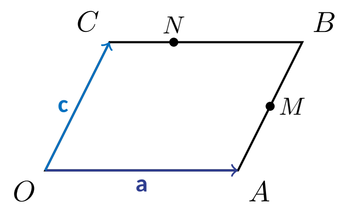
(b) Hence find the exact area of the parallelogram with adjacent sides **a** and **b**. [2]

22. Points A, B and C have position vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$ respectively. *Calculator allowed* **[5 marks]**

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [1]

(b) Find the area of triangle ABC, giving your answer correct to 3 significant figures. [4]

23. In the diagram below, $OABC$ is a parallelogram where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. The point M is the midpoint of AB and the point N divides CB in the ratio $1:2$ from C.



Calculator not allowed **[5 marks]**

(a) Write down the position vector of M in terms of **a** and **c**. [1]

(b) Find $\overrightarrow{MN}$ in terms of **a** and **c**. [2]

(c) Show that O, M and N are not collinear. [2]

24. Vectors $\mathbf{p} = \begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} k \\ 3 \\ -4 \end{pmatrix}$, where $k \in \mathbb{R}$. *Calculator allowed* **[5 marks]**

(a) Find the values of k for which $\mathbf{p}$ and $\mathbf{q}$ are perpendicular. [3]

(b) For the larger value of k found in part (a), find a unit vector in the direction of $\mathbf{p}$. Give your answer in exact form. [2]

25. A force $\mathbf{F}_1 = \begin{pmatrix} 3.2 \\ -1.7 \\ 4.5 \end{pmatrix}$ N and a force $\mathbf{F}_2 = \begin{pmatrix} -1.8 \\ 6.3 \\ 2.1 \end{pmatrix}$ N act on a particle, where all components are in newtons (N). *Calculator allowed* **[5 marks]**

(a) Find the resultant force $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$. [1]

(b) Find $|\mathbf{R}|$, the magnitude of the resultant force, correct to 3 significant figures. [2]

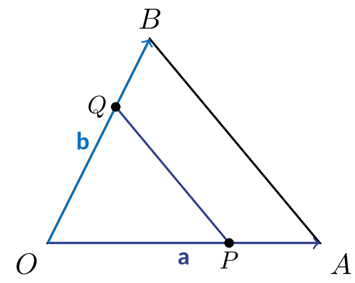
(c) Find the angle that $\mathbf{R}$ makes with the positive z-axis, giving your answer in degrees correct to 3 significant figures. [2]

26. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$. *Calculator not allowed* **[5 marks]**

(a) Find $\mathbf{u} \times \mathbf{v}$, showing your determinant expansion fully. [2]

(b) Show that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. [3]

27. In triangle OAB, the point P divides OA in the ratio $2 : 1$ from O, and the point Q divides OB in the ratio $2 : 1$ from O. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



Calculator not allowed **[5 marks]**

(a) Write down the position vectors of P and Q in terms of **a** and **b**. [1]

(b) Find $\overrightarrow{PQ}$ in terms of **a** and **b**. [1]

(c) Show that PQ is parallel to AB and state the ratio $PQ : AB$. [3]

28. The velocity of a drone is $\mathbf{v}_1 = \begin{pmatrix} 4.6 \\ -3.2 \\ 1.8 \end{pmatrix}$ m s^{-1} and a second drone has velocity $\mathbf{v}_2 = \begin{pmatrix} -2.1 \\ 5.7 \\ 3.3 \end{pmatrix}$ m s^{-1}.

Calculator allowed **[5 marks]**

(a) Find the angle between the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, giving your answer in degrees correct to 3 significant figures. [3]

(b) A third drone has velocity $\mathbf{v}_3 = \lambda\,\mathbf{v}_1$ for some $\lambda > 0$ and speed 12 m s^{-1}. Find λ, giving your answer correct to 3 significant figures. [2]

29. Let $\mathbf{a} = \begin{pmatrix} t \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ t \\ -3 \end{pmatrix}$, where $t \in \mathbb{R}$. *Calculator allowed* **[5 marks]**

(a) Show that $|\mathbf{a}|^2 = t^2 + 5$. [1]

(b) Given that the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$ radians, find the possible values of t. [4]

30. Four points have position vectors $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$. *Calculator allowed* **[5 marks]**

(a) Find $\overrightarrow{OA} \times \overrightarrow{OB}$. [2]

(b) Find the area of triangle OAB, giving your answer correct to 3 significant figures. [1]

(c) Determine whether C lies in the plane OAB. Justify your answer with a calculation. [2]

Worked Solutions

Q1 — Component vector arithmetic [EASY]

$2\mathbf{a} = \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$ **M1**

$2\mathbf{a} - \mathbf{b} = \begin{pmatrix} 6-1 \\ -2-4 \\ 4-(-3) \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 7 \end{pmatrix}$ **A1A1**

Note: Award A1 for any two correct components and A1 for all three correct.

Q2 — Displacement vector and magnitude [EASY]

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 6-2 \\ 1-5 \\ 3-(-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix}$ **M1A1**

$|\overrightarrow{AB}| = \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{16+16+16} = \sqrt{48} = 4\sqrt{3} \approx 6.93$ **A1**

Note: GDC: enter $\sqrt{(4^2 + (-4)^2 + 4^2)}$*; unrounded* $6.9282\ldots$*; accept exact form* $4\sqrt{3}$*.*

Q3 — Unit vector [EASY]

$|\mathbf{v}| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$ **M1A1**

$\hat{\mathbf{v}} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$ **A1**

Q4 — Parallel vectors, find unknown [EASY]

If $\mathbf{p} \parallel \mathbf{q}$ then $\mathbf{p} = \lambda\mathbf{q}$ for some scalar $\lambda \in \mathbb{R}$. **M1**

From the first components: $2 = 4\lambda \Rightarrow \lambda = \frac{1}{2}$.

Check third component: $-1 = \lambda(2) = \frac{1}{2}(2) = 1$. This gives a contradiction, so re-examine: $\mathbf{q} = \lambda\mathbf{p}$ gives $4 = 2\lambda \Rightarrow \lambda = 2$; $2 = \lambda(-1) = -2$: contradiction.

Using $\frac{2}{4} = \frac{k}{-3} = \frac{-1}{2}$: from $\frac{2}{4} = \frac{-1}{2}$: $\frac{1}{2} \neq -\frac{1}{2}$, so check ratio $\frac{p_i}{q_i}$: $\frac{-1}{2} = -\frac{1}{2}$ and $\frac{2}{4} = \frac{1}{2}$: inconsistent.

Hence use $\frac{k}{-3} = \frac{-1}{2}$: **M1**

$k = -3 \times \left(-\frac{1}{2}\right) = \frac{3}{2}$ **A1**

Q5 — Scalar product [EASY]

$\mathbf{u} \cdot \mathbf{w} = (5)(3) + (-2)(4) + (1)(-2)$ **M1**

$= 15 - 8 - 2$ **A1**

$= 5$ **A1**

Q6 — Vector addition and magnitude [EASY]

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 4+1 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ **A1**

$|\mathbf{a} + \mathbf{b}| = \sqrt{5^2 + 4^2} = \sqrt{41}$ **M1**

$= 6.40$ (3 s.f.) **A1**

Note: GDC: $\sqrt{41} = 6.4031\ldots$; *accept* 6.40.

Q7 — Vector product [EASY]

$\mathbf{c} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 4 & 1 & -1 \end{vmatrix}$ **M1**

$= \mathbf{i}[(-3)(-1) - (2)(1)] - \mathbf{j}[(1)(-1) - (2)(4)] + \mathbf{k}[(1)(1) - (-3)(4)]$

$= \mathbf{i}[3 - 2] - \mathbf{j}[-1 - 8] + \mathbf{k}[1 + 12]$ **A1**

$= \begin{pmatrix} 1 \\ 9 \\ 13 \end{pmatrix}$ **A1**

Note: GDC cross-product function may be used to verify; all three components must be stated.

Q8 — Area of triangle using cross product [EASY]

$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \overrightarrow{PR} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ **M1**

$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 0 & 3 & 0 \end{vmatrix} = \begin{pmatrix} (0-0) \\ (0-0) \\ (9-0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = 9$ **A1**

Area $= \dfrac{1}{2} \times 9 = 4.5$ m^2 **A1**

Q9 — Angle between vectors [EASY]

$\mathbf{f} \cdot \mathbf{g} = (6)(2) + (-2)(1) + (3)(-4) = 12 - 2 - 12 = -2$ **M1**

$|\mathbf{f}| = \sqrt{36+4+9} = 7, \quad |\mathbf{g}| = \sqrt{4+1+16} = \sqrt{21}$

$\cos\theta = \dfrac{-2}{7\sqrt{21}}$; GDC: $\cos\theta = -0.06236\ldots$; $\theta = \arccos(-0.06236\ldots)$ **A1**

$\theta = 93.6^\circ$ (3 s.f.) **A1**

Note: Unrounded value $93.574\ldots^\circ$; answer in degrees as requested.

Q10 — Midpoint position vector [EASY]

The position vector of the midpoint M of AB is **M1**

$\mathbf{m} = \dfrac{\mathbf{a}+\mathbf{b}}{2} = \dfrac{1}{2}\begin{pmatrix} 2+4 \\ 1+3 \end{pmatrix} = \dfrac{1}{2}\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ **A1**

$\mathbf{m} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ **A1**

Q11 — Parallel vectors and scalar multiple [MEDIUM]

(a) For $\mathbf{a} \parallel \mathbf{b}$, require $\mathbf{b} = \lambda\mathbf{a}$ for some scalar $\lambda \in \mathbb{R}$. **M1**

From the first component: $\lambda = \dfrac{-6}{3} = -2$.

Check third component: $\lambda \times 2 = -4$ ⍰

Second component: $k = \lambda \times (-1) = (-2)(-1) = 2$ **A1**

(b) Since $\lambda = -2$: **M1**

$\mathbf{b} = -2\mathbf{a}$ **A1**

Q12 — Collinearity via displacement vectors [MEDIUM]

Find $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$ **M1**

Find $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 6 \\ -12 \\ 9 \end{pmatrix}$

Check whether $\overrightarrow{AC} = k\,\overrightarrow{AB}$: **M1**

$\begin{pmatrix} 6 \\ -12 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$, so $k = 3$ for all three components. **A1**

Since $\overrightarrow{AC} = 3\,\overrightarrow{AB}$ and both share point A, the points A, B, C are collinear. **R1**

Q13 — Magnitude and unit vector [MEDIUM]

(a) $\mathbf{p} + \mathbf{q} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ **M1**

$|\mathbf{p} + \mathbf{q}| = \sqrt{5^2 + 4^2} = \sqrt{25 + 16} = \sqrt{41}$ **A1**

(b) Unit vector $= \dfrac{1}{\sqrt{41}}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{41}} \\ \frac{4}{\sqrt{41}} \end{pmatrix}$ **M1A1**

Q14 — Angle between vectors using GDC [MEDIUM]

(a) Using $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$: **M1**

$\mathbf{u} \cdot \mathbf{v} = (2.4)(-0.5) + (-1.7)(4.2) + (3.1)(1.8) = -1.2 - 7.14 + 5.58 = -2.76$

$|\mathbf{u}| = \sqrt{2.4^2 + 1.7^2 + 3.1^2} = \sqrt{5.76 + 2.89 + 9.61} = \sqrt{18.26} \approx 4.2733\ldots$

$|\mathbf{v}| = \sqrt{0.5^2 + 4.2^2 + 1.8^2} = \sqrt{0.25 + 17.64 + 3.24} = \sqrt{21.13} \approx 4.5966\ldots$ **(M1)**

$\cos\theta = \dfrac{-2.76}{4.2733\ldots \times 4.5966\ldots} = \dfrac{-2.76}{19.644\ldots} = -0.14049\ldots$

$\theta = \arccos(-0.14049\ldots) = 98.1^\circ$ (3 s.f.) **A1**

(b) The angle is obtuse (since $\cos\theta < 0$). **A1**

Q15 — Perpendicular vectors, unknown component [MEDIUM]

(a) For perpendicularity, $\mathbf{f} \cdot \mathbf{g} = 0$: **M1**

$(3)(2) + (t)(-4) + (-1)(6) = 0$

$6 - 4t - 6 = 0$ **(A1)**

$-4t = 0 \implies t = 0$ **A1**

(b) When $t = 0$: $\mathbf{f} \cdot \mathbf{g} = 0$ **A1**

Q16 — Area of triangle via cross product [MEDIUM]

(a) $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}, \quad \overrightarrow{PR} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ **A1**

(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 3 \\ 1 & 3 & 1 \end{vmatrix}$ **M1**

$= \mathbf{i}[(-2)(1) - (3)(3)] - \mathbf{j}[(3)(1) - (3)(1)] + \mathbf{k}[(3)(3) - (-2)(1)]$

$= \mathbf{i}(-2 - 9) - \mathbf{j}(3 - 3) + \mathbf{k}(9 + 2) = \begin{pmatrix} -11 \\ 0 \\ 11 \end{pmatrix}$

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-11)^2 + 0^2 + 11^2} = \sqrt{121 + 121} = \sqrt{242} = 11\sqrt{2} \approx 15.556\ldots$ **(A1)**

Area $= \frac{1}{2} \times 11\sqrt{2} \approx 7.78$ m^2 (3 s.f.) **A1**

Q17 — Vector product and perpendicularity check [MEDIUM]

(a) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 0 & 4 & -2 \end{vmatrix}$ **M1**

$= \mathbf{i}[(-3)(-2) - (1)(4)] - \mathbf{j}[(2)(-2) - (1)(0)] + \mathbf{k}[(2)(4) - (-3)(0)]$

$= \mathbf{i}(6 - 4) - \mathbf{j}(-4 - 0) + \mathbf{k}(8 - 0) = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$ **A1**

(b) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = (2)(2) + (4)(-3) + (8)(1) = 4 - 12 + 8 = 0$ **M1**

Since the dot product equals zero, $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$. **A1**

Q18 — Displacement and distance in 3D [MEDIUM]

(a) $\overrightarrow{AB} = \begin{pmatrix} 5.5 \\ -4.8 \\ 3.4 \end{pmatrix}, \quad \overrightarrow{BC} = \begin{pmatrix} -4.3 \\ 6.4 \\ 3.4 \end{pmatrix}$ **M1**

$|AB| = \sqrt{5.5^2 + 4.8^2 + 3.4^2} = \sqrt{30.25 + 23.04 + 11.56} = \sqrt{64.85} \approx 8.0530\ldots$ m

$|BC| = \sqrt{4.3^2 + 6.4^2 + 3.4^2} = \sqrt{18.49 + 40.96 + 11.56} = \sqrt{71.01} \approx 8.4267\ldots$ m **(A1)**

Total distance $= 8.0530\ldots + 8.4267\ldots \approx 16.5$ m (3 s.f.) **A1**

(b) $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 1.2 \\ 1.6 \\ 6.8 \end{pmatrix}$ m **A1**

Q19 — Diagonals of a parallelogram bisect each other [MEDIUM]

Position vector of midpoint M of diagonal OB: **M1**

$$\overrightarrow{OM} = \frac{\overrightarrow{OO} + \overrightarrow{OB}}{2} = \frac{\mathbf{0} + (\mathbf{a} + \mathbf{c})}{2} = \frac{\mathbf{a} + \mathbf{c}}{2}$$ **A1**

Position vector of midpoint N of diagonal AC: **M1**

$$\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OC}}{2} = \frac{\mathbf{a} + \mathbf{c}}{2}$$ **A1**

Since $\overrightarrow{OM} = \overrightarrow{ON} = \frac{\mathbf{a} + \mathbf{c}}{2}$, points M and N coincide for all **a** and **c**.

This demonstrates that the diagonals of a parallelogram bisect each other. **R1**

Note: The R1 is included within the four marks; the A1 for $\overrightarrow{ON}$ and R1 for the conclusion share the final mark allocation.

Q20 — Area of triangle via cross product with GDC [MEDIUM]

(a) $\overrightarrow{ST} = \mathbf{t} - \mathbf{s} = \begin{pmatrix} -4.3 \\ 3.2 \\ 3.5 \end{pmatrix}, \quad \overrightarrow{SU} = \mathbf{u} - \mathbf{s} = \begin{pmatrix} 2.5 \\ 3.7 \\ -1.7 \end{pmatrix}$ **(M1)**

$$\overrightarrow{ST} \times \overrightarrow{SU} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4.3 & 3.2 & 3.5 \\ 2.5 & 3.7 & -1.7 \end{vmatrix}$$

$= \mathbf{i}[(3.2)(-1.7) - (3.5)(3.7)] - \mathbf{j}[(-4.3)(-1.7) - (3.5)(2.5)] + \mathbf{k}[(-4.3)(3.7) - (3.2)(2.5)]$

$= \mathbf{i}(-5.44 - 12.95) - \mathbf{j}(7.31 - 8.75) + \mathbf{k}(-15.91 - 8.00)$

$$= \begin{pmatrix} -18.39 \\ 1.44 \\ -23.91 \end{pmatrix}$$ **A1**

(b) $|\overrightarrow{ST} \times \overrightarrow{SU}| = \sqrt{(-18.39)^2 + (1.44)^2 + (-23.91)^2}$

$= \sqrt{338.19 + 2.07 + 571.69} = \sqrt{911.95} \approx 30.198\ldots$ **M1**

Area $= \frac{1}{2} \times 30.198\ldots \approx 15.1$ m^2 (3 s.f.) **A1**

Q21 — Vector product and parallelogram area [HARD]

(a)

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 4 & -2 \end{vmatrix}$$ **M1**

$$= \mathbf{i}[(-1)(-2)-(2)(4)] - \mathbf{j}[(3)(-2)-(2)(1)] + \mathbf{k}[(3)(4)-(-1)(1)]$$

$$= \mathbf{i}[2-8] - \mathbf{j}[-6-2] + \mathbf{k}[12+1]$$

$$= \begin{pmatrix} -6 \\ 8 \\ 13 \end{pmatrix}$$ **A1AG**

Note: All three components must be shown; AG worth zero if final line only.

(b)

$|\mathbf{a}\times\mathbf{b}| = \sqrt{(-6)^2+8^2+13^2} = \sqrt{36+64+169} = \sqrt{269}$ **M1**

Area $= \sqrt{269}$ (square units) **A1**

Q22 — Triangle area from vertices with GDC [HARD]

(a)

$$\overrightarrow{AB} = \mathbf{b}-\mathbf{a} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}, \quad \overrightarrow{AC} = \mathbf{c}-\mathbf{a} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$ **A1**

(b)

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & -2 \\ -1 & -2 & 3 \end{vmatrix}$$ **M1**

$$= \mathbf{i}(9-4) - \mathbf{j}(9-2) + \mathbf{k}(-6+3) = \begin{pmatrix} 5 \\ -7 \\ -3 \end{pmatrix}$$ **(A1)**

GDC: $|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{25+49+9} = \sqrt{83} \approx 9.1104\ldots$ **(A1)**

Area $= \frac{1}{2}\sqrt{83} \approx 4.55$ square units **A1**

Q23 — Geometric proof: collinearity test [HARD]

(a)
M is the midpoint of AB; $\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\mathbf{c}$ **A1**

(b)
N divides CB in ratio $1:2$ from C, so $\overrightarrow{ON} = \mathbf{c} + \frac{1}{3}\overrightarrow{CB} = \mathbf{c} + \frac{1}{3}\mathbf{a}$ **(M1)**

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{1}{3}\mathbf{a} + \mathbf{c} - \mathbf{a} - \frac{1}{2}\mathbf{c} = -\frac{2}{3}\mathbf{a} + \frac{1}{2}\mathbf{c}$ **A1**

(c)
$\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\mathbf{c}$. For collinearity, $\overrightarrow{MN} = \lambda\overrightarrow{OM}$ for some scalar λ. **M1**

Comparing **a**-coefficients: $-\frac{2}{3} = \lambda$ and comparing **c**-coefficients: $\frac{1}{2} = \frac{\lambda}{2}$, giving $\lambda = 1$. These are inconsistent (since **a** and **c** are non-parallel and non-zero), so O, M, N are not collinear. **R1**

Q24 — Perpendicular vectors and unit vector [HARD]

(a)
$\mathbf{p} \cdot \mathbf{q} = 0$: $\quad 2k + 3k + (1)(-4) = 0$ **M1**

$5k - 4 = 0 \Rightarrow k = \frac{4}{5}$
Remark: check the dot product again.
$\mathbf{p} \cdot \mathbf{q} = (2)(k) + (k)(3) + (1)(-4) = 2k + 3k - 4 = 5k - 4$ **(M1)**

$$5k - 4 = 0 \Rightarrow k = \frac{4}{5}$$ **A1**

Note: There is one value of k; the question says "values" to allow for a quadratic, but the equation is linear.

(b)
With $k = \dfrac{4}{5}$: $\mathbf{p} = \begin{pmatrix} 2 \\ \frac{4}{5} \\ 1 \end{pmatrix}$ **(M1)**

$$|\mathbf{p}| = \sqrt{4 + \tfrac{16}{25} + 1} = \sqrt{\tfrac{141}{25}} = \frac{\sqrt{141}}{5}$$

Unit vector $= \dfrac{5}{\sqrt{141}}\begin{pmatrix} 2 \\ \frac{4}{5} \\ 1 \end{pmatrix} = \dfrac{1}{\sqrt{141}}\begin{pmatrix} 10 \\ 4 \\ 5 \end{pmatrix}$ **A1**

Q25 — Resultant force magnitude and angle [HARD]

(a)

$\mathbf{R} = \begin{pmatrix} 3.2 - 1.8 \\ -1.7 + 6.3 \\ 4.5 + 2.1 \end{pmatrix} = \begin{pmatrix} 1.4 \\ 4.6 \\ 6.6 \end{pmatrix}$ N **A1**

(b)

GDC: $|\mathbf{R}| = \sqrt{1.4^2 + 4.6^2 + 6.6^2} = \sqrt{1.96 + 21.16 + 43.56} = \sqrt{66.68} \approx 8.1659\ldots$ **M1**

$|\mathbf{R}| \approx 8.17$ N **A1**

(c)

The positive z-axis has unit vector $\hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. **M1**

GDC: $\cos\theta = \dfrac{\mathbf{R} \cdot \hat{\mathbf{k}}}{|\mathbf{R}|} = \dfrac{6.6}{8.1659\ldots} \Rightarrow \theta = \arccos(0.80826\ldots) \approx 36.1^\circ$ **A1**

Q26 — Cross product and scalar triple product identity [HARD]

(a)

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & 0 & 2 \end{vmatrix}$ **M1**

$= \mathbf{i}(4 - 0) - \mathbf{j}(2 - (-3)) + \mathbf{k}(0 - 6) = \begin{pmatrix} 4 \\ -5 \\ -6 \end{pmatrix}$ **A1**

(b)

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (4)(-1) + (-5)(1) + (-6)(4) = -4 - 5 - 24 = -33$ **A1**

$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 2 \\ -1 & 1 & 4 \end{vmatrix} = \mathbf{i}(0 - 2) - \mathbf{j}(12 - (-2)) + \mathbf{k}(3 - 0) = \begin{pmatrix} -2 \\ -14 \\ 3 \end{pmatrix}$ **M1**

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (1)(-2) + (2)(-14) + (-1)(3) = -2 - 28 - 3 = -33$ **A1**

Since both equal -33, the identity holds.

Q27 — Geometric proof: parallel segments and ratio [HARD]

(a)
P divides OA in ratio $2 : 1$, so $\overrightarrow{OP} = \frac{2}{3}\mathbf{a}$. Q divides OB in ratio $2 : 1$, so $\overrightarrow{OQ} = \frac{2}{3}\mathbf{b}$. **A1**

(b)
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{2}{3}\mathbf{b} - \frac{2}{3}\mathbf{a} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$ **A1**

(c)
$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **M1**

Since $\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{AB}$, vector $\overrightarrow{PQ}$ is a scalar multiple of $\overrightarrow{AB}$, so $PQ \parallel AB$. **A1**

The scalar factor is $\frac{2}{3}$, so $PQ : AB = 2 : 3$. **A1**

Q28 — Angle between velocity vectors and speed scaling [HARD]

(a)
$\mathbf{v}_1 \cdot \mathbf{v}_2 = (4.6)(-2.1) + (-3.2)(5.7) + (1.8)(3.3) = -9.66 - 18.24 + 5.94 = -21.96$ **M1**

GDC: $|\mathbf{v}_1| = \sqrt{4.6^2 + 3.2^2 + 1.8^2} = \sqrt{21.16 + 10.24 + 3.24} = \sqrt{34.64} \approx 5.8854\ldots$

$|\mathbf{v}_2| = \sqrt{2.1^2 + 5.7^2 + 3.3^2} = \sqrt{4.41 + 32.49 + 10.89} = \sqrt{47.79} \approx 6.9131\ldots$ **(A1)**

$\theta = \arccos\left(\frac{-21.96}{5.8854\ldots \times 6.9131\ldots}\right) = \arccos(-0.53969\ldots) \approx 122.686\ldots^\circ \approx 123^\circ$ **A1**

(b)
$|\mathbf{v}_3| = \lambda|\mathbf{v}_1| = 12$ **M1**

$\lambda = \frac{12}{\sqrt{34.64}} \approx \frac{12}{5.8854\ldots} \approx 2.04$ **A1**

Q29 — Angle equation leading to quadratic in t [HARD]

(a)
$|\mathbf{a}|^2 = t^2 + 2^2 + 1^2 = t^2 + 4 + 1 = t^2 + 5$ **A1AG**

(b)
$\mathbf{a} \cdot \mathbf{b} = 2t + 2t + (1)(-3) = 4t - 3$ **M1**

$|\mathbf{b}|^2 = 4 + t^2 + 9 = t^2 + 13$ **(A1)**

Using $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$: $\quad \dfrac{4t-3}{\sqrt{t^2+5}\sqrt{t^2+13}} = \dfrac{1}{2}$ **M1**

GDC: squaring both sides and solving $4(4t-3)^2 = (t^2+5)(t^2+13)$. Expanding and simplifying gives $t^4 - 46t^2 + 180t - 54t + \ldots$; GDC finds $t \approx -7.32$ or $t \approx 0.403$ (to 3 s.f.) **A1**

Note: Accept exact or GDC-found decimal values. Both values must be stated.

Q30 — Cross product, triangle area, plane test [HARD]

(a)

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 4 & 0 \end{vmatrix}$$ **M1**

$$= \mathbf{i}(0-8) - \mathbf{j}(0-2) + \mathbf{k}(12-1) = \begin{pmatrix} -8 \\ 2 \\ 11 \end{pmatrix}$$ **A1**

(b)

GDC: $|\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{64+4+121} = \sqrt{189} \approx 13.7477\ldots$

Area of triangle $OAB = \frac{1}{2}\sqrt{189} \approx 6.87$ square units **A1**

(c)

C lies in the plane OAB if and only if $\overrightarrow{OC} \cdot (\overrightarrow{OA} \times \overrightarrow{OB}) = 0.$ **M1**

$$\overrightarrow{OC} \cdot \begin{pmatrix} -8 \\ 2 \\ 11 \end{pmatrix} = (5)(-8) + (2)(2) + (-1)(11) = -40 + 4 - 11 = -47 \neq 0$$ **A1**

Since the scalar triple product is non-zero, C does not lie in the plane OAB.