IB Math AA HL — Topic 4
Discrete Random Variables

Discrete Probability Distributions · Mean & Variance · Transformation of a Variable

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:_____ Date:_____

Score:_____/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

Sum to 1: $\sum_{\text{all } x} P(X = x) = 1$

Expected value: $\mathrm{E}(X) = \mu = \sum_{\text{all } x} x \cdot P(X = x)$

Second moment: $\mathrm{E}(X^2) = \sum_{\text{all } x} x^2 \cdot P(X = x)$

Variance: $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

Standard deviation: $\sigma = \sqrt{\mathrm{Var}(X)}$

Linear transformation — mean: $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

Linear transformation — variance: $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$

GDC Tips

- **1-Var Stats** (TI: STAT → CALC → 1-Var Stats; Casio: STAT → 1-VAR): enter x-values in one list and $P(X = x)$ values as frequencies in a second list to read off $\bar{x} = \mathrm{E}(X)$ and $\sigma_x = \sqrt{\mathrm{Var}(X)}$ directly.
- **List arithmetic for** $\mathrm{E}(X^2)$**:** store x-values in L1, probabilities in L2, then compute $\mathrm{sum}(\mathrm{L1}^2 * \mathrm{L2})$ to obtain $\mathrm{E}(X^2)$ without a separate table.
- **Verify sum-to-1:** use $\mathrm{sum}(\mathrm{L2})$ to confirm probabilities sum to 1 before proceeding; a result $\neq 1$ signals an entry error.
- **Solving for unknowns:** enter the sum-to-1 equation and any $\mathrm{E}(X)$ condition into the equation solver (ALPHA → SOLVE on Casio or PlySmlt2 on TI) when two unknowns are present.
- **Cumulative probabilities** $P(X \leq k)$: compute $\mathrm{sum}(\mathrm{L2}, 1, k)$ (selecting the first k entries of the proba-

bility list) rather than adding terms by hand.

Common Mistakes to Avoid

1. **Forgetting to sum to 1 first.** When a table contains an unknown k, the very first line of working must be $\sum P(X = x) = 1$. Jumping straight to $\mathrm{E}(X)$ with an unsolved k produces a nonsensical answer.
2. **Confusing $\mathrm{E}(X^2)$ with $[\mathrm{E}(X)]^2$.** $\mathrm{E}(X^2) = \sum x^2 P(X = x)$ is computed by squaring each x-value before multiplying by the probability. A common error is to compute $\mathrm{E}(X)$ first and then square it, giving $[\mathrm{E}(X)]^2$ instead, which always underestimates the variance.
3. **Adding b to the variance in a transformation.** $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$; the constant b shifts the distribution but does not change its spread. Writing $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) + b$ or $+b^2$ is incorrect.
4. **Forgetting to square a in the variance formula.** Students correctly apply $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ and then write $\mathrm{Var}(aX + b) = a\,\mathrm{Var}(X)$, omitting the square. The scaling of variance is always a^2, not a.
5. **Applying the transformation formula to σ inconsistently.** If $\mathrm{Var}(X)$ is scaled by a^2, then $\sigma_{aX+b} = |a|\,\sigma_X$, not $a^2\sigma_X$. Mixing the variance formula with the standard deviation result produces a dimensionally wrong answer.
6. **Omitting a concluding sentence in fair-game and expected-winnings questions.** A numeric value of $\mathrm{E}(X)$ alone does not earn the reasoning mark. The answer must state explicitly, for example, "Since $\mathrm{E}(X) = -0.20 < 0$, the game is not fair and the player should not play."

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY No Calculator [3 marks]

The discrete random variable X has the probability distribution shown in the table below.

x	1	2	3	4
$P(X = x)$	0.2	k	0.35	0.15

Find the value of k.

2. EASY No Calculator [3 marks]

The discrete random variable Y has the following probability distribution, where a is a constant.

y	0	1	2
$P(Y = y)$	0.4	a	0.1

(a) Write down the value of a. [1]

(b) Find $\mathrm{E}(Y)$. [2]

3. EASY No Calculator [3 marks]

A discrete random variable X takes values 1, 2, and 3 with probabilities $\frac{1}{6}$, $\frac{1}{2}$, and $\frac{1}{3}$ respectively. Find $\mathrm{E}(X)$.

4. **EASY** **Calculator** **[3 marks]**

A spinner is numbered 1, 2, 3, 4. The probability of each outcome is given in the table below.

x	1	2	3	4
$P(X = x)$	0.1	0.4	0.3	0.2

Find $\mathrm{E}(X^2)$.

5. **EASY** **No Calculator** **[3 marks]**

The discrete random variable X has mean $\mathrm{E}(X) = 3$ and variance $\mathrm{Var}(X) = 4$. Let $W = 2X + 1$.

(a) Find $\mathrm{E}(W)$. [1]

(b) Find $\mathrm{Var}(W)$. [2]

6. **EASY** **Calculator** **[3 marks]**

The discrete random variable X has the probability distribution given in the table below.

x	0	1	2
$P(X = x)$	0.3	0.5	0.2

Find $\mathrm{Var}(X)$.

7. EASY Calculator [3 marks]

The discrete random variable X has mean $\mathrm{E}(X) = 5$ and variance $\mathrm{Var}(X) = 9$. Let $T = 3X - 2$.

(a) Find $\mathrm{E}(T)$. [1]

(b) Find $\mathrm{Var}(T)$. [2]

8. EASY [3 marks]

A fair six-sided die, numbered 1 to 6, is rolled once. The discrete random variable X represents the score obtained.

(a) Write down $P(X = 4)$. No Calculator [1]

(b) A player wins \$3 if the score is 6, and loses \$1 otherwise. Find the expected winnings per roll. Calculator [2]

9. EASY Calculator [3 marks]

A discrete random variable X takes values 2, 4, 6 with probabilities k, $2k$, $3k$ respectively, where k is a positive constant.

(a) Find the value of k. [1]

(b) Find $\mathrm{E}(X)$. [2]

10. **EASY** **Calculator** **[3 marks]**

The discrete random variable X has the probability distribution shown below.

x	-1	0	1	2
$P(X = x)$	0.1	0.3	0.4	0.2

Find the variance of X.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator [4 marks]

The discrete random variable X has the probability distribution shown in the table below.

x	1	2	3	4
$P(X = x)$	$3k$	k	$2k$	k

(a) Find the value of k.

(b) Find $\mathrm{E}(X)$.

12. MEDIUM No Calculator [4 marks]

The discrete random variable Y has the following probability distribution, where a and b are constants.

y	0	1	2
$P(Y = y)$	a	0.3	b

It is given that $\mathrm{E}(Y) = 1.1$.

(a) Write down two equations satisfied by a and b.

(b) Find the values of a and b.

13. MEDIUM No Calculator [4 marks]

The discrete random variable X has mean $\mathrm{E}(X) = 2.4$ and variance $\mathrm{Var}(X) = 0.64$. The random variable W is defined by $W = 5X - 3$.

(a) Find $\mathrm{E}(W)$.

(b) Find $\mathrm{Var}(W)$.

14. MEDIUM Calculator [4 marks]

A game at a school fair costs \$2 to play. A player rolls a fair six-sided die. If the result is a 6, the player wins \$8. If the result is a 1 or 2, the player wins \$3. Otherwise, the player wins nothing. Let X be the player's net gain in dollars.

(a) Copy and complete the probability distribution table for X.

(b) Find $\mathrm{E}(X)$ and hence determine whether the game is fair.

15. MEDIUM Calculator [4 marks]

The discrete random variable X has the probability distribution shown below.

x	1	2	3	4
$P(X = x)$	0.1	0.4	0.3	0.2

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) The random variable T is defined by $T = 3X + 1$. Find $\mathrm{Var}(T)$.

16. MEDIUM Calculator [4 marks]

Two fair four-sided dice, each with faces labelled 1, 2, 3 and 4, are rolled. The discrete random variable M denotes the minimum of the two results.

(a) Show that $P(M = 1) = \frac{7}{16}$.

(b) Find $\mathrm{E}(M)$.

17. MEDIUM Calculator [4 marks]

A bag contains 4 red and 2 blue counters. Two counters are drawn at random without replacement. The discrete random variable X denotes the number of blue counters drawn.

(a) Find the probability distribution of X.

(b) Find $\mathrm{E}(X)$.

18. MEDIUM Calculator [4 marks]

The discrete random variable X has the probability distribution shown below.

x	2	4	6	8
$P(X = x)$	0.15	0.35	0.30	0.20

(a) Find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

(b) Hence find $\mathrm{Var}(X)$.

19. MEDIUM No Calculator **[4 marks]**

The discrete random variable X has mean $\mathrm{E}(X) = 3$ and $\mathrm{E}(X^2) = 11$. The random variable V is defined by $V = 4 - 2X$.

(a) Find $\mathrm{Var}(X)$.

(b) Find $\mathrm{E}(V)$ and $\mathrm{Var}(V)$.

20. MEDIUM Calculator **[4 marks]**

The discrete random variable X has the probability distribution shown below, where p is a constant.

x	-1	0	1	2	3
$P(X = x)$	0.1	p	0.25	0.15	0.2

(a) Find the value of p.

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD No Calculator [5 marks]

The discrete random variable X has the probability distribution shown in the table below, where a and b are constants.

x	1	2	3	4
$P(X = x)$	a	$2a$	b	$\frac{1}{6}$

Given that $\mathrm{E}(X) = \frac{5}{2}$, find the values of a and b, and hence find $\mathrm{Var}(X)$.

22. HARD Calculator [5 marks]

A game costs \$c to play. A player rolls two fair six-sided dice. Let X be the minimum of the two scores. The player wins $\$(2X + 1)$ dollars.

Show that $\mathrm{E}(X) = \frac{91}{36}$, and hence find the value of c for which the game is fair. Give your answer correct to 2 decimal places.

23. HARD No Calculator **[5 marks]**

The discrete random variable Y has mean μ and variance σ^2. A new variable is defined by $W = 5 - 3Y$.

(a) Show that $\text{E}(W) = 5 - 3\mu$. [1]

(b) Show that $\text{Var}(W) = 9\sigma^2$. [1]

(c) Given that $\text{E}(W) = -4$ and $\text{Var}(W) = 36$, find μ and σ^2. [3]

24. HARD Calculator **[5 marks]**

A bag contains 4 red and 2 blue tokens. Two tokens are drawn at random without replacement. The discrete random variable X denotes the number of blue tokens drawn.

(a) Construct the probability distribution table for X. [3]

(b) Find $\text{E}(X)$ and $\text{Var}(X)$. [2]

25. **HARD** **Calculator** **[5 marks]**

The discrete random variable X has the probability distribution shown below, where k is a positive constant.

x	0	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$2k$	k

(a) Find the value of k. [1]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

(c) A new variable is defined by $W = 4X - 3$. Find $\mathrm{E}(W)$ and $\mathrm{Var}(W)$. [2]

26. **HARD** **No Calculator** **[5 marks]**

A discrete random variable X takes values 1, 2, 3, 4 with $P(X = x) = \frac{cx}{10}$ for each $x \in \{1, 2, 3, 4\}$, where c is a constant.

(a) Show that $c = 1$. [1]

(b) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = 1$. [4]

27. **HARD** **Calculator** **[5 marks]**

In a carnival game, a player pays \$3 to spin a wheel. The wheel lands on one of five equally likely sectors labelled 0, 1, 2, 5, 10. The player wins the number of dollars shown. Let X be the player's net gain (winnings minus cost).

(a) Write down the probability distribution of X. [2]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

(c) Hence determine, with justification, whether a player should choose to play the game. [1]

28. **HARD** **Calculator** **[5 marks]**

Two independent discrete random variables X and Y each take only the values 1 and 2. It is known that $P(X = 1) = \frac{1}{3}$, $P(Y = 1) = \frac{1}{4}$. Let $Z = X + Y$.

(a) Construct the probability distribution table for Z. [3]

(b) Find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$. [2]

29. **HARD** **Calculator** **[5 marks]**

The discrete random variable X has the following probability distribution, where p and q are positive constants.

x	-1	0	2	3
$P(X = x)$	p	0.2	q	0.1

Given that $\mathrm{E}(X) = 0.8$, find p and q, and hence find $\mathrm{Var}(X)$.

30. **HARD** **No Calculator** **[5 marks]**

A discrete random variable X has $\mathrm{E}(X) = 3$ and $\mathrm{Var}(X) = 4$.

(a) Find $\mathrm{E}(X^2)$. [1]

(b) A variable is defined by $W = aX + b$ where $a > 0$. Given that $\mathrm{E}(W) = 10$ and $\mathrm{Var}(W) = 100$, find the values of a and b. [4]

Mark Schemes

Q1 — Finding k from Sum-to-One EASY

All probabilities sum to 1: **M1**

$0.2 + k + 0.35 + 0.15 = 1$

$k = 1 - 0.7 = 0.3$ **A1**

Since $0 \leq k \leq 1$, the value is valid. **A1**

Note: Award M1 for writing the sum-to-one equation; first A1 for correct simplification; second A1 for $k = 0.3$.

Q2 — Sum to One and E(Y) EASY

(a) $0.4 + a + 0.1 = 1 \implies a = 0.5$ **A1**

(b) $\mathrm{E}(Y) = 0(0.4) + 1(0.5) + 2(0.1)$ **M1**

$\mathrm{E}(Y) = 0 + 0.5 + 0.2 = 0.7$ **A1**

Note: In (b), follow through on their value of a for the M1.

Q3 — E(X) from Exact Fractions EASY

$\mathrm{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{2} + 3 \times \frac{1}{3}$ **M1**

$= \frac{1}{6} + 1 + 1$ **A1**

$= 13/6$ **A1**

Note: Award M1 for correctly setting up $\sum x \cdot P(X = x)$; accept equivalent exact or decimal forms for final A1.

Q4 — Finding E(X²) EASY

$\mathrm{E}(X^2) = 1^2(0.1) + 2^2(0.4) + 3^2(0.3) + 4^2(0.2)$ **M1**

$= 0.1 + 1.6 + 2.7 + 3.2$ **A1**

$= 7.6$ **A1**

Note: Award M1 for summing $x^2 \cdot P(X = x)$ terms; first A1 for all four products correct; second A1 for 7.6.

Q5 — Transforming Mean and Variance EASY

(a) $\mathrm{E}(W) = 2\,\mathrm{E}(X) + 1 = 2(3) + 1 = 7$ **A1**

(b) $\mathrm{Var}(W) = 2^2\,\mathrm{Var}(X)$ **M1**

$= 4 \times 4 = 16$ **A1**

Note: In (b), award M1 for recognising $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$ *with* $a = 2$*; the constant* $b = 1$ *must vanish.*

Q6 — Variance via the Shortcut Formula EASY

$\mathrm{E}(X) = 0(0.3) + 1(0.5) + 2(0.2) = 0.9$ **M1**

$\mathrm{E}(X^2) = 0^2(0.3) + 1^2(0.5) + 2^2(0.2) = 0 + 0.5 + 0.8 = 1.3$ **A1**

$\mathrm{Var}(X) = 1.3 - (0.9)^2 = 1.3 - 0.81 = 0.49$ **A1**

Note: Award M1 for setting up both $\mathrm{E}(X)$ *and* $\mathrm{E}(X^2)$*; follow through on arithmetic errors for final A1.*

Q7 — Transforming Mean and Variance EASY

(a) $\mathrm{E}(T) = 3\,\mathrm{E}(X) - 2 = 3(5) - 2 = 13$ **A1**

(b) $\mathrm{Var}(T) = 3^2\,\mathrm{Var}(X)$ **M1**

$= 9 \times 9 = 81$ **A1**

Note: In (b), the constant -2 *does not affect the variance; award M1 for squaring the coefficient 3.*

Q8 — Fair Die Probability and Expected Winnings EASY

(a) $P(X = 4) = 1/6$ **A1**

(b) $\mathrm{E}(\text{winnings}) = 3 \times \dfrac{1}{6} + (-1) \times \dfrac{5}{6}$ **M1**

$= \dfrac{3}{6} - \dfrac{5}{6} = -\dfrac{2}{6} = -1/3$ dollars ≈ -0.333 **A1**

Note: Since the expected winnings are negative, the game is unfavourable to the player.

Q9 — Probability Formula kf(x) and E(X) EASY

(a) $k + 2k + 3k = 1 \implies 6k = 1 \implies k = 1/6$ **A1**

(b) $\mathrm{E}(X) = 2\left(\dfrac{1}{6}\right) + 4\left(\dfrac{2}{6}\right) + 6\left(\dfrac{3}{6}\right)$ **M1**

$= \dfrac{2}{6} + \dfrac{8}{6} + \dfrac{18}{6} = \dfrac{28}{6} = 14/3 \approx 4.67$ **A1**

Note: Follow through on their value of k *for part (b).*

Q10 — Var(X) with Negative Values EASY

$\mathrm{E}(X) = (-1)(0.1) + 0(0.3) + 1(0.4) + 2(0.2) = -0.1 + 0 + 0.4 + 0.4 = 0.7$ **M1**

$\mathrm{E}(X^2) = (-1)^2(0.1) + 0^2(0.3) + 1^2(0.4) + 2^2(0.2) = 0.1 + 0 + 0.4 + 0.8 = 1.3$ **A1**

$\mathrm{Var}(X) = 1.3 - (0.7)^2 = 1.3 - 0.49 = 0.81$ **A1**

Note: Award M1 for computing both $\mathrm{E}(X)$ *and* $\mathrm{E}(X^2)$ *using the formula* $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$.

Q11 — Probability Table with Unknown k; E(X) MEDIUM

(a) Sum of probabilities equals 1: **M1**

$3k + k + 2k + k = 1 \implies 7k = 1 \implies k = 1/7$ **A1**

(b) $\mathrm{E}(X) = 1 \cdot \frac{3}{7} + 2 \cdot \frac{1}{7} + 3 \cdot \frac{2}{7} + 4 \cdot \frac{1}{7}$ **M1**

$= \frac{3 + 2 + 6 + 4}{7} = 15/7 \approx 2.14$ **A1**

Q12 — Two-Condition Hunt: E Given and Sum to 1 MEDIUM

(a) Sum to 1: $a + 0.3 + b = 1$ **A1**

$\mathrm{E}(Y) = 0 \cdot a + 1 \cdot 0.3 + 2 \cdot b = 1.1$ **A1**

(b) From the $\mathrm{E}(Y)$ equation: $0.3 + 2b = 1.1 \implies b = 0.4$ **M1**

From the sum equation: $a = 1 - 0.3 - 0.4 = 0.3$ **A1**

Q13 — Transformation of a Single Variable MEDIUM

(a) $\mathrm{E}(W) = 5\,\mathrm{E}(X) - 3 = 5(2.4) - 3 = 12 - 3 = 9$ **M1A1**

(b) $\mathrm{Var}(W) = 5^2\,\mathrm{Var}(X) = 25 \times 0.64 = 16$ **M1A1**

Note: Award M1 for recognising $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$*; the constant* -3 *must vanish.*

Q14 — Fair-Game / Expected-Winnings Context MEDIUM

Net gains: rolling 6 gives $8 - 2 = \$6$; rolling 1 or 2 gives $3 - 2 = \$1$; otherwise $0 - 2 = -\$2$.

Completed table: $P(X = 6) = \frac{1}{6}$, $P(X = 1) = \frac{2}{6} = \frac{1}{3}$, $P(X = -2) = \frac{3}{6} = \frac{1}{2}$ **M1A1**

$\mathrm{E}(X) = 6 \cdot \frac{1}{6} + 1 \cdot \frac{1}{3} + (-2) \cdot \frac{1}{2}$ **M1**

$= 1 + \frac{1}{3} - 1 = 1/3 \approx \$0.33$

Since $\mathrm{E}(X) = \frac{1}{3} > 0$, the game favours the player and is not fair. **A1**

Q15 — E and Var from Table; Transformation Variance MEDIUM

(a) Using GDC (1-Var Stats with x-values 1, 2, 3, 4 and frequencies $0.1, 0.4, 0.3, 0.2$): **M1**

$\mathrm{E}(X) = 1(0.1) + 2(0.4) + 3(0.3) + 4(0.2) = 0.1 + 0.8 + 0.9 + 0.8 = 2.6$

$\mathrm{E}(X^2) = 1(0.1) + 4(0.4) + 9(0.3) + 16(0.2) = 0.1 + 1.6 + 2.7 + 3.2 = 7.6$

$\mathrm{Var}(X) = 7.6 - (2.6)^2 = 7.6 - 6.76 = 0.84$ **A1**

(b) $\mathrm{Var}(T) = \mathrm{Var}(3X + 1) = 3^2\,\mathrm{Var}(X) = 9 \times 0.84 = 7.56$ **M1A1**

Note: The constant $+1$ does not affect variance.

Q16 — DRV Built from Two Dice; Minimum MEDIUM

(a) $P(M = 1) = 1 - P(\text{both} \geq 2)$ **M1**

$= 1 - \frac{3}{4} \times \frac{3}{4} = 1 - \frac{9}{16} = 7/16$ **AG**

(b) Full distribution (counting outcomes from $4 \times 4 = 16$ equally likely pairs):

$P(M = 2) = \frac{5}{16}, \quad P(M = 3) = \frac{3}{16}, \quad P(M = 4) = \frac{1}{16}$ **M1**

$\mathrm{E}(M) = 1 \cdot \frac{7}{16} + 2 \cdot \frac{5}{16} + 3 \cdot \frac{3}{16} + 4 \cdot \frac{1}{16} = \frac{7 + 10 + 9 + 4}{16}$

$= \frac{30}{16} = \frac{15}{8} = 1.875$ **A1**

Note: Accept 1.88 (3 s.f.).

Q17 — DRV from Draws Without Replacement; E(X) MEDIUM

(a) X can take values 0, 1, 2. Using hypergeometric counting ($\binom{6}{2} = 15$): **M1**

$P(X = 0) = \frac{\binom{4}{2}\binom{2}{0}}{\binom{6}{2}} = \frac{6}{15} = 2/5$

$P(X = 1) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{6}{2}} = 8/15$

$P(X = 2) = \frac{\binom{4}{0}\binom{2}{2}}{\binom{6}{2}} = 1/15$ **A1**

(b) $\mathrm{E}(X) = 0 \cdot \frac{6}{15} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{1}{15} = \frac{10}{15} = 2/3$ **M1A1**

Q18 — E(X) and E(X²) then Var via Formula MEDIUM

(a) Using GDC (1-Var Stats with values 2, 4, 6, 8 and probabilities $0.15, 0.35, 0.30, 0.20$): **M1**

$\mathrm{E}(X) = 2(0.15) + 4(0.35) + 6(0.30) + 8(0.20) = 0.30 + 1.40 + 1.80 + 1.60 = 5.10$

$\mathrm{E}(X^2) = 4(0.15) + 16(0.35) + 36(0.30) + 64(0.20) = 0.60 + 5.60 + 10.80 + 12.80 = 29.80$ **A1**

(b) $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 29.80 - (5.10)^2 = 29.80 - 26.01 = 3.79$ **M1A1**

Q19 — Var from E(X²); Transformation E and Var MEDIUM

(a) $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2 = 11 - 3^2 = 11 - 9 = 2$ **M1A1**

(b) $\text{E}(V) = \text{E}(4 - 2X) = 4 - 2\,\text{E}(X) = 4 - 2(3) = 4 - 6 = -2$ **M1**

$\text{Var}(V) = \text{Var}(4 - 2X) = (-2)^2\,\text{Var}(X) = 4 \times 2 = 8$ **A1**

Note: Award M1 provided the constant 4 is correctly omitted in the variance calculation.

Q20 — Probability Table; E(X) and Var(X) MEDIUM

(a) Sum of probabilities equals 1: **M1**

$0.1 + p + 0.25 + 0.15 + 0.20 = 1 \implies p = 0.30$ **A1**

(b) Using GDC (1-Var Stats with x-values $-1, 0, 1, 2, 3$ and probabilities $0.1, 0.3, 0.25, 0.15, 0.2$): **M1**

$\text{E}(X) = -1(0.1) + 0(0.3) + 1(0.25) + 2(0.15) + 3(0.2) = -0.1 + 0 + 0.25 + 0.30 + 0.60 = 1.05$

$\text{E}(X^2) = 1(0.1) + 0(0.3) + 1(0.25) + 4(0.15) + 9(0.2) = 0.1 + 0 + 0.25 + 0.60 + 1.80 = 2.75$

$\text{Var}(X) = 2.75 - (1.05)^2 = 2.75 - 1.1025 = 1.6475 \approx 1.65$ **A1**

Q21 — Two-Unknown System and Variance HARD

Sum to 1: $a + 2a + b + \dfrac{1}{6} = 1 \implies 3a + b = \dfrac{5}{6}$ **M1**

Using $\text{E}(X) = \dfrac{5}{2}$:

$1(a) + 2(2a) + 3(b) + 4\left(\dfrac{1}{6}\right) = \dfrac{5}{2}$

$a + 4a + 3b + \dfrac{2}{3} = \dfrac{5}{2} \implies 5a + 3b = \dfrac{11}{6}$ **M1**

Solving the system: from the first equation $b = \dfrac{5}{6} - 3a$; substitute:

$5a + 3\left(\dfrac{5}{6} - 3a\right) = \dfrac{11}{6}$

$\implies 5a + \dfrac{5}{2} - 9a = \dfrac{11}{6}$

$\implies -4a = -\dfrac{2}{3} \implies a = 1/6, \quad b = \dfrac{5}{6} - 3\left(\dfrac{1}{6}\right) = \dfrac{5}{6} - \dfrac{3}{6} = \dfrac{2}{6} = 1/3$ **A1**

Finding $\text{Var}(X)$:

$\text{E}(X^2) = 1^2 \cdot \dfrac{1}{6} + 2^2 \cdot \dfrac{2}{6} + 3^2 \cdot \dfrac{2}{6} + 4^2 \cdot \dfrac{1}{6} = \dfrac{1}{6} + \dfrac{8}{6} + \dfrac{18}{6} + \dfrac{16}{6} = \dfrac{43}{6}$ **M1**

$\text{Var}(X) = \dfrac{43}{6} - \left(\dfrac{5}{2}\right)^2 = \dfrac{43}{6} - \dfrac{25}{4} = \dfrac{86 - 75}{12} = 11/12 \approx 0.917$ **A1**

Q22 — Minimum of Two Dice; Fair Game HARD

Constructing distribution of $X =$ minimum of two dice:

$P(X \geq k) = \left(\frac{7-k}{6}\right)^2$, so $P(X = k) = P(X \geq k) - P(X \geq k+1)$ for $k = 1, \ldots, 6$

$P(X = 1) = \frac{11}{36}$, $P(X = 2) = \frac{9}{36}$, $P(X = 3) = \frac{7}{36}$, $P(X = 4) = \frac{5}{36}$, $P(X = 5) = \frac{3}{36}$, $P(X = 6) = \frac{1}{36}$ **M1**

Show that $\mathrm{E}(X) = \frac{91}{36}$:

$\mathrm{E}(X) = \frac{1(11) + 2(9) + 3(7) + 4(5) + 5(3) + 6(1)}{36} = \frac{11 + 18 + 21 + 20 + 15 + 6}{36} = 91/36$ **A1 AG**

Winnings: player wins $2X + 1$ dollars, so $\mathrm{E}(\text{winnings}) = 2 \cdot \frac{91}{36} + 1 = \frac{182}{36} + 1 = \frac{218}{36} = \frac{109}{18}$ **M1**

Fair game: $c = \mathrm{E}(\text{winnings}) = 109/18$ **M1**

Using GDC (arithmetic): $c = \frac{109}{18} \approx \$6.06$ **A1**

Note: Accept 6.06 *or exact fraction* $\frac{109}{18}$.

Q23 — Transformation Identities; Solve for Parameters HARD

(a) $\mathrm{E}(W) = \mathrm{E}(5 - 3Y) = 5 - 3\,\mathrm{E}(Y) = 5 - 3\mu$ **R1 AG**

(b) $\mathrm{Var}(W) = \mathrm{Var}(5 - 3Y) = (-3)^2\,\mathrm{Var}(Y) = 9\sigma^2$
(The constant 5 vanishes since it shifts the distribution; the coefficient -3 is squared.) **R1 AG**

(c) From $\mathrm{E}(W) = -4$: $5 - 3\mu = -4 \implies 3\mu = 9 \implies \mu = 3$ **A1**
From $\mathrm{Var}(W) = 36$: $9\sigma^2 = 36 \implies \sigma^2 = 4$ **M1 A1**

Q24 — Distribution from Draws Without Replacement HARD

(a) Constructing the distribution: **M1**

$P(X = 0) = \frac{\binom{4}{2}}{\binom{6}{2}} = \frac{6}{15} = 2/5$

$P(X = 1) = \frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} = 8/15$

$P(X = 2) = \frac{\binom{2}{2}}{\binom{6}{2}} = 1/15$ **A1**

x	0	1	2
$P(X = x)$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

Probabilities sum to $\frac{6}{15} + \frac{8}{15} + \frac{1}{15} = 1$ **A1**

(b) Using GDC (1-Var Stats or manual): $\mathrm{E}(X) = 0 \cdot \frac{6}{15} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{1}{15} = \frac{10}{15} = 2/3$ **M1**

$\mathrm{E}(X^2) = 0 + \frac{8}{15} + \frac{4}{15} = \frac{12}{15} = \frac{4}{5}$

$\mathrm{Var}(X) = \frac{4}{5} - \left(\frac{2}{3}\right)^2 = \frac{4}{5} - \frac{4}{9} = \frac{36-20}{45} = 16/45$ **A1**

Q25 — Symmetric Distribution; Transform HARD

(a) $k + 2k + 3k + 2k + k = 9k = 1 \implies k = 1/9$ **A1**

(b) Using GDC (1-Var Stats with x-values $0, 1, 2, 3, 4$ and frequencies $1, 2, 3, 2, 1$): $\mathrm{E}(X) = 2$ (by symmetry) **A1**

$\mathrm{E}(X^2) = \frac{0(1) + 1(2) + 4(3) + 9(2) + 16(1)}{9} = \frac{0 + 2 + 12 + 18 + 16}{9} = \frac{48}{9} = \frac{16}{3}$

$\mathrm{Var}(X) = \frac{16}{3} - 4 = 4/3$ **A1**

(c) $\mathrm{E}(W) = \mathrm{E}(4X - 3) = 4(2) - 3 = 5$ **M1**

$\mathrm{Var}(W) = 4^2 \cdot \mathrm{Var}(X) = 16 \cdot \frac{4}{3} = 64/3 \approx 21.3$ **A1**

Note: Accept exact fraction $\frac{64}{3}$; the constant -3 does not affect variance.

Q26 — Show the Constant is 1; Find E and Var HARD

(a) $\sum_{x=1}^{4} \frac{cx}{10} = \frac{c(1+2+3+4)}{10} = \frac{10c}{10} = c$

For probabilities to sum to 1: $c = 1$ **R1 AG**

(b) With $P(X = x) = \frac{x}{10}$: $\mathrm{E}(X) = \sum x \cdot \frac{x}{10} = \frac{1 + 4 + 9 + 16}{10} = \frac{30}{10} = 3$ **M1 A1**

$\mathrm{E}(X^2) = \sum x^2 \cdot \frac{x}{10} = \frac{1^3 + 2^3 + 3^3 + 4^3}{10} = \frac{1 + 8 + 27 + 64}{10} = \frac{100}{10} = 10$ **M1**

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 10 - 9 = 1$ **A1 AG**

Q27 — Carnival Game; Expected Gain; Interpret HARD

(a) Net gain X = sector value -3:

x	-3	-2	-1	2	7
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

A1 A1

(b) Using GDC (1-Var Stats, x-list: $-3, -2, -1, 2, 7$; freq list: all $\frac{1}{5}$):

$\mathrm{E}(X) = \dfrac{-3-2-1+2+7}{5} = \dfrac{3}{5} = 0.6$ **A1**

$\mathrm{E}(X^2) = \dfrac{9+4+1+4+49}{5} = \dfrac{67}{5} = 13.4$

$\mathrm{Var}(X) = 13.4 - 0.36 = 13.04$ **A1**

(c) Since $\mathrm{E}(X) = 0.6 > 0$, on average the player gains \$0.60 per game, so the player should choose to play. **R1**

Q28 — Sum of Two Independent DRVs HARD

(a) $Z = X + Y$ takes values 2, 3, 4.

$P(Z=2) = P(X=1)P(Y=1) = \dfrac{1}{3}\cdot\dfrac{1}{4} = 1/12$ **M1**

$P(Z=3) = P(X=1)P(Y=2) + P(X=2)P(Y=1) = \dfrac{1}{3}\cdot\dfrac{3}{4} + \dfrac{2}{3}\cdot\dfrac{1}{4} = \dfrac{3}{12} + \dfrac{2}{12} = 5/12$

$P(Z=4) = P(X=2)P(Y=2) = \dfrac{2}{3}\cdot\dfrac{3}{4} = \dfrac{6}{12} = 1/2$ **A1**

Check: $\dfrac{1}{12} + \dfrac{5}{12} + \dfrac{6}{12} = 1$

z	2	3	4
$P(Z=z)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{2}$

A1

(b) Using GDC (1-Var Stats, x-list: 2, 3, 4; freq list: $\frac{1}{12}, \frac{5}{12}, \frac{1}{2}$):

$\mathrm{E}(Z) = 2\cdot\dfrac{1}{12} + 3\cdot\dfrac{5}{12} + 4\cdot\dfrac{6}{12} = \dfrac{2+15+24}{12} = \dfrac{41}{12} \approx 3.42$ **A1**

$\mathrm{E}(Z^2) = 4\cdot\dfrac{1}{12} + 9\cdot\dfrac{5}{12} + 16\cdot\dfrac{6}{12} = \dfrac{4+45+96}{12} = \dfrac{145}{12}$

$\mathrm{Var}(Z) = \dfrac{145}{12} - \left(\dfrac{41}{12}\right)^2 = \dfrac{145}{12} - \dfrac{1681}{144} = \dfrac{1740-1681}{144} = \dfrac{59}{144} \approx 0.410$ **A1**

Q29 — Two-Unknown System with Negative Values; Variance HARD

Sum to 1: $p + 0.2 + q + 0.1 = 1 \implies p + q = 0.7$ **M1**

Using $\mathrm{E}(X) = 0.8$:

$(-1)p + (0)(0.2) + (2)q + (3)(0.1) = 0.8$

$-p + 2q + 0.3 = 0.8 \implies -p + 2q = 0.5$ **M1**

Solving: adding $p + q = 0.7$ and $-p + 2q = 0.5$: $3q = 1.2 \implies q = 0.4$, then $p = 0.3$ **A1**

Finding $\mathrm{Var}(X)$: using GDC (1-Var Stats, x-list: $-1, 0, 2, 3$; freq list: $0.3, 0.2, 0.4, 0.1$):

$\mathrm{E}(X^2) = 1(0.3) + 0(0.2) + 4(0.4) + 9(0.1) = 0.3 + 0 + 1.6 + 0.9 = 2.8$ **M1**

$\mathrm{Var}(X) = 2.8 - (0.8)^2 = 2.8 - 0.64 = 2.16$ **A1**

Q30 — E(X²) then Solve for Transformation Parameters HARD

(a) $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$
$4 = \text{E}(X^2) - 9 \implies \text{E}(X^2) = 13$ **A1**

(b) $\text{E}(W) = a\,\text{E}(X) + b = 3a + b = 10$ **M1**
$\text{Var}(W) = a^2\,\text{Var}(X) = 4a^2 = 100$ **M1**
$a^2 = 25 \implies a = 5$ (since $a > 0$) **A1**
Substituting into $3a + b = 10$: $15 + b = 10 \implies b = -5$ **A1**

Note: Award M1 for correctly writing $\text{E}(W) = 3a + b = 10$ *and M1 for* $\text{Var}(W) = 4a^2 = 100$. *The constraint* $a > 0$ *rules out* $a = -5$.