IB Math AA HL — Topic 4
Discrete Random Variables

Discrete Probability Distributions · Mean & Variance · Transformation of a Variable

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Sum to 1:** $\sum_{\text{all } x} P(X = x) = 1$
- **Expected value:** $E(X) = \mu = \sum_{\text{all } x} x \cdot P(X = x)$
- **Second moment:** $E(X^2) = \sum_{\text{all } x} x^2 \cdot P(X = x)$
- **Variance:** $\text{Var}(X) = E(X^2) - [E(X)]^2$
- **Standard deviation:** $\sigma = \sqrt{\text{Var}(X)}$
- **Linear transformation — mean:** $E(aX + b) = a\,E(X) + b$
- **Linear transformation — variance:** $\text{Var}(aX + b) = a^2\,\text{Var}(X)$

GDC Tips

- **1-Var Stats** (TI: `STAT` $\to$ `CALC` $\to$ `1-Var Stats`; Casio: `STAT` $\to$ `1-VAR`): enter x-values in one list and $P(X=x)$ values as frequencies in a second list to read off $\bar{x} = E(X)$ and $\sigma_x = \sqrt{\text{Var}(X)}$ directly.
- **List arithmetic for $E(X^2)$**: store x-values in `L1`, probabilities in `L2`, then compute `sum(L1^2 * L2)` to obtain $E(X^2)$ without a separate table.
- **Verify sum-to-1**: use `sum(L2)` to confirm probabilities sum to 1 before proceeding; a result $\neq 1$ signals an entry error.
- **Solving for unknowns**: enter the sum-to-1 equation and any $E(X)$ condition into the equation solver (`ALPHA` $\to$ `SOLVE` on Casio or `PlySmlt2` on TI) when two unknowns are present.
- **Cumulative probabilities** $P(X \leq k)$: compute `sum(L2,1,k)` (selecting the first k entries of the probability list) rather than adding terms by hand.

Common Mistakes to Avoid

1. **Forgetting to sum to 1 first.** When a table contains an unknown k, the very first line of working must be $\sum P(X=x) = 1$. Jumping straight to $E(X)$ with an unsolved k produces a nonsensical answer.
2. **Confusing $E(X^2)$ with $[E(X)]^2$.** $E(X^2) = \sum x^2 P(X=x)$ is computed by squaring each x-value *before* multiplying by the probability. A common error is to compute $E(X)$ first and then square it, giving $[E(X)]^2$ instead, which always underestimates the variance.
3. **Adding b to the variance in a transformation.** $\text{Var}(aX+b) = a^2\text{Var}(X)$; the constant b shifts the distribution but does *not* change its spread. Writing $\text{Var}(aX+b) = a^2\text{Var}(X) + b$ or $+b^2$ is incorrect.
4. **Forgetting to square a in the variance formula.** Students correctly apply $E(aX+b) = aE(X)+b$ and then write $\text{Var}(aX+b) = a\,\text{Var}(X)$, omitting the square. The scaling of variance is always a^2, not a.
5. **Applying the transformation formula to σ inconsistently.** If $\text{Var}(X)$ is scaled by a^2, then $\sigma_{aX+b} = |a|\,\sigma_X$, not $a^2\sigma_X$. Mixing the variance formula with the standard deviation result produces a dimensionally wrong answer.
6. **Omitting a concluding sentence in fair-game and expected-winnings questions.** A numeric value of $E(X)$ alone does not earn the reasoning mark. The answer must state explicitly, for example, "Since $E(X) = -0.20 < 0$, the game is not fair and the player should not play."

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Section A — Easy

EASY

1. The discrete random variable X has the probability distribution shown in the table below.

x	1	2	3	4
$P(X = x)$	0.2	k	0.35	0.15

Find the value of k. *Calculator not allowed* **[3 marks]**

2. The discrete random variable Y has the following probability distribution, where a is a constant.

y	0	1	2
$P(Y = y)$	0.4	a	0.1

(a) Write down the value of a. *Calculator not allowed* **[1 marks]**

(b) Find E(Y). *Calculator not allowed* **[2 marks]**

3. A discrete random variable X takes values $1, 2,$ and 3 with probabilities $\frac{1}{6}, \frac{1}{2},$ and $\frac{1}{3}$ respectively.
Find $\mathrm{E}(X)$. *Calculator not allowed* **[3 marks]**

4. A spinner is numbered $1, 2, 3, 4$. The probability of each outcome is given in the table below.

x	1	2	3	4
$P(X = x)$	0.1	0.4	0.3	0.2

Find $\mathrm{E}(X^2)$. *Calculator allowed* **[3 marks]**

5. The discrete random variable X has mean $\mathrm{E}(X) = 3$ and variance $\mathrm{Var}(X) = 4$.
Let $W = 2X + 1$.

(a) Find $\mathrm{E}(W)$. *Calculator not allowed* **[1 marks]**

(b) Find $\mathrm{Var}(W)$. *Calculator not allowed* **[2 marks]**

6. The discrete random variable X has the probability distribution given in the table below.

x	0	1	2
$P(X = x)$	0.3	0.5	0.2

Find Var(X). *Calculator allowed* **[3 marks]**

7. The discrete random variable X has mean E$(X) = 5$ and variance Var$(X) = 9$.
Let $T = 3X - 2$.

(a) Find E(T). *Calculator allowed* **[1 marks]**

(b) Find Var(T). *Calculator allowed* **[2 marks]**

8. A fair six-sided die, numbered 1 to 6, is rolled once. The discrete random variable X represents the score obtained.

(a) Write down $P(X = 4)$. *Calculator not allowed* **[1 marks]**

(b) A player wins \$3 if the score is 6, and loses \$1 otherwise. Find the expected winnings per roll. *Calculator allowed* **[2 marks]**

9. A discrete random variable X takes values $2, 4, 6$ with probabilities $k, 2k, 3k$ respectively, where k is a positive constant.

(a) Find the value of k. *Calculator allowed* **[1 marks]**

(b) Find E(X). *Calculator allowed* **[2 marks]**

10. The discrete random variable X has the probability distribution shown below.

x	-1	0	1	2
$P(X = x)$	0.1	0.3	0.4	0.2

Find the variance of X. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. The discrete random variable X has the probability distribution shown in the table below. *Calculator not allowed*
[4 marks]

x	1	2	3	4
$P(X = x)$	$3k$	k	$2k$	k

(a) Find the value of k.

(b) Find $\mathrm{E}(X)$.

12. The discrete random variable Y has the following probability distribution, where a and b are constants.
Calculator not allowed **[4 marks]**

y	0	1	2
$P(Y = y)$	a	0.3	b

It is given that E$(Y) = 1.1$.

(a) Write down two equations satisfied by a and b.

(b) Find the values of a and b.

13. The discrete random variable X has mean E$(X) = 2.4$ and variance Var$(X) = 0.64$. The random variable W is defined by $W = 5X - 3$. *Calculator not allowed* **[4 marks]**

(a) Find E(W).

(b) Find Var(W).

14. A game at a school fair costs \$2 to play. A player rolls a fair six-sided die. If the result is a 6, the player wins \$8. If the result is a 1 or 2, the player wins \$3. Otherwise, the player wins nothing. Let X be the player's net gain in dollars. *Calculator allowed* **[4 marks]**

(a) Copy and complete the probability distribution table for X.

(b) Find E(X) and hence determine whether the game is fair.

15. The discrete random variable X has the probability distribution shown below. *Calculator allowed* **[4 marks]**

x	1	2	3	4
$P(X = x)$	0.1	0.4	0.3	0.2

(a) Find E(X) and Var(X).

(b) The random variable T is defined by $T = 3X + 1$. Find Var(T).

16. Two fair four-sided dice, each with faces labelled 1, 2, 3 and 4, are rolled. The discrete random variable M denotes the minimum of the two results. *Calculator allowed* **[4 marks]**

(a) Show that $P(M = 1) = \dfrac{7}{16}$.

(b) Find E(M).

17. A bag contains 4 red and 2 blue counters. Two counters are drawn at random without replacement. The discrete random variable X denotes the number of blue counters drawn. *Calculator allowed* **[4 marks]**

(a) Find the probability distribution of X.

(b) Find E(X).

18. The discrete random variable X has the probability distribution shown below. *Calculator allowed* **[4 marks]**

x	2	4	6	8
$P(X=x)$	0.15	0.35	0.30	0.20

(a) Find E(X) and E(X^2).

(b) Hence find Var(X).

19. The discrete random variable X has mean E$(X)=3$ and E$(X^2)=11$. The random variable V is defined by $V=4-2X$. *Calculator not allowed* **[4 marks]**

(a) Find Var(X).

(b) Find E(V) and Var(V).

20. The discrete random variable X has the probability distribution shown below, where p is a constant. *Calculator allowed*
[4 marks]

x	-1	0	1	2	3
$P(X = x)$	0.1	p	0.25	0.15	0.2

(a) Find the value of p.

(b) Find $\text{E}(X)$ and $\text{Var}(X)$.

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Section C — Hard

HARD

21. The discrete random variable X has the probability distribution shown in the table below, where a and b are constants. *Calculator not allowed* **[5 marks]**

x	1	2	3	4
$P(X = x)$	a	$2a$	b	$\frac{1}{6}$

Given that $\mathrm{E}(X) = \frac{5}{2}$, find the values of a and b, and hence find $\mathrm{Var}(X)$.

22. A game costs $\$c$ to play. A player rolls two fair six-sided dice. Let X be the minimum of the two scores. The player wins $\$(2X+1)$ dollars. *Calculator allowed* **[5 marks]**

Show that $\mathrm{E}(X) = \frac{91}{36}$, and hence find the value of c for which the game is fair. Give your answer correct to 2 decimal places.

23. The discrete random variable Y has mean μ and variance σ^2. A new variable is defined by $W = 5 - 3Y$. *Calculator not allowed* **[5 marks]**

(a) Show that $\mathrm{E}(W) = 5 - 3\mu$. [1]

(b) Show that $\mathrm{Var}(W) = 9\sigma^2$. [1]

(c) Given that $\mathrm{E}(W) = -4$ and $\mathrm{Var}(W) = 36$, find μ and σ^2. [3]

24. A bag contains 4 red and 2 blue tokens. Two tokens are drawn at random without replacement. The discrete random variable X denotes the number of blue tokens drawn. *Calculator allowed* **[5 marks]**

(a) Construct the probability distribution table for X. [3]

(b) Find E(X) and Var(X). [2]

25. The discrete random variable X has the probability distribution shown below, where k is a positive constant. *Calculator allowed* **[5 marks]**

x	0	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$2k$	k

(a) Find the value of k. [1]

(b) Find E(X) and Var(X). [2]

(c) A new variable is defined by $W = 4X - 3$. Find E(W) and Var(W). [2]

26. A discrete random variable X takes values $1, 2, 3, 4$ with $P(X = x) = \dfrac{cx}{10}$ for each $x \in \{1, 2, 3, 4\}$, where c is a constant. *Calculator not allowed* **[5 marks]**

(a) Show that $c = 1$. [1]

(b) Find E(X) and show that Var$(X) = 1$. [4]

27. In a carnival game, a player pays $\$3$ to spin a wheel. The wheel lands on one of five equally likely sectors labelled $0, 1, 2, 5, 10$. The player wins the number of dollars shown. Let X be the player's net gain (winnings minus cost). *Calculator allowed* **[5 marks]**

(a) Write down the probability distribution of X. [2]

(b) Find E(X) and Var(X). [2]

(c) Hence determine, with justification, whether a player should choose to play the game. [1]

28. Two independent discrete random variables X and Y each take only the values 1 and 2. It is known that $P(X=1)=\dfrac{1}{3}$, $P(Y=1)=\dfrac{1}{4}$. Let $Z=X+Y$. *Calculator allowed* **[5 marks]**

(a) Construct the probability distribution table for Z. [3]

(b) Find E(Z) and Var(Z). [2]

29. The discrete random variable X has the following probability distribution, where p and q are positive constants. *Calculator allowed* **[5 marks]**

x	-1	0	2	3
$P(X=x)$	p	0.2	q	0.1

Given that E$(X)=0.8$, find p and q, and hence find Var(X).

30. A discrete random variable X has E$(X) = 3$ and Var$(X) = 4$. *Calculator not allowed* **[5 marks]**

(a) Find E(X^2). [1]

(b) A variable is defined by $W = aX + b$ where $a > 0$. Given that E$(W) = 10$ and Var$(W) = 100$, find the values of a and b. [4]

Worked Solutions

Q1 — Finding k from sum-to-one [EASY]

All probabilities sum to 1: **M1**

$0.2 + k + 0.35 + 0.15 = 1$

$k = 1 - 0.7 = 0.3$ **A1**

Since $0 \le k \le 1$, the value is valid. **A1**

Note: Award M1 for writing the sum-to-one equation; first A1 for correct simplification; second A1 for $k = 0.3$.

Q2 — Sum to one and E(Y) [EASY]

(a)

$0.4 + a + 0.1 = 1 \Rightarrow a = 0.5$ **A1**

(b)

E$(Y) = 0(0.4) + 1(0.5) + 2(0.1)$ **M1**

E$(Y) = 0 + 0.5 + 0.2 = 0.7$ **A1**

Note: In (b), follow through on their value of a for the M1.

Q3 — E(X) from exact fractions [EASY]

E$(X) = 1 \times \dfrac{1}{6} + 2 \times \dfrac{1}{2} + 3 \times \dfrac{1}{3}$ **M1**

$= \dfrac{1}{6} + 1 + 1$ **A1**

$= \dfrac{13}{6}$ **A1**

Note: Award M1 for correctly setting up $\sum x \cdot P(X = x)$; accept equivalent exact or decimal forms for final A1.

Q4 — Finding E(X^2) [EASY]

E$(X^2) = 1^2(0.1) + 2^2(0.4) + 3^2(0.3) + 4^2(0.2)$ **M1**

$= 0.1 + 1.6 + 2.7 + 3.2$ **A1**

$= 7.6$ **A1**

Note: Award M1 for summing $x^2 \cdot P(X = x)$ terms; first A1 for all four products correct; second A1 for 7.6.

Q5 — Transforming mean and variance [EASY]

(a)

$\mathsf{E}(W) = 2\,\mathsf{E}(X) + 1 = 2(3) + 1 = 7$ **A1**

(b)

$\mathsf{Var}(W) = 2^2\,\mathsf{Var}(X)$ **M1**

$= 4 \times 4 = 16$ **A1**

Note: In (b), award M1 for recognising $\mathsf{Var}(aX + b) = a^2\mathsf{Var}(X)$ *with* $a = 2$*; the constant* $b = 1$ *must vanish.*

Q6 — Var(X) via $\mathsf{E}(X^2) - [\mathsf{E}(X)]^2$ [EASY]

$\mathsf{E}(X) = 0(0.3) + 1(0.5) + 2(0.2) = 0.9$ **M1**

$\mathsf{E}(X^2) = 0^2(0.3) + 1^2(0.5) + 2^2(0.2) = 0 + 0.5 + 0.8 = 1.3$ **A1**

$\mathsf{Var}(X) = 1.3 - (0.9)^2 = 1.3 - 0.81 = 0.49$ **A1**

Note: Award M1 for setting up both $\mathsf{E}(X)$ *and* $\mathsf{E}(X^2)$*; follow through on arithmetic errors for final A1.*

Q7 — Transforming mean and variance [EASY]

(a)

$\mathsf{E}(T) = 3\,\mathsf{E}(X) - 2 = 3(5) - 2 = 13$ **A1**

(b)

$\mathsf{Var}(T) = 3^2\,\mathsf{Var}(X)$ **M1**

$= 9 \times 9 = 81$ **A1**

Note: In (b), the constant -2 *does not affect the variance; award M1 for squaring the coefficient* 3*.*

Q8 — Fair die probability and expected winnings [EASY]

(a)

$P(X = 4) = \dfrac{1}{6}$ **A1**

(b)

$\mathsf{E}(\text{winnings}) = 3 \times \dfrac{1}{6} + (-1) \times \dfrac{5}{6}$ **M1**

$= \dfrac{3}{6} - \dfrac{5}{6} = -\dfrac{2}{6} = -\$\dfrac{1}{3} \approx -\$0.333$ **A1**

Note: Since the expected winnings are negative, the game is unfavourable to the player.

Q9 — Probability formula $kf(x)$ and E(X) [EASY]

(a)

$k + 2k + 3k = 1 \Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$ **A1**

(b)

$\mathrm{E}(X) = 2\left(\frac{1}{6}\right) + 4\left(\frac{2}{6}\right) + 6\left(\frac{3}{6}\right)$ **M1**

$= \frac{2}{6} + \frac{8}{6} + \frac{18}{6} = \frac{28}{6} = \frac{14}{3} \approx 4.67$ **A1**

Note: Follow through on their value of k for part (b).

Q10 — Var(X) with negative values [EASY]

$\mathrm{E}(X) = (-1)(0.1) + 0(0.3) + 1(0.4) + 2(0.2) = -0.1 + 0 + 0.4 + 0.4 = 0.7$ **M1**

$\mathrm{E}(X^2) = (-1)^2(0.1) + 0^2(0.3) + 1^2(0.4) + 2^2(0.2) = 0.1 + 0 + 0.4 + 0.8 = 1.3$ **A1**

$\mathrm{Var}(X) = 1.3 - (0.7)^2 = 1.3 - 0.49 = 0.81$ **A1**

Note: Award M1 for computing both $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$ using the formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$.

Q11 — Probability table with unknown k; E(X) [MEDIUM]

(a) Sum of probabilities equals 1: **M1**

$3k + k + 2k + k = 1 \implies 7k = 1 \implies k = \frac{1}{7}$ **A1**

(b) $\mathrm{E}(X) = 1 \cdot \frac{3}{7} + 2 \cdot \frac{1}{7} + 3 \cdot \frac{2}{7} + 4 \cdot \frac{1}{7}$ **M1**

$= \frac{3 + 2 + 6 + 4}{7} = \frac{15}{7} \approx 2.14$ **A1**

Q12 — Two-condition hunt: E given, sum to 1 [MEDIUM]

(a) Sum to 1: $a + 0.3 + b = 1$ **A1**

$\mathrm{E}(Y) = 0 \cdot a + 1 \cdot 0.3 + 2 \cdot b = 1.1$ **A1**

(b) From the $\mathrm{E}(Y)$ equation: $0.3 + 2b = 1.1 \implies b = 0.4$ **M1**

From the sum equation: $a = 1 - 0.3 - 0.4 = 0.3$ **A1**

Q13 — Transformation of a single variable [MEDIUM]

(a) $\mathsf{E}(W) = 5\,\mathsf{E}(X) - 3 = 5(2.4) - 3 = 12 - 3 = 9$ **M1A1**

(b) $\mathsf{Var}(W) = 5^2\,\mathsf{Var}(X) = 25 \times 0.64 = 16$ **M1A1**

Note: Award M1 for recognising $\mathsf{Var}(aX + b) = a^2\mathsf{Var}(X)$*; the constant* -3 *must vanish.*

Q14 — Fair-game / expected-winnings context [MEDIUM]

Net gains: rolling 6 gives $8 - 2 = \$6$; rolling 1 or 2 gives $3 - 2 = \$1$; otherwise $0 - 2 = -\$2$.

Completed table: $P(X = 6) = \frac{1}{6}$, $P(X = 1) = \frac{2}{6} = \frac{1}{3}$, $P(X = -2) = \frac{3}{6} = \frac{1}{2}$ **M1A1**

$\mathsf{E}(X) = 6 \cdot \dfrac{1}{6} + 1 \cdot \dfrac{1}{3} + (-2) \cdot \dfrac{1}{2}$ **M1**

$= 1 + \dfrac{1}{3} - 1 = \dfrac{1}{3} \approx \$0.33$

Since $\mathsf{E}(X) = \dfrac{1}{3} > 0$, the game favours the player and is **not fair**. **A1**

Q15 — E, Var from table; transformation variance [MEDIUM]

(a) Using GDC (1-Var Stats with x-values 1,2,3,4 and frequencies 0.1,0.4,0.3,0.2): **M1**

$\mathsf{E}(X) = 1(0.1) + 2(0.4) + 3(0.3) + 4(0.2) = 0.1 + 0.8 + 0.9 + 0.8 = 2.6$

$\mathsf{E}(X^2) = 1(0.1) + 4(0.4) + 9(0.3) + 16(0.2) = 0.1 + 1.6 + 2.7 + 3.2 = 7.6$

$\mathsf{Var}(X) = 7.6 - (2.6)^2 = 7.6 - 6.76 = 0.84$ **A1**

(b) $\mathsf{Var}(T) = \mathsf{Var}(3X + 1) = 3^2\,\mathsf{Var}(X) = 9 \times 0.84 = 7.56$ **M1A1**

Note: The constant $+1$ *does not affect variance.*

Q16 — DRV built from two dice; minimum [MEDIUM]

(a) $P(M = 1)$ = $1 - P(\text{both} \geq 2)$ **M1**

$= 1 - \dfrac{3}{4} \times \dfrac{3}{4} = 1 - \dfrac{9}{16} = \dfrac{7}{16}$ **AG**

(b) Full distribution (counting outcomes from $4 \times 4 = 16$ equally likely pairs):

$P(M = 2) = \frac{5}{16}$, $P(M = 3) = \frac{3}{16}$, $P(M = 4) = \frac{1}{16}$ **M1**

$\mathsf{E}(M) = 1 \cdot \dfrac{7}{16} + 2 \cdot \dfrac{5}{16} + 3 \cdot \dfrac{3}{16} + 4 \cdot \dfrac{1}{16} = \dfrac{7 + 10 + 9 + 4}{16} = \dfrac{30}{16} = \dfrac{15}{8} = 1.875$ **A1**

Note: Accept 1.88 *(3 s.f.).*

Q17 — DRV from draws without replacement; E(X) [MEDIUM]

(a) X can take values 0, 1, 2. Using hypergeometric counting ($\binom{6}{2} = 15$): **M1**

$$P(X=0) = \frac{\binom{4}{2}\binom{2}{0}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5}$$

$$P(X=1) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{6}{2}} = \frac{8}{15}$$

$$P(X=2) = \frac{\binom{4}{0}\binom{2}{2}}{\binom{6}{2}} = \frac{1}{15}$$ **A1**

(b) $\mathsf{E}(X) = 0 \cdot \frac{6}{15} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{1}{15} = \frac{10}{15} = \frac{2}{3}$ **M1A1**

Q18 — E(X), E(X^2), Var via formula [MEDIUM]

(a) Using GDC (1-Var Stats with values 2,4,6,8 and probabilities 0.15,0.35,0.30,0.20): **M1**

$\mathsf{E}(X) = 2(0.15) + 4(0.35) + 6(0.30) + 8(0.20) = 0.30 + 1.40 + 1.80 + 1.60 = 5.10$

$\mathsf{E}(X^2) = 4(0.15) + 16(0.35) + 36(0.30) + 64(0.20) = 0.60 + 5.60 + 10.80 + 12.80 = 29.80$ **A1**

(b) $\mathsf{Var}(X) = \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 = 29.80 - (5.10)^2 = 29.80 - 26.01 = 3.79$ **M1A1**

Q19 — Var from E(X^2); transformation E and Var [MEDIUM]

(a) $\mathsf{Var}(X) = \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 = 11 - 3^2 = 11 - 9 = 2$ **M1A1**

(b) $\mathsf{E}(V) = \mathsf{E}(4 - 2X) = 4 - 2\,\mathsf{E}(X) = 4 - 2(3) = 4 - 6 = -2$ **M1**

$\mathsf{Var}(V) = \mathsf{Var}(4 - 2X) = (-2)^2\,\mathsf{Var}(X) = 4 \times 2 = 8$ **A1**

Note: Award M1 provided the constant 4 is correctly omitted in the variance calculation.

Q20 — Probability table; E(X) and Var(X) [MEDIUM]

(a) Sum of probabilities equals 1: **M1**

$0.1 + p + 0.25 + 0.15 + 0.20 = 1 \implies p = 0.30$ **A1**

(b) Using GDC (1-Var Stats with x-values $-1, 0, 1, 2, 3$ and probabilities $0.1, 0.3, 0.25, 0.15, 0.2$): **M1**

$\mathsf{E}(X) = -1(0.1) + 0(0.3) + 1(0.25) + 2(0.15) + 3(0.2) = -0.1 + 0 + 0.25 + 0.30 + 0.60 = 1.05$

$\mathsf{E}(X^2) = 1(0.1) + 0(0.3) + 1(0.25) + 4(0.15) + 9(0.2) = 0.1 + 0 + 0.25 + 0.60 + 1.80 = 2.75$

$\mathsf{Var}(X) = 2.75 - (1.05)^2 = 2.75 - 1.1025 = 1.6475 \approx 1.65$ **A1**

Q21 — Two-unknown system and variance [HARD]

Sum to 1:

$$a + 2a + b + \frac{1}{6} = 1 \implies 3a + b = \frac{5}{6}$$ **M1**

Using E$(X) = \frac{5}{2}$**:**

$$1(a) + 2(2a) + 3(b) + 4\left(\tfrac{1}{6}\right) = \frac{5}{2}$$

$$a + 4a + 3b + \frac{2}{3} = \frac{5}{2} \implies 5a + 3b = \frac{11}{6}$$ **M1**

Solving the system: From first equation $b = \frac{5}{6} - 3a$; substitute:

$$5a + 3\left(\frac{5}{6} - 3a\right) = \frac{11}{6} \implies 5a + \frac{5}{2} - 9a = \frac{11}{6} \implies -4a = -\frac{2}{3} \implies a = \frac{1}{6},\ b = \frac{3}{6} = \frac{1}{2}$$ **A1**

Finding Var(X)**:**

$$\mathrm{E}(X^2) = 1^2 \cdot \tfrac{1}{6} + 2^2 \cdot \tfrac{2}{6} + 3^2 \cdot \tfrac{1}{2} + 4^2 \cdot \tfrac{1}{6} = \frac{1}{6} + \frac{8}{6} + \frac{9}{2} + \frac{16}{6} = \frac{25}{6} + \frac{27}{6} = \frac{52}{6} = \frac{26}{3}$$ **M1**

$$\mathrm{Var}(X) = \frac{26}{3} - \left(\frac{5}{2}\right)^2 = \frac{26}{3} - \frac{25}{4} = \frac{104 - 75}{12} = \frac{29}{12}$$ **A1**

Q22 — Minimum of two dice; fair game [HARD]

Constructing distribution of $X = \min$ **of two dice:**

$P(X \geq k) = \left(\frac{7-k}{6}\right)^2$, so $P(X = k) = P(X \geq k) - P(X \geq k+1)$ for $k = 1, \ldots, 6$

$P(X = 1) = \frac{11}{36},\ P(X = 2) = \frac{9}{36},\ P(X = 3) = \frac{7}{36},\ P(X = 4) = \frac{5}{36},\ P(X = 5) = \frac{3}{36},\ P(X = 6) = \frac{1}{36}$ **M1**

Show that E$(X) = \frac{91}{36}$**:**

$$\mathrm{E}(X) = \frac{1(11) + 2(9) + 3(7) + 4(5) + 5(3) + 6(1)}{36} = \frac{11 + 18 + 21 + 20 + 15 + 6}{36} = \frac{91}{36}$$ **A1 AG**

Winnings: Player wins $2X + 1$ dollars, so E(winnings) $= 2 \cdot \frac{91}{36} + 1 = \frac{182}{36} + 1 = \frac{218}{36} = \frac{109}{18}$ **M1**

Fair game: $c =$ E(winnings) $= \frac{109}{18}$ **M1**

Using GDC (arithmetic): $c = \frac{109}{18} \approx \$6.06$ **A1**

Note: Accept 6.06 *or exact fraction* $\frac{109}{18}$.

Q23 — Transformation identities; solve for parameters [HARD]

(a) $\mathsf{E}(W) = \mathsf{E}(5 - 3Y) = 5 - 3\mathsf{E}(Y) = 5 - 3\mu$ **R1 AG**

(b) $\mathsf{Var}(W) = \mathsf{Var}(5 - 3Y) = (-3)^2\mathsf{Var}(Y) = 9\sigma^2$

(The constant 5 vanishes since it shifts the distribution; the coefficient -3 is squared.) **R1 AG**

(c) From $\mathsf{E}(W) = -4$:

$5 - 3\mu = -4 \implies 3\mu = 9 \implies \mu = 3$ **A1**

From $\mathsf{Var}(W) = 36$:

$9\sigma^2 = 36 \implies \sigma^2 = 4$ **M1 A1**

Q24 — Distribution from draws without replacement [HARD]

(a) Constructing the distribution:

$$P(X = 0) = \frac{\binom{4}{2}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5}$$ **M1**

$$P(X = 1) = \frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} = \frac{8}{15}$$

$$P(X = 2) = \frac{\binom{2}{2}}{\binom{6}{2}} = \frac{1}{15}$$ **A1**

x	0	1	2
$P(X = x)$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

Probabilities sum to $\frac{6}{15} + \frac{8}{15} + \frac{1}{15} = 1$ **A1**

(b) Using GDC (1-Var Stats or manual): $\mathsf{E}(X) = 0 \cdot \frac{6}{15} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{1}{15} = \dfrac{10}{15} = \dfrac{2}{3}$ **M1**

$\mathsf{E}(X^2) = 0 + \frac{8}{15} + \frac{4}{15} = \dfrac{12}{15} = \dfrac{4}{5}$

$\mathsf{Var}(X) = \dfrac{4}{5} - \left(\dfrac{2}{3}\right)^2 = \dfrac{4}{5} - \dfrac{4}{9} = \dfrac{36 - 20}{45} = \dfrac{16}{45}$ **A1**

Q25 — Symmetric distribution; transform [HARD]

(a) $k + 2k + 3k + 2k + k = 9k = 1 \implies k = \dfrac{1}{9}$ **A1**

(b) Using GDC (1-Var Stats with x-values $0, 1, 2, 3, 4$ and frequencies $1, 2, 3, 2, 1$): $\mathsf{E}(X) = 2$ (by symmetry) **A1**

$\mathrm{E}(X^2) = \dfrac{0(1) + 1(2) + 4(3) + 9(2) + 16(1)}{9} = \dfrac{0 + 2 + 12 + 18 + 16}{9} = \dfrac{48}{9} = \dfrac{16}{3}$

$\mathrm{Var}(X) = \dfrac{16}{3} - 4 = \dfrac{4}{3}$ **A1**

(c) $\mathrm{E}(W) = \mathrm{E}(4X - 3) = 4(2) - 3 = 5$ **M1**

$\mathrm{Var}(W) = 4^2 \cdot \mathrm{Var}(X) = 16 \cdot \dfrac{4}{3} = \dfrac{64}{3} \approx 21.3$ **A1**

Note: Accept exact fraction $\frac{64}{3}$; the constant -3 does not affect variance.

Q26 — Show c=1; find E and show Var=1 [HARD]

(a) $\displaystyle\sum_{x=1}^{4} \frac{cx}{10} = \frac{c(1+2+3+4)}{10} = \frac{10c}{10} = c$

For probabilities to sum to 1: $c = 1$ **R1 AG**

(b) With $P(X = x) = \dfrac{x}{10}$: $\mathrm{E}(X) = \sum x \cdot \dfrac{x}{10} = \dfrac{1 + 4 + 9 + 16}{10} = \dfrac{30}{10} = 3$ **M1 A1**

$\mathrm{E}(X^2) = \sum x^2 \cdot \dfrac{x}{10} = \dfrac{1^3 + 2^3 + 3^3 + 4^3}{10} = \dfrac{1 + 8 + 27 + 64}{10} = \dfrac{100}{10} = 10$ **M1**

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 10 - 9 = 1$ **A1 AG**

Q27 — Carnival game; expected gain; interpret [HARD]

(a) Net gain $X =$ sector value $- 3$:

x	-3	-2	-1	2	7
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

A1 A1

(b) Using GDC (1-Var Stats, x-list: $-3, -2, -1, 2, 7$; freq list: all $\frac{1}{5}$):

$\mathrm{E}(X) = \dfrac{-3 - 2 - 1 + 2 + 7}{5} = \dfrac{3}{5} = 0.6$ **A1**

$\mathrm{E}(X^2) = \dfrac{9 + 4 + 1 + 4 + 49}{5} = \dfrac{67}{5} = 13.4$

$\mathrm{Var}(X) = 13.4 - 0.36 = 13.04$ **A1**

(c) Since $\mathrm{E}(X) = 0.6 > 0$, on average the player gains $\$0.60$ per game, so the player should choose to play. **R1**

Q28 — Sum of two independent DRVs [HARD]

(a) $Z = X + Y$ takes values $2, 3, 4$.

$P(Z=2) = P(X=1)P(Y=1) = \dfrac{1}{3} \cdot \dfrac{1}{4} = \dfrac{1}{12}$ **M1**

$P(Z=3) = P(X=1)P(Y=2) + P(X=2)P(Y=1) = \dfrac{1}{3} \cdot \dfrac{3}{4} + \dfrac{2}{3} \cdot \dfrac{1}{4} = \dfrac{3}{12} + \dfrac{2}{12} = \dfrac{5}{12}$

$P(Z=4) = P(X=2)P(Y=2) = \dfrac{2}{3} \cdot \dfrac{3}{4} = \dfrac{6}{12} = \dfrac{1}{2}$ **A1**

Check: $\frac{1}{12} + \frac{5}{12} + \frac{6}{12} = 1$ ✓

z	2	3	4
$P(Z=z)$	$\dfrac{1}{12}$	$\dfrac{5}{12}$	$\dfrac{1}{2}$

A1

(b) Using GDC (1-Var Stats, x-list: $2, 3, 4$; freq list: $\frac{1}{12}, \frac{5}{12}, \frac{1}{2}$):

$\mathsf{E}(Z) = 2 \cdot \frac{1}{12} + 3 \cdot \frac{5}{12} + 4 \cdot \frac{6}{12} = \frac{2+15+24}{12} = \frac{41}{12} \approx 3.42$ **A1**

$\mathsf{E}(Z^2) = 4 \cdot \frac{1}{12} + 9 \cdot \frac{5}{12} + 16 \cdot \frac{6}{12} = \frac{4+45+96}{12} = \frac{145}{12}$

$\mathrm{Var}(Z) = \frac{145}{12} - \left(\frac{41}{12}\right)^2 = \frac{145}{12} - \frac{1681}{144} = \frac{1740-1681}{144} = \frac{59}{144} \approx 0.410$ **A1**

Q29 — Two-unknown system with negative values; variance [HARD]

Sum to 1:

$p + 0.2 + q + 0.1 = 1 \implies p + q = 0.7$ **M1**

Using $\mathsf{E}(X) = 0.8$**:**

$(-1)p + (0)(0.2) + (2)q + (3)(0.1) = 0.8$

$-p + 2q + 0.3 = 0.8 \implies -p + 2q = 0.5$ **M1**

Solving: Adding $p + q = 0.7$ and $-p + 2q = 0.5$: $3q = 1.2 \implies q = 0.4$, then $p = 0.3$ **A1**

Finding $\mathrm{Var}(X)$**:** Using GDC (1-Var Stats, x-list: $-1, 0, 2, 3$; freq list: $0.3, 0.2, 0.4, 0.1$):

$\mathsf{E}(X^2) = 1(0.3) + 0(0.2) + 4(0.4) + 9(0.1) = 0.3 + 0 + 1.6 + 0.9 = 2.8$ **M1**

$\mathrm{Var}(X) = 2.8 - (0.8)^2 = 2.8 - 0.64 = 2.16$ **A1**

Q30 — E(X²) then solve for transformation parameters [HARD]

(a) $\mathrm{Var}(X) = \mathsf{E}(X^2) - [\mathsf{E}(X)]^2$

$4 = \mathsf{E}(X^2) - 9 \implies \mathsf{E}(X^2) = 13$ **A1**

(b) $\mathsf{E}(W) = a\mathsf{E}(X) + b = 3a + b = 10$ **M1**

$\mathsf{Var}(W) = a^2\mathsf{Var}(X) = 4a^2 = 100$ **M1**

$a^2 = 25 \implies a = 5$ (since $a > 0$) **A1**

Substituting into $3a + b = 10$:

$15 + b = 10 \implies b = -5$ **A1**

Note: Award M1 for correctly writing $\mathsf{E}(W) = 3a + b = 10$ *and M1 for* $\mathsf{Var}(W) = 4a^2 = 100$*. The constraint* $a > 0$ *rules out* $a = -5$.