IB Math AA HL — Topic 4
Continuous Random Variables

Probability Density Functions · Median & Mode · Mean & Variance

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Grand Total: 120 marks

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Name:_____ Date:_____

Score:_____/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

Validity of a pdf: $f(x) \geq 0$ for all x in the support and $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Probability: $P(a < X < b) = \displaystyle\int_{a}^{b} f(x)\,dx$; note $P(X = a) = 0$ for any continuous X.

CDF and pdf link: $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\,dt$ and $f(x) = F'(x)$.

Median m: $\displaystyle\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$, equivalently $F(m) = \frac{1}{2}$; reject roots outside the support.

Mode: value of x that maximises $f(x)$ on its support (set $f'(x) = 0$ or inspect endpoints/vertex).

Mean and Variance: $\mathrm{E}(X) = \displaystyle\int_{-\infty}^{\infty} x f(x)\,dx$; $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ where $\mathrm{E}(X^2) = \displaystyle\int_{-\infty}^{\infty} x^2 f(x)\,dx$.

Linear transformation: $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$; $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.

GDC Tips. *Definite integral for probability:* Use fnInt(f(x),x,a,b) (TI) or the $\int_a^b$ template in the Math menu. Always set up the integral algebraically first, then evaluate. *Solving for the median numerically:* Store $F(x) - 0.5$ and use solve(or the Equation Solver; supply an initial guess inside the support. Reject any solution outside the support. *Computing* $\mathrm{E}(X)$ *and* $\mathrm{E}(X^2)$*:* Enter fnInt(x*f(x),x,a,b) and fnInt(x^2*f(x),x,a,b) separately; subtract $[\mathrm{E}(X)]^2$ for the variance. *Piecewise pdfs:* Split the integral at every join; use two separate fnInt calls and add the results. Verify the total equals 1 before proceeding. *Finding k numerically:* Define $g(k) = \int f(x)\,dx - 1$ and use solve(g(k)=0,k); confirm $f(x) \geq 0$ on the support for the value found.

Section A — Easy EASY (Q1–Q10, 3 marks each = 30 marks)

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1. EASY No Calculator [3 marks]

The continuous random variable X has probability density function

$$f(x) = kx^2, \qquad 0 \leq x \leq 3,$$

and $f(x) = 0$ otherwise. Find the value of k.

2. EASY No Calculator [3 marks]

The continuous random variable X has probability density function

$$f(x) = \frac{3}{8}x^2, \qquad 0 \leq x \leq 2,$$

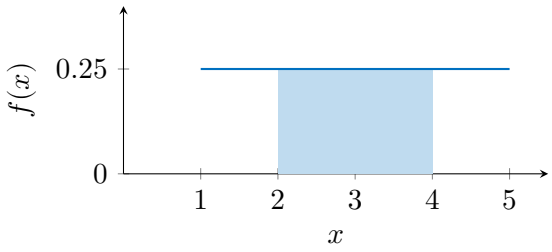
and $f(x) = 0$ otherwise. Find the mode of X, justifying your answer.

3. **EASY** **Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{1}{4}, \qquad 1 \leq x \leq 5,$$

and $f(x) = 0$ otherwise. The diagram below shows $f(x)$ with the shaded region representing $P(2 \leq X \leq 4)$.



Find $P(2 \leq X \leq 4)$.

4. **EASY** **Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{3}{26}x^2, \qquad 1 \leq x \leq 3,$$

and $f(x) = 0$ otherwise. Find $\mathrm{E}(X)$.

5. **EASY** **No Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{1}{2}, \qquad 0 \leq x \leq 2,$$

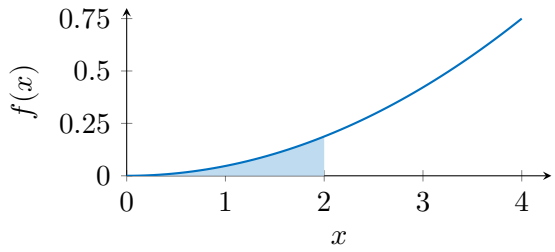
and $f(x) = 0$ otherwise. Find the median of X.

6. **EASY** **Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{3}{64}x^2, \qquad 0 \leq x \leq 4,$$

and $f(x) = 0$ otherwise. The diagram below shows $f(x)$ with the shaded region representing $P(0 \leq X \leq 2)$.



Find $P(0 \leq X \leq 2)$.

7. **EASY** **No Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{3}{2}x(2-x), \qquad 0 \leq x \leq 1,$$

and $f(x) = 0$ otherwise. Find the mean $\mathrm{E}(X)$.

8. **EASY** **Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{1}{20}(x+2), \qquad 1 \leq x \leq 5,$$

and $f(x) = 0$ otherwise. Find $\mathrm{E}(X)$.

9. **EASY** **Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = kx, \qquad 0 \leq x \leq 4,$$

and $f(x) = 0$ otherwise, where k is a positive constant.

(a) Write down the value of k. [1]

(b) Hence find $P(1 \leq X \leq 3)$. [2]

10. **EASY** **Calculator** **[3 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{2}{9}(5x - x^2 - 4), \qquad 1 \leq x \leq 4,$$

and $f(x) = 0$ otherwise. Find the median m of X.

Section B — Medium MEDIUM (Q11–Q20, 4 marks each = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

The continuous random variable X has probability density function

$$f(x) = k(4x - x^2), \qquad 0 \leq x \leq 4,$$

and $f(x) = 0$ otherwise. Find the value of k.

12. MEDIUM No Calculator **[4 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{3}{16}x^2, \qquad 0 \leq x \leq c,$$

and $f(x) = 0$ otherwise, where c is a positive constant.

(a) Write down the value of c. [1]

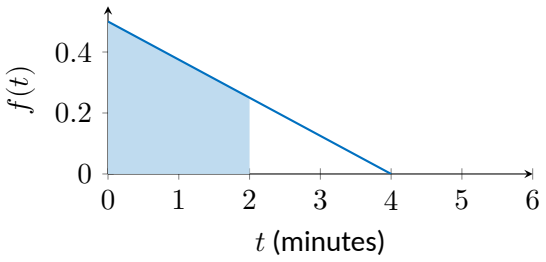
(b) Find $\mathrm{E}(X)$. [3]

13. MEDIUM Calculator **[4 marks]**

The continuous random variable T (in minutes) models the waiting time at a coffee shop. Its probability density function is

$$f(t) = \frac{1}{2} - \frac{1}{8}t, \qquad 0 \leq t \leq 4,$$

and $f(t) = 0$ otherwise. The diagram below shows $f(t)$ with the shaded region representing $P(T < 2)$.



(a) Find $P(T < 2)$. **[2]**

(b) Find the median waiting time. **[2]**

14. MEDIUM No Calculator **[4 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{3}{2}x(2 - x), \qquad 0 \leq x \leq 1,$$

and $f(x) = 0$ otherwise. Find $\text{Var}(X)$.

15. MEDIUM Calculator **[4 marks]**

The continuous random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ ax^3 + bx^2 & 0 \leq x \leq 2 \\ 1 & x > 2, \end{cases}$$

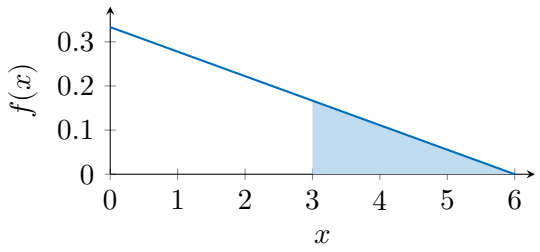
where a and b are constants. Given that $f(x)$, the probability density function of X, satisfies $f(1) = \frac{3}{4}$, find the values of a and b.

16. MEDIUM Calculator **[4 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{1}{18}(6 - x), \qquad 0 \leq x \leq 6,$$

and $f(x) = 0$ otherwise. The probability density function is shown below, with the shaded region representing $P(X > 3)$.



(a) Find $P(X > 3)$. **[2]**

(b) Find the mode of X. **[1]**

(c) Find $\mathrm{E}(X)$. **[1]**

17. MEDIUM Calculator [4 marks]

The continuous random variable X has probability density function

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0,$$

where $\lambda = 0.4$, and $f(x) = 0$ otherwise.

(a) Find $P(1 \leq X \leq 3)$. [2]

(b) Find the median of X, giving your answer to three significant figures. [2]

18. MEDIUM No Calculator [4 marks]

The continuous random variable X has probability density function

$$f(x) = kx(9 - x^2), \qquad 0 \leq x \leq 3,$$

and $f(x) = 0$ otherwise.

(a) Show that $k = \frac{4}{81}$. [2]

(b) Find the mode of X. [2]

19. MEDIUM Calculator **[4 marks]**

The continuous random variable X has probability density function

$$f(x) = \frac{1}{15}(x^2 + 2), \qquad 0 \leq x \leq 3,$$

and $f(x) = 0$ otherwise. Using your GDC, find

(a) $\mathrm{E}(X)$, [2]

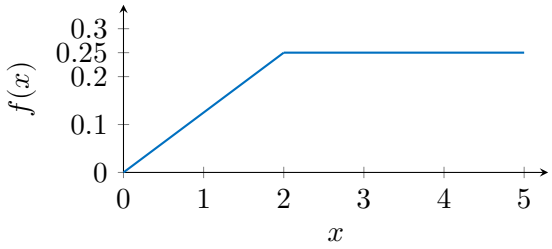
(b) $\mathrm{Var}(X)$. [2]

20. MEDIUM Calculator **[4 marks]**

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{x}{8} & 0 \leq x \leq 2 \\ \frac{1}{4} & 2 < x \leq 5, \end{cases}$$

and $f(x) = 0$ otherwise.



(a) Verify that f is a valid probability density function. [2]

(b) Find the median of X. [2]

Section C — Hard HARD (Q21–Q30, 5 marks each = 50 marks)

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21. HARD No Calculator **[5 marks]**

The continuous random variable X has probability density function

$$f(x) = \begin{cases} k(4x - x^3) & 0 \leq x \leq 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a positive constant.

(a) Show that $k = \frac{1}{4}$. [2]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [3]

22. **HARD** **Calculator** **[5 marks]**

The continuous random variable T (in minutes) models the time a server takes to process a request. Its probability density function is

$$f(t) = \begin{cases} at^2 & 0 \le t \le 1 \\ b(2-t) & 1 < t \le 2 \\ 0 & \text{otherwise,} \end{cases}$$

where a and b are positive constants. Given that f is continuous at $t = 1$ and that f is a valid pdf, find the values of a and b. Then find $P(T > 1.5)$.

23. **HARD** **No Calculator** **[5 marks]**

The continuous random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2(3-2x) & 0 \le x \le 1 \\ 1 & x > 1. \end{cases}$$

(a) Write down $P(X \le 0.5)$. [1]

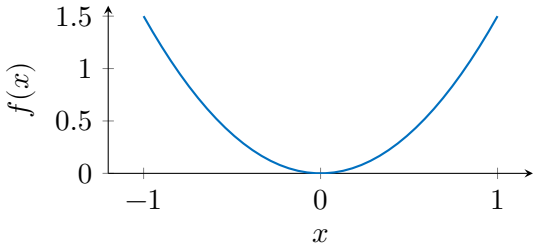
(b) Find the probability density function $f(x)$. [2]

(c) Find the mode of X. [2]

24. **HARD** **No Calculator** **[5 marks]**

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{3}{2}x^2 & -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$



(a) Find the median m of X. [3]

(b) Find Var(X). [2]

25. **HARD** **Calculator** **[5 marks]**

The continuous random variable W has probability density function

$$f(w) = \begin{cases} cw(3-w)^2 & 0 \leq w \leq 3 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a positive constant. Using your GDC:

(a) Show that $c = \frac{4}{27}$. [2]

(b) Find E(W) and E(W^2), and hence find Var(W). [3]

26. HARD Calculator **[5 marks]**

A factory produces components whose lifetime L (in hundreds of hours) has probability density function

$$f(l) = \begin{cases} kle^{-l^2/2} & l \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a positive constant.

(a) Show that $k = 1$. [2]

(b) Find the median lifetime. Give your answer in hundreds of hours. [2]

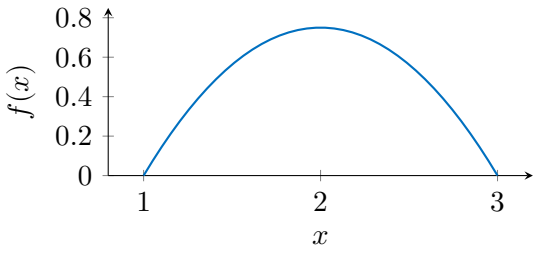
(c) Interpret your answer to part (b) in context. [1]

27. HARD Calculator **[5 marks]**

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \dfrac{3(x-1)(3-x)}{4} & 1 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

The figure below shows the graph of f.



(a) Write down the mode of X. [1]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4]

28. HARD No Calculator **[5 marks]**

The continuous random variable X has probability density function $f(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$ and $f(x) = 0$ otherwise.

(a) Build the cumulative distribution function $F(x)$ for all $x \in \mathbb{R}$. [2]

(b) Find the median m of X, giving an exact answer. [2]

(c) Hence find $P(X > m)$. [1]

29. HARD No Calculator **[5 marks]**

The continuous random variable Y has probability density function

$$f(y) = \begin{cases} \frac{1}{2}\sin y & 0 \leq y \leq \pi \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that f is a valid probability density function. [2]

(b) Find $\mathrm{E}(Y)$. [2]

(c) Find $P\left(\frac{\pi}{4} < Y < \frac{3\pi}{4}\right)$, giving an exact answer. [1]

30. HARD Calculator **[5 marks]**

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{3}{26}x^2 & 1 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = 5X - 2$.

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [3]

(b) Find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$. [2]

Mark Schemes

Q1 — Find k for a Polynomial pdf EASY

For a valid pdf, both $f(x) \geq 0$ on $[0, 3]$ (satisfied since $k > 0$) and $\int_0^3 kx^2\,dx = 1$.

$k\int_0^3 x^2\,dx = k\left[\frac{x^3}{3}\right]_0^3 = k \cdot 9 = 1$ **M1**

$9k = 1$ **(A1)**

$k = 1/9$ **A1**

Q2 — Mode of a CRV by Calculus EASY

Since $f(x) = \frac{3}{8}x^2$ is an increasing function on $[0, 2]$, $f'(x) = \frac{3}{4}x > 0$ for $x \in (0, 2)$. **M1**

Therefore f is strictly increasing on the support $[0, 2]$, so the maximum occurs at $x = 2$. **R1**

Mode $= 2$ **A1**

Q3 — Probability for a Uniform CRV EASY

$P(2 \leq X \leq 4) = \int_2^4 \frac{1}{4}\,dx$ **M1**

$= \frac{1}{4}(4 - 2) = \frac{1}{4} \times 2$ **(A1)**

$P(2 \leq X \leq 4) = 1/2 = 0.5$ **A1**

Note: Accept 0.5.

Q4 — Mean of a Polynomial CRV EASY

Using GDC to evaluate $\mathrm{E}(X) = \int_1^3 x \cdot \frac{3}{26}x^2\,dx = \frac{3}{26}\int_1^3 x^3\,dx$ **M1**

GDC: fnInt$\left(\frac{3}{26}x^3, x, 1, 3\right)$ gives $\frac{3}{26}\left[\frac{x^4}{4}\right]_1^3 = \frac{3}{26} \cdot \frac{80}{4} = \frac{240}{104}$ **(A1)**

$\mathrm{E}(X) = 30/13 \approx 2.31$ **A1**

Note: Accept $30/13$ *or* 2.31.

Q5 — Median of a Uniform CRV EASY

The median m satisfies $\int_0^m \frac{1}{2}\,dx = \frac{1}{2}$ **M1**

$\frac{1}{2}m = \frac{1}{2}$ **(A1)**

$m = 1$ **A1**

Note: $m = 1 \in [0, 2]$, *so it lies within the support.*

Q6 — Probability for a Quadratic CRV EASY

Using GDC to evaluate $P(0 \le X \le 2) = \int_0^2 \frac{3}{64}x^2\,dx$ **M1**

GDC: fnInt$\left(\frac{3}{64}x^2, x, 0, 2\right)$ gives $\frac{3}{64}\left[\frac{x^3}{3}\right]_0^2 = \frac{3}{64}\cdot\frac{8}{3} = \frac{8}{64}$ **(A1)**

$P(0 \le X \le 2) = 1/8 = 0.125$ **A1**

Note: Accept 0.125.

Q7 — Mean of a Product Polynomial CRV EASY

$\mathrm{E}(X) = \int_0^1 x \cdot \frac{3}{2}x(2-x)\,dx = \frac{3}{2}\int_0^1 (2x^2 - x^3)\,dx$ **M1**

$= \frac{3}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{3}{2}\cdot\frac{5}{12}$ **(A1)**

$\mathrm{E}(X) = 5/8 = 0.625$ **A1**

Q8 — Mean of a Linear CRV EASY

With $f(x) = 1/20\,(x+2)$ on $[1, 5]$, using GDC to evaluate $\mathrm{E}(X) = \int_1^5 x \cdot \frac{1}{20}(x+2)\,dx = \frac{1}{20}\int_1^5 (x^2 + 2x)\,dx$ **M1**

GDC: fnInt$\left(\frac{1}{20}(x^2+2x), x, 1, 5\right)$ gives $\frac{1}{20}\left[\frac{x^3}{3} + x^2\right]_1^5 = \frac{1}{20}\left(\frac{124}{3}\right)$ **(A1)**

$\mathrm{E}(X) = 49/15 \approx 3.27$ **A1**

Note: Accept $49/15$ *or* 3.27.

Q9 — Find k Then Probability for a Linear pdf EASY

(a) $\displaystyle\int_0^4 kx\,dx = k\left[\frac{x^2}{2}\right]_0^4 = 8k = 1$, so $k = 1/8$ **A1**

(b) Using GDC: fnInt$\left(\frac{1}{8}x, x, 1, 3\right)$ **M1**

$$P(1 \le X \le 3) = \frac{1}{8}\left[\frac{x^2}{2}\right]_1^3 = \frac{1}{8}\left(\frac{9}{2} - \frac{1}{2}\right) = \frac{1}{8} \times 4 = 1/2$$ **A1**

Note: Accept 0.5.

Q10 — Median of a Quadratic CRV EASY

With $f(x) = 2/9\,(5x - x^2 - 4)$ on $[1, 4]$, the median m satisfies $\displaystyle\int_1^m \frac{2}{9}(5x - x^2 - 4)\,dx = \frac{1}{2}$ **M1**

Using GDC: solve fnInt$\left(\frac{2}{9}(5x - x^2 - 4), x, 1, m\right) = 0.5$ for $m \in [1, 4]$ **(A1)**

$m = 5/2$ **A1**

Note: $m = 5/2 = 2.5 \in [1, 4]$, so it lies within the support. By symmetry of f about $x = 2.5$, the answer may also be verified analytically.

Q11 — Finding k for a Polynomial pdf MEDIUM

For f to be a valid pdf, $\displaystyle\int_0^4 k(4x - x^2)\,dx = 1$ and $f(x) \ge 0$ on $[0, 4]$. **M1**

Integrate: $\displaystyle k\left[2x^2 - \frac{x^3}{3}\right]_0^4 = 1$

$$k\left(2(16) - \frac{64}{3}\right) = 1 \implies k\left(32 - \frac{64}{3}\right) = 1$$ **A1**

$$k \cdot \frac{32}{3} = 1$$ **A1**

$k = 3/32$ **A1**

Note: Award full marks only if the total integral condition is stated. The condition $f(x) \ge 0$ on $[0, 4]$ must be acknowledged.

Q12 — Value of c and E(X) MEDIUM

(a) For a valid pdf: $\displaystyle\int_0^c \frac{3}{16}x^2\,dx = 1$

$\frac{3}{16} \cdot \frac{c^3}{3} = 1 \implies \frac{c^3}{16} = 1 \implies c^3 = 16$

$c = \sqrt[3]{16}$ **A1**

(b) $\mathrm{E}(X) = \int_0^c \frac{3}{16}x^3\,dx = \frac{3}{16} \cdot \frac{c^4}{4} = \frac{3c^4}{64}$ **M1**

$c^4 = (c^3) \cdot c = 16\sqrt[3]{16} = 16^{4/3}$ **A1**

$\mathrm{E}(X) = \frac{3 \cdot 16^{4/3}}{64} = \frac{3}{4} \cdot 16^{1/3} = \frac{3\sqrt[3]{16}}{4}$ **A1**

Note: Accept equivalent exact forms such as $\frac{3}{4}c$ where $c = \sqrt[3]{16}$.

Q13 — Probability and Median of a Linear pdf MEDIUM

(a) $P(T < 2) = \int_0^2 \left(\frac{1}{2} - \frac{1}{8}t\right) dt$ **M1**

Using GDC (numerical integration, limits 0 to 2, integrand $0.5 - 0.125t$):

$= \left[\frac{t}{2} - \frac{t^2}{16}\right]_0^2 = 1 - \frac{4}{16}$

$P(T < 2) = 3/4$ **A1**

(b) Set $\int_0^m \left(\frac{1}{2} - \frac{1}{8}t\right) dt = \frac{1}{2}$ **M1**

$\frac{m}{2} - \frac{m^2}{16} = \frac{1}{2}$

Using GDC to solve: $m \approx 1.17$ minutes (to 3 s.f.) **A1**

Note: Exact solution: $m^2 - 8m + 8 = 0 \implies m = 4 - 2\sqrt{2} \approx 1.17$; accept exact or decimal form. The root $m = 4 + 2\sqrt{2} \approx 6.83$ lies outside $[0, 4]$ and must be rejected.

Q14 — Variance of a Quadratic pdf MEDIUM

$\mathrm{E}(X) = \int_0^1 \frac{3}{2}x^2(2 - x)\,dx = \frac{3}{2}\int_0^1 (2x^2 - x^3)\,dx$ **M1**

$= \frac{3}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{3}{2} \cdot \frac{5}{12} = 5/8$ **A1**

$\mathrm{E}(X^2) = \int_0^1 \frac{3}{2}x^3(2 - x)\,dx = \frac{3}{2}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2} \cdot \frac{3}{10} = 9/20$ **M1**

$\mathrm{Var}(X) = \frac{9}{20} - \left(\frac{5}{8}\right)^2 = \frac{9}{20} - \frac{25}{64} = \frac{288}{640} - \frac{250}{640}$

$\mathrm{Var}(X) = 19/320$ **A1**

Q15 — CDF with Unknown Constants MEDIUM

Differentiate: $f(x) = F'(x) = 3ax^2 + 2bx$ for $0 \le x \le 2$ **M1**

Condition $f(1) = \frac{3}{4}$: $3a + 2b = 3/4$ **A1**

Condition $F(2) = 1$: $8a + 4b = 1$ **M1**

Using GDC to solve the system $\begin{cases} 3a + 2b = 3/4 \\ 8a + 4b = 1 \end{cases}$:

From the second equation: $2a + b = \frac{1}{4}$, so $b = \frac{1}{4} - 2a$. Substituting: $3a + 2\left(\frac{1}{4} - 2a\right) = \frac{3}{4} \implies$

$3a + \frac{1}{2} - 4a = \frac{3}{4} \implies -a = \frac{1}{4} \implies a = -1/4, b = 3/4$ **A1**

Q16 — Probability Mode and Mean of a Linear pdf MEDIUM

(a) $P(X > 3) = \int_3^6 \frac{1}{18}(6 - x)\,dx$ **M1**

Using GDC (numerical integration, limits 3 to 6, integrand $(6 - x)/18$):

$= \frac{1}{18}\left[6x - \frac{x^2}{2}\right]_3^6 = \frac{1}{18}\left[(36 - 18) - \left(18 - \frac{9}{2}\right)\right] = \frac{1}{18} \cdot \frac{9}{2}$

$P(X > 3) = 1/4$ **A1**

(b) $f(x) = \frac{1}{18}(6 - x)$ is strictly decreasing on $[0, 6]$, so the mode is $x = 0$. **A1**

(c) $\mathrm{E}(X) = \int_0^6 \frac{x(6 - x)}{18}\,dx$. Using GDC: $\mathrm{E}(X) = 2$ **A1**

Note: For part (c), GDC command: numerical integration, integrand $x(6 - x)/18$, limits 0 to 6, giving 2.

Q17 — Exponential pdf: Probability and Median MEDIUM

(a) $P(1 \le X \le 3) = \int_1^3 0.4e^{-0.4x}\,dx$ **M1**

Using GDC (numerical integration, integrand $0.4e^{-0.4x}$, limits 1 to 3): unrounded $e^{-0.4} - e^{-1.2} = 0.369126\ldots$

$P(1 \le X \le 3) \approx 0.369$ **A1**

Note: The exact value is $e^{-0.4} - e^{-1.2}$; do not round to $e^{-1.2}$ alone.

(b) Set $\int_0^m 0.4e^{-0.4x}\,dx = 0.5$:

$1 - e^{-0.4m} = 0.5$ **M1**

Solving with GDC (equation solver or algebra): $e^{-0.4m} = 0.5 \implies m = \frac{\ln 2}{0.4}$, unrounded $1.7328\ldots$

Median ≈ 1.73 minutes **A1**

Q18 — Show k and Find Mode MEDIUM

(a) For a valid pdf: $\int_0^3 kx(9-x^2)\,dx = 1$ **M1**

$$k\left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = k\left(\frac{81}{2} - \frac{81}{4}\right) = k \cdot \frac{81}{4} = 1$$

$k = 4/81$ **AG**

Note: Both conditions ($\int = 1$ and $f(x) \geq 0$ on $[0,3]$, since $9 - x^2 \geq 0$ for $x \in [0,3]$) must be cited for full credit on (a).

(b) To find the mode, maximise $f(x) = \frac{4}{81}x(9-x^2) = \frac{4}{81}(9x - x^3)$. **M1**

$f'(x) = \frac{4}{81}(9 - 3x^2) = 0 \implies x^2 = 3 \implies x = \sqrt{3}$ (since $x \in [0,3]$)

Mode $= \sqrt{3}$ **A1**

Note: sqrt(3) ≈ 1.73.

Q19 — E(X) and Var(X) by GDC MEDIUM

With $f(x) = 1/15\,(x^2+2)$ on $[0,3]$:

(a) $\mathrm{E}(X) = \int_0^3 \frac{1}{15}x(x^2+2)\,dx$ **M1**

Using GDC (numerical integration, integrand $\frac{1}{15}x(x^2+2)$, limits 0 to 3): unrounded 1.95 exactly.

$\mathrm{E}(X) = 1.95$ **A1**

Note: Exact value $39/20 = 1.95$.

(b) $\mathrm{E}(X^2) = \int_0^3 \frac{1}{15}x^2(x^2+2)\,dx$ **M1**

Using GDC (numerical integration, integrand $\frac{1}{15}x^2(x^2+2)$, limits 0 to 3): unrounded 4.44.

$\mathrm{Var}(X) = 4.44 - (1.95)^2 = 4.44 - 3.8025 = 0.6375 \approx 0.638$ **A1**

Note: Exact value $\mathrm{Var}(X) = 51/80 = 0.6375$, *which rounds to 0.638 (3 s.f.).*

Q20 — Validity and Median of a Piecewise pdf MEDIUM

(a) Check $f(x) \geq 0$: both pieces are non-negative on their respective domains ($x/8 \geq 0$ for $x \in [0,2]$; $1/4 > 0$ for $x \in (2,5]$). **M1**

Total area $= \int_0^2 \frac{x}{8}\,dx + \int_2^5 \frac{1}{4}\,dx = \left[\frac{x^2}{16}\right]_0^2 + \left[\frac{x}{4}\right]_2^5 = \frac{4}{16} + \left(\frac{5}{4} - \frac{2}{4}\right) = \frac{1}{4} + \frac{3}{4} = 1$

Since $f(x) \geq 0$ on its support and the total area is exactly 1, f is a valid pdf. **A1**

(b) Since $\int_0^2 \frac{x}{8}\,dx = \frac{1}{4} < \frac{1}{2}$, the median lies in $(2, 5]$. **M1**

$\frac{1}{4} + \frac{m-2}{4} = \frac{1}{2} \implies \frac{m-2}{4} = \frac{1}{4} \implies m - 2 = 1 \implies m = 3$

Median $= 3$ **A1**

Note: The two pieces agree at the join ($f(2^-) = f(2^+) = 1/4$), so f is also continuous, though continuity is not required for validity.

Q21 — Show k and Find E and Var HARD

(a) $\int_0^2 k(4x - x^3)\,dx = k\left[2x^2 - \frac{x^4}{4}\right]_0^2 = k(8 - 4) = 4k$ **M1**

Setting $4k = 1$ gives $k = 1/4$.

Since $4x - x^3 = x(4 - x^2) \geq 0$ for $0 \leq x \leq 2$, $f \geq 0$. **AG**

(b) $\mathrm{E}(X) = \int_0^2 x \cdot \frac{1}{4}(4x - x^3)\,dx = \frac{1}{4}\int_0^2 (4x^2 - x^4)\,dx = \frac{1}{4}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$ **M1**

$= \frac{1}{4}\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{1}{4} \cdot \frac{64}{15}$

$\mathrm{E}(X) = 16/15$ **A1**

$\mathrm{E}(X^2) = \frac{1}{4}\int_0^2 (4x^3 - x^5)\,dx = \frac{1}{4}\left[x^4 - \frac{x^6}{6}\right]_0^2 = \frac{1}{4}\left(16 - \frac{64}{6}\right) = \frac{1}{4} \cdot \frac{32}{6} = 4/3$

$\mathrm{Var}(X) = \frac{4}{3} - \left(\frac{16}{15}\right)^2 = \frac{4}{3} - \frac{256}{225} = \frac{300 - 256}{225}$

$\mathrm{Var}(X) = 44/225$ **A1**

Q22 — Piecewise pdf: Continuity and Validity HARD

Continuity at $t = 1$: $f(1^-) = a(1)^2 = a$ and $f(1^+) = b(2 - 1) = b$, so $b = a$. **M1**

Total integral equals 1: $\int_0^1 at^2\,dt + \int_1^2 a(2 - t)\,dt = a\left[\frac{t^3}{3}\right]_0^1 + a\left[2t - \frac{t^2}{2}\right]_1^2$ **M1**

$= a \cdot \frac{1}{3} + a\left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right] = \frac{a}{3} + a \cdot \frac{1}{2} = \frac{5a}{6} = 1 \implies a = 6/5, b = 6/5$ **A1**

$P(T > 1.5) = \int_{1.5}^2 \frac{6}{5}(2 - t)\,dt$

Using GDC (fnInt, $\frac{6}{5}(2-t)$, t, 1.5, 2): result $= 0.15$ **M1**

$P(T > 1.5) = 0.150$ **A1**

Q23 — CDF to pdf and Mode HARD

(a) $P(X \leq 0.5) = F(0.5) = (0.5)^2(3 - 2(0.5)) = 0.25 \times 2 = 1/2$ **A1**

(b) $f(x) = F'(x) = \dfrac{d}{dx}\left[x^2(3-2x)\right] = 2x(3-2x) + x^2(-2) = 6x - 6x^2$ **M1**

$$f(x) = \begin{cases} 6x(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$ **A1**

(c) $f'(x) = 6(1-2x) = 0 \implies x = 1/2$ **M1**

$f''(x) = -12 < 0$, confirming a maximum. Mode $= 1/2$ **A1**

Q24 — Median and Variance of a Symmetric pdf HARD

(a) The median m satisfies $\displaystyle\int_{-1}^{m} \frac{3}{2}x^2\,dx = \frac{1}{2}$. **M1**

$$\left[\frac{x^3}{2}\right]_{-1}^{m} = \frac{1}{2} \implies \frac{m^3}{2} + \frac{1}{2} = \frac{1}{2} \implies m^3 = 0 \implies m = 0$$ **M1A1**

Note: $m = 0$ lies in $[-1, 1]$, so it is valid.

(b) $\mathrm{E}(X) = \displaystyle\int_{-1}^{1} x \cdot \frac{3}{2}x^2\,dx = \frac{3}{2}\int_{-1}^{1} x^3\,dx = 0$ (odd integrand on symmetric interval)

$\mathrm{E}(X^2) = \displaystyle\int_{-1}^{1} \frac{3}{2}x^4\,dx = \frac{3}{2} \cdot \frac{2}{5} = 3/5$ **M1**

$\mathrm{Var}(X) = \dfrac{3}{5} - 0^2 = 3/5$ **A1**

Q25 — Show c; E and Var via GDC HARD

(a) $\displaystyle\int_0^3 cw(3-w)^2\,dw = c\int_0^3 (9w - 6w^2 + w^3)\,dw$ **M1**

Using GDC (fnInt, $w(3-w)^2$, w, 0, 3) $= \dfrac{27}{4}$.

$c \cdot \dfrac{27}{4} = 1 \implies c = 4/27$.

Since $w(3-w)^2 \geq 0$ for $0 \leq w \leq 3$, $f \geq 0$. **AG**

(b) Using GDC (fnInt, $\frac{4}{27} \cdot w^2(3-w)^2$, w, 0, 3): $\mathrm{E}(W) = 1.2 = 6/5$ **M1A1**

Using GDC (fnInt, $\frac{4}{27} \cdot w^3(3-w)^2$, w, 0, 3): $\mathrm{E}(W^2) = 1.8 = 9/5$
$\mathrm{Var}(W) = \frac{9}{5} - \left(\frac{6}{5}\right)^2 = \frac{9}{5} - \frac{36}{25} = \frac{45-36}{25}$
$\mathrm{Var}(W) = 9/25 = 0.360$ **A1**

Q26 — Exponential-type pdf: Show k Median and Interpret HARD

(a) $\int_0^\infty le^{-l^2/2}\,dl$: let $u = l^2/2$, $du = l\,dl$, so integral $= \int_0^\infty e^{-u}\,du = 1$. **M1**

Therefore $k \cdot 1 = 1 \implies k = 1$.
Since $le^{-l^2/2} \geq 0$ for $l \geq 0$, $f \geq 0$. **AG**

(b) Median m satisfies $\int_0^m le^{-l^2/2}\,dl = \frac{1}{2}$.

$\left[-e^{-l^2/2}\right]_0^m = 1 - e^{-m^2/2} = \frac{1}{2}$ **M1**

$e^{-m^2/2} = \frac{1}{2} \implies m^2 = 2\ln 2 \implies m = \sqrt{2\ln 2} \approx 1.18$ hundreds of hours **A1**

(c) Half of all components last fewer than approximately 118 hours before failing. **R1**

Q27 — Mode by Symmetry and E and Var via GDC HARD

(a) The pdf is symmetric about $x = 2$ on $[1, 3]$, so the mode $= 2$. **A1**

(b) $\mathrm{E}(X)$: Using GDC (fnInt, $x \cdot \frac{3}{4}(x-1)(3-x)$, x, 1, 3): result $= 2.00$ **M1A1**

$\mathrm{E}(X^2)$: Using GDC (fnInt, $x^2 \cdot \frac{3}{4}(x-1)(3-x)$, x, 1, 3): result $= 4.20$
$\mathrm{Var}(X) = 4.20 - (2.00)^2 = 0.200$ **M1A1**

Note: Exact values are $\mathrm{E}(X) = 2$, $\mathrm{Var}(X) = 1/5$.

Q28 — CDF Exact Median and Symmetry HARD

(a) For $0 \leq x \leq 2$: $F(x) = \int_0^x \frac{3}{8}t^2\,dt = x^3/8$ **M1**

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3/8 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$ **A1**

(b) $F(m) = \frac{m^3}{8} = \frac{1}{2} \implies m^3 = 4 \implies m = \sqrt[3]{4} = 4^{1/3}$, i.e. $4^(1/3)$ **M1A1**

Note: $4^{1/3} \approx 1.587 \in [0, 2]$, *so valid.*

(c) By definition of the median, $P(X > m) = 1/2$. **A1**

Q29 — Sinusoidal pdf: Verify E and Exact Probability HARD

(a) $f(y) = \frac{1}{2}\sin y \geq 0$ for $0 \leq y \leq \pi$ since $\sin y \geq 0$ on $[0, \pi]$. **R1**

$$\int_0^{\pi} \frac{1}{2}\sin y\, dy = \frac{1}{2}\left[-\cos y\right]_0^{\pi} = \frac{1}{2}(1+1) = 1$$ **A1**

(b) $\mathrm{E}(Y) = \int_0^{\pi} \frac{y}{2}\sin y\, dy$. Integrating by parts: $u = y$, $dv = \frac{1}{2}\sin y\, dy$. **M1**

$$= \frac{1}{2}\left[-y\cos y + \sin y\right]_0^{\pi} = \frac{1}{2}\left[(-\pi\cos\pi + \sin\pi) - 0\right] = \frac{1}{2}(\pi)$$

$\mathrm{E}(Y) = \pi/2$ **A1**

(c) $P\left(\frac{\pi}{4} < Y < \frac{3\pi}{4}\right) = \frac{1}{2}\left[-\cos y\right]_{\pi/4}^{3\pi/4} = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$

$= sqrt(2)/2$ **A1**

Q30 — Linear Transform: E and Var of Z

(a) Using GDC (fnInt, $\frac{3}{26}x^3$, x, 1, 3): $\mathrm{E}(X) = \frac{3}{26}\cdot\frac{80}{4} = \frac{60}{26} = \frac{30}{13} \approx 2.31$ **M1A1**

Using GDC (fnInt, $\frac{3}{26}x^4$, x, 1, 3): $\mathrm{E}(X^2) = \frac{3}{26}\cdot\frac{242}{5} = \frac{726}{130} = \frac{363}{65} \approx 5.585$

$\mathrm{Var}(X) = \frac{363}{65} - \left(\frac{30}{13}\right)^2 = \frac{363}{65} - \frac{900}{169} = \frac{363\times 13 - 900\times 5}{845} = \frac{4719-4500}{845} = \frac{219}{845} \approx 0.259$ **A1**

(b) $\mathrm{E}(Z) = 5\,\mathrm{E}(X) - 2 = 5\times\frac{30}{13} - 2 = \frac{150}{13} - \frac{26}{13} = \frac{124}{13} \approx 9.54$ **M1A1**

$\mathrm{Var}(Z) = 5^2\,\mathrm{Var}(X) = 25\times\frac{219}{845} = \frac{5475}{845} = \frac{1095}{169} \approx 6.48$

Note: $\mathrm{Var}(Z) = 1095/169 \approx 6.48$.

Common Mistakes to Avoid

1. **Checking only one validity condition.** Students verify $\int f(x)\,dx = 1$ but forget to confirm $f(x) \geq 0$ on the support (or vice versa). Both conditions must be stated and checked for full marks.
2. **Forgetting to reject extraneous median roots.** Solving $F(m) = \frac{1}{2}$ often yields more than one root; every root outside the stated support must be explicitly rejected with the reason "outside the support".

3. **Computing variance as $\mathrm{E}(X^2)$ alone.** A common error is writing $\mathrm{Var}(X) = \mathrm{E}(X^2)$ and omitting $-[\mathrm{E}(X)]^2$. Both integrals must be shown and the subtraction performed.
4. **Applying the variance formula for linear transformations incorrectly.** Students write $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X) + b$ instead of $a^2\mathrm{Var}(X)$; the additive constant b has no effect on variance.
5. **Omitting the split at joins in piecewise pdfs.** When a pdf is defined in pieces, the total integral must be separated at every join. Treating the whole domain as a single expression gives an incorrect value.
6. **Confusing mode with mean.** The mode is the x-value that maximises $f(x)$, found by differentiation or inspection; it is not the same as $\mathrm{E}(X)$. Always maximise f, not $xf(x)$.