IB Math AA HL — Topic 5

Limits & Convergence

Idea of a Limit · Infinite Limits & Limits at Infinity · Continuity

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:________________________ Date:______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Limit at a point:** $\lim_{x \to a} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by taking x sufficiently close (but not equal) to a.
- **Factor-and-cancel (0/0 form):** For $x \neq a$, cancel the common factor, then substitute; e.g. $\lim_{x \to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x \to a}(x+a) = 2a$.
- **Limit at infinity of a rational function:** Divide numerator and denominator by the highest power of x; terms of the form $\frac{c}{x^n} \to 0$, giving a horizontal asymptote $y = \frac{a_n}{b_n}$.
- **l'Hôpital's Rule:** If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided the latter limit exists.
- **Continuity at $x = a$:** f is continuous at $x = a$ if and only if (i) $\lim_{x \to a^-} f(x)$ exists, (ii) $\lim_{x \to a^+} f(x)$ exists, (iii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.
- **One-sided infinite limits:** $\lim_{x \to a^+} \frac{1}{x-a} = +\infty$ and $\lim_{x \to a^-} \frac{1}{x-a} = -\infty$; determined by the sign of numerator and denominator near a.
- **Rationalising a surd limit:** Multiply by the conjugate $\frac{\sqrt{f(x)}+c}{\sqrt{f(x)}+c}$ to remove the $0/0$ indeterminate form before cancelling.

GDC Tips

- **Table of values near a point:** Use `Table` (or `2nd GRAPH` on TI-84; `G-Solv` → `Table` on Casio) with x values such as $a \pm 0.1,\ a \pm 0.01,\ a \pm 0.001$ to conjecture $\lim_{x \to a} f(x)$ numerically.
- **Graph-and-trace for end behaviour:** Plot $f(x)$ over a very large window (e.g. $x \in [-1000, 1000]$) and use `Trace` to read off the value the function approaches, confirming the horizontal asymptote equation $y = L$.
- **Checking one-sided limits near a vertical asymptote:** Graph f and zoom in from each side of the suspected asymptote $x = a$; observe the sign of $f(x)$ to confirm $+\infty$ or $-\infty$ on each side.
- **Verifying a continuity parameter:** After finding a parameter algebraically, substitute into `Y=` and use `Table` to confirm left and right values agree at the join point to at least 6 significant figures.
- **Numerical `nSolve` / `solve(` for parameter:** When continuity requires solving an equation with decimal coefficients, use `nSolve(equation, variable)` (TI-Nspire) or `Solve` (Casio CAS) to obtain the value to 3 s.f.

Common Mistakes to Avoid

1. **Applying l'Hôpital without verifying the indeterminate form.** l'Hôpital's Rule is only valid when the limit gives $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. Always substitute first and write the form explicitly; applying the rule to, say, a $\frac{3}{0}$ form produces a completely wrong answer and loses the verification mark.

2. **Stating asymptotes as bare numbers instead of equations.** Writing "the asymptote is 2" instead of "$y = 2$" or "$x = -3$" is incorrect notation in IB. Asymptotes must always be written as equations of lines.

3. **Cancelling without noting $x \neq a$.** When a common factor $(x - a)$ is cancelled to resolve a $0/0$ form, the working must state $x \neq a$ before the cancellation; omitting this is a logical error and can cost an accuracy mark.

4. **Incomplete continuity argument.** Checking only that the two one-sided limits are equal is not sufficient. All three ingredients must be stated and compared explicitly: $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, and $f(a)$, and shown to be equal.

5. **Confusing a removable discontinuity (hole) with a vertical asymptote.** A factor that cancels produces a hole at that x-value (removable discontinuity), not an asymptote. An asymptote arises only when the denominator factor does *not* cancel. The classification must be justified with a reason.

6. **Incorrect sign on a one-sided infinite limit.** Writing $\lim_{x\to a^+} \dfrac{-2}{x - a} = +\infty$ without checking the sign of the numerator is a common error. Always argue the sign of numerator and denominator separately as $x \to a$ from each side before concluding $+\infty$ or $-\infty$.

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Section A — Easy

EASY

1. The function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is defined for $x \neq 3$.

Complete the table of values below and hence write down $\lim_{x \to 3} f(x)$. *Calculator allowed* **[3 marks]**

x	2.9	2.99	3.01	3.1
$f(x)$				

2. Find $\lim_{x \to 4} \dfrac{x^2 - 16}{x - 4}$, showing all algebraic working. *Calculator not allowed* **[3 marks]**

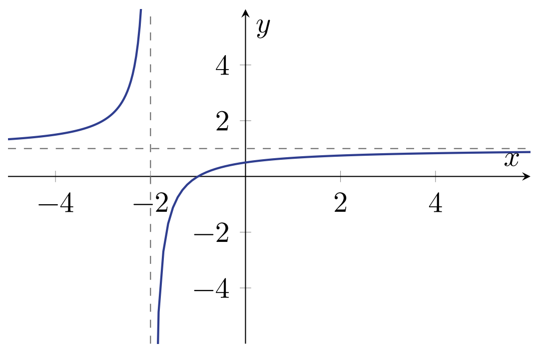
3. Find $\lim_{x \to \infty} \dfrac{3x^2 + 5x}{2x^2 - 1}$, justifying your answer by dividing numerator and denominator by the highest power of x. *Calculator not allowed* **[3 marks]**

4. The graph below shows a rational function $g(x)$. Using the graph, write down:

(a) the equation of the vertical asymptote,

(b) the equation of the horizontal asymptote.

Calculator allowed **[3 marks]**



5. A piecewise function is defined by

$$h(x) = \begin{cases} x^2 + 1, & x < 2 \\ 5, & x = 2 \\ 3x - 1, & x > 2. \end{cases}$$

Determine whether h is continuous at $x = 2$. Justify your answer by evaluating the left-hand limit, the right-hand limit, and the function value at $x = 2$. *Calculator allowed* **[3 marks]**

6. Find $\lim\limits_{x \to -3} \dfrac{x^2 + 2x - 3}{x + 3}$, showing all algebraic working. *Calculator not allowed* **[3 marks]**

7. The concentration (in mg/L) of a medicine in the bloodstream is modelled by

$$C(t) = \frac{4t+2}{t+1}, \quad t \geq 0,$$

where t is the time in hours. Use your GDC to find $\lim_{t\to\infty} C(t)$ and interpret the result in context. *Calculator allowed* **[3 marks]**

8. Consider the following table of values for a function $p(x)$ as x approaches 1.

x	0.9	0.99	0.999	1.001	1.01	1.1
$p(x)$	2.710	2.970	2.997	3.003	3.030	3.310

Write down $\lim_{x\to 1} p(x)$ and state whether p could be continuous at $x = 1$ if $p(1) = 3$, giving a reason. *Calculator allowed* **[3 marks]**

9. The function $q(x) = \dfrac{2x^2-8}{x-2}$ has a discontinuity at $x = 2$.
Classify the discontinuity as removable or non-removable, giving a reason. State the value to which $q(x)$ could be defined at $x = 2$ to make it continuous. *Calculator not allowed* **[3 marks]**

10. Use your GDC to investigate the one-sided limits of $r(x) = \dfrac{1}{x-5}$ as $x \to 5$.
Write down $\lim_{x\to 5^-} r(x)$ and $\lim_{x\to 5^+} r(x)$, and hence state whether $\lim_{x\to 5} r(x)$ exists. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. The function f is defined by

$$f(x) = \frac{x^2 - 9}{x - 3}, \quad x \neq 3.$$

(a) Show that $f(x) = x + 3$ for $x \neq 3$. *Calculator not allowed* **[1 marks]**

(b) Hence find $\lim_{x \to 3} f(x)$, stating the type of discontinuity at $x = 3$. *Calculator not allowed* **[3 marks]**

12. The table below shows values of $g(x)$ as x approaches 2 from each side. *Calculator allowed* **[4 marks]**

x	1.9	1.99	1.999	2.001	2.01	2.1
$g(x)$	4.61	4.9601	4.9960	5.0040	5.0401	5.41

(a) Write down the value of $\lim_{x \to 2} g(x)$ conjectured from the table. **[1 marks]**

(b) Given that $g(x) = x^2 + 1$, verify your answer algebraically and state the value of $g(2)$. **[2 marks]**

(c) Hence state whether g is continuous at $x = 2$, justifying your answer. **[1 marks]**

13. Consider the piecewise function

$$h(x) = \begin{cases} 3x + k & x < 1 \\ x^2 + 4 & x \geq 1 \end{cases}$$

where k is a real constant.

(a) Find $\lim_{x \to 1^-} h(x)$ in terms of k, and find $\lim_{x \to 1^+} h(x)$. *Calculator not allowed* **[2 marks]**

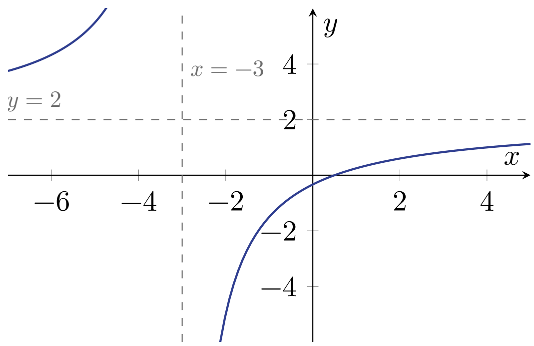
(b) Find the value of k for which h is continuous at $x = 1$. *Calculator not allowed* **[2 marks]**

14. Let $R(x) = \dfrac{4x^2 - 3x + 1}{2x^2 + 5}$.

(a) Find $\lim_{x \to \infty} R(x)$ by dividing numerator and denominator by the highest power of x. *Calculator not allowed* **[3 marks]**

(b) State the equation of the horizontal asymptote of R. *Calculator not allowed* **[1 marks]**

15. The graph of $y = \dfrac{2x-1}{x+3}$ is shown below, with a dashed vertical asymptote. *Calculator allowed* **[4 marks]**



(a) Determine $\lim_{x \to -3^-} \dfrac{2x-1}{x+3}$ and $\lim_{x \to -3^+} \dfrac{2x-1}{x+3}$, showing a sign argument for each. **[3 marks]**

(b) Write down $\lim_{x \to \infty} \dfrac{2x-1}{x+3}$ and state the equation of the horizontal asymptote. **[1 marks]**

16. Use l'Hôpital's rule to evaluate $\lim_{x \to 0} \dfrac{\sin(3x)}{5x}$. *Calculator not allowed* **[4 marks]**

17. A drug concentration model gives

$$C(t) = \frac{8t}{t^2 + 4}, \quad t \geq 0,$$

where C is in mg L^{-1} and t is in hours.

(a) Use your GDC to complete the table of values of $C(t)$ for the given values of t, rounding to three significant figures. *Calculator allowed* **[2 marks]**

t (hours)	0	5	20	50	100
C (mg L^{-1})					

(b) Hence conjecture $\lim_{t \to \infty} C(t)$ and confirm algebraically. **[2 marks]**

18. The function p is defined by

$$p(x) = \frac{x^2 - 5x + 6}{x^2 - 4}.$$

(a) Factorise the numerator and denominator of $p(x)$. *Calculator not allowed* **[1 marks]**

(b) Find $\lim_{x \to 2} p(x)$, stating why the result is valid even though $p(2)$ is undefined. *Calculator not allowed* **[2 marks]**

(c) Classify the discontinuity of p at $x = 2$, giving a reason. *Calculator not allowed* **[1 marks]**

19. Use your GDC to investigate the end behaviour of

$$f(x) = \frac{3x^3 - 2x + 7}{x^3 + 4x^2 - 1}$$

by evaluating $f(x)$ for large values of x. *Calculator allowed* **[4 marks]**

(a) Complete the table below using your GDC, rounding to four decimal places. **[2 marks]**

x	100	1000	10000	100000
$f(x)$				

(b) Hence state the conjectured value of $\lim_{x\to\infty} f(x)$. **[1 marks]**

(c) Confirm your answer by dividing numerator and denominator by x^3. **[1 marks]**

20. Apply l'Hôpital's rule to evaluate $\lim_{x\to\infty} \frac{x^2 + 3x}{e^x}$. *Calculator allowed* **[4 marks]**

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Section C — Hard

HARD

21. A function is defined by

$$f(x) = \frac{x^3 - 8}{x^2 - 4}, \quad x \neq \pm 2.$$

(a) Show that $\lim_{x \to 2} f(x) = 3$. **[2 marks]**

(b) Find $\lim_{x \to \infty} f(x)$, justifying your answer by dividing by the highest power of x. **[2 marks]**

(c) State, with a reason, whether the discontinuity at $x = -2$ is removable or non-removable. **[1 marks]**

22. The piecewise function g is defined by

$$g(x) = \begin{cases} \dfrac{\sin(kx)}{x} & x < 0 \\ 4x + k^2 & x \geq 0 \end{cases}$$

where k is a real constant. Find all values of k for which g is continuous at $x = 0$. *Calculator not allowed* **[5 marks]**

23. Let $h(x) = \dfrac{e^{3x} - 1 - 3x}{x^2}$.

(a) Show that $\lim_{x \to 0} h(x)$ is an indeterminate form of type $\dfrac{0}{0}$. **[1 marks]**

(b) Apply l'Hôpital's rule twice to evaluate $\lim_{x \to 0} h(x)$. **[4 marks]**

Calculator not allowed **[5 marks]**

24. A drug concentration in the bloodstream (in μg L^{-1}) is modelled by

$$C(t) = \frac{12t}{t^2+9}, \quad t \geq 0,$$

where t is time in hours. The table below shows values of $C(t)$ for large t.

t (hours)	10	50	100	500
$C(t)$ (μg L^{-1})	1.031	0.240	0.120	0.024

(a) Use the table to conjecture the value of $\lim_{t\to\infty} C(t)$. **[1 marks]**

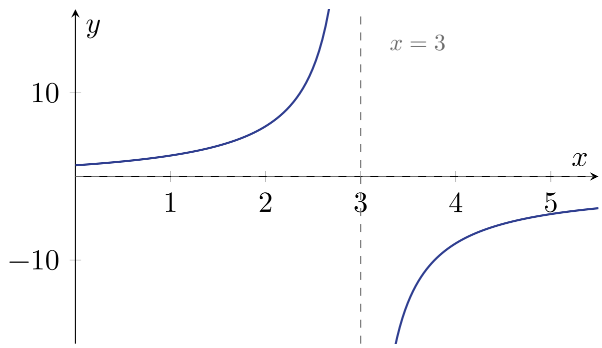
(b) Confirm your conjecture algebraically by dividing numerator and denominator by t^2. **[2 marks]**

(c) Using a GDC, find the time t at which $C(t)$ first equals 0.05, giving your answer to 3 significant figures. **[2 marks]**

Calculator allowed **[5 marks]**

25. Consider $\lim_{x\to 3^+} \frac{2x-1}{(x-3)^2}$ and $\lim_{x\to 3^-} \frac{x+4}{3-x}$.

The graph below shows the behaviour of $p(x) = \frac{x+4}{3-x}$ near $x = 3$.



(a) Evaluate $\lim_{x\to 3^+} \frac{2x-1}{(x-3)^2}$, giving a sign argument. **[2 marks]**

(b) Evaluate $\lim_{x\to 3^-} \frac{x+4}{3-x}$, giving a sign argument and using the graph to verify. **[2 marks]**

(c) Justify whether $\lim_{x\to 3} \frac{x+4}{3-x}$ exists. **[1 marks]**

Calculator allowed **[5 marks]**

26. Let $L = \lim_{x\to\infty} \dfrac{5x^3 - 2x + 1}{3x^3 + 7x^2 - 4}$.

(a) Show that $L = \dfrac{5}{3}$ by dividing every term by x^3. **[3 marks]**

(b) Hence state the equation of the horizontal asymptote of $f(x) = \dfrac{5x^3 - 2x + 1}{3x^3 + 7x^2 - 4}$. **[1 marks]**

(c) Using a GDC, verify the result by graphing $f(x)$ and confirming that the curve approaches the horizontal asymptote as $x \to \infty$. State the GDC command used. **[1 marks]**

Calculator allowed **[5 marks]**

27. The function $q(x)$ is defined by

$$q(x) = \begin{cases} ax^2 + b & x \leq 1 \\ \dfrac{3x^3 - 3}{x - 1} & x > 1 \end{cases}$$

where a and b are real constants. Given that q is continuous at $x = 1$ and $\lim\limits_{x \to -\infty} q(x) = 7$, find the values of a and b. *Calculator not allowed* **[5 marks]**

28. Consider $\lim_{x \to 0} \dfrac{\sqrt{x+4}-2}{x}$.

(a) Show that this limit is of the form $\dfrac{0}{0}$ as $x \to 0$. **[1 marks]**

(b) By rationalising the numerator, evaluate the limit exactly. **[3 marks]**

(c) Using a GDC, complete the table of values below and verify your answer numerically. State the GDC operation used. **[1 marks]**

x	-0.1	-0.01	0.01	0.1
$\dfrac{\sqrt{x+4}-2}{x}$				

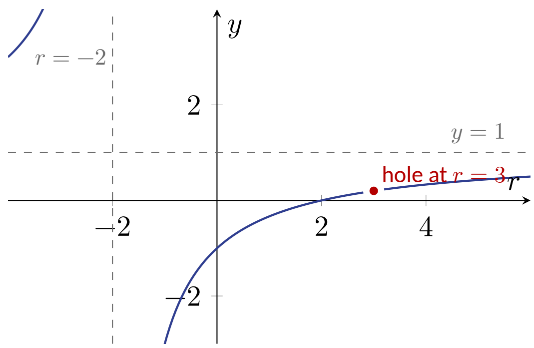
Calculator allowed **[5 marks]**

29. Using l'Hôpital's rule, evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$, showing all steps including verification of the indeterminate form at each application of the rule. *Calculator not allowed* **[5 marks]**

30. A population model gives the fraction of a habitat occupied as

$$F(r) = \frac{r^2 - 5r + 6}{r^2 - r - 6}, \quad r \neq -2,\ r \neq 3,$$

where r is a growth-rate parameter. The graph below shows $F(r)$.



(a) Factorise the numerator and denominator of $F(r)$ completely. **[1 marks]**

(b) Hence find $\lim_{r \to 3} F(r)$, noting that $r \neq 3$ during cancellation. **[2 marks]**

(c) Classify the discontinuity at $r = 3$ and the discontinuity at $r = -2$, giving a reason for each. **[2 marks]**

Calculator allowed **[5 marks]**

Worked Solutions

Q1 — Table of values limit [EASY]

Evaluating $f(x) = \dfrac{x^2 - 9}{x - 3}$ at each given x-value (GDC table command): **M1**

x	2.9	2.99	3.01	3.1
$f(x)$	5.9	5.99	6.01	6.1

A1

$\lim_{x \to 3} f(x) = 6$ **A1**

Q2 — Factor-and-cancel limit [EASY]

Factorise the numerator, noting $x \neq 4$:

$$\frac{x^2 - 16}{x - 4} = \frac{(x-4)(x+4)}{x-4} = x + 4, \quad x \neq 4$$

M1

Cancel the common factor $(x - 4)$: **A1**

$$\lim_{x \to 4} (x + 4) = 4 + 4 = 8$$

A1

Q3 — Limit at infinity by highest power [EASY]

Divide numerator and denominator by x^2:

$$\frac{3x^2 + 5x}{2x^2 - 1} = \frac{3 + \frac{5}{x}}{2 - \frac{1}{x^2}}$$

M1

As $x \to \infty$, $\dfrac{5}{x} \to 0$ and $\dfrac{1}{x^2} \to 0$: **A1**

$$\lim_{x \to \infty} \frac{3x^2 + 5x}{2x^2 - 1} = \frac{3 + 0}{2 - 0} = \frac{3}{2}$$

A1

Q4 — Reading asymptotes from a graph [EASY]

(a) Reading the graph, the vertical asymptote is **A1**

$$x = -2$$

(b) Reading the graph, the horizontal asymptote is **A1**

$$y = 1$$

Both asymptotes stated as equations. **A1**

Note: Award A1A1 for both equations correct; award A1 for one correct equation only. Bare numbers without equation form earn no marks.

Q5 — Continuity of a piecewise function [EASY]

Left-hand limit: $\lim_{x\to 2^-} h(x) = 2^2 + 1 = 5$ **A1**

Right-hand limit: $\lim_{x\to 2^+} h(x) = 3(2) - 1 = 5$ **A1**

Function value: $h(2) = 5$.

Since $\lim_{x\to 2^-} h(x) = \lim_{x\to 2^+} h(x) = h(2) = 5$, the function h is continuous at $x = 2$. **A1**

Note: All three ingredients (left limit, right limit, function value) must be stated and compared to earn the final A1.

Q6 — Factor-and-cancel limit [EASY]

Factorise the numerator, noting $x \neq -3$:

$$\frac{x^2 + 2x - 3}{x + 3} = \frac{(x+3)(x-1)}{x+3} = x - 1, \quad x \neq -3$$

M1

Cancel the common factor $(x + 3)$: **A1**

$$\lim_{x\to -3}(x - 1) = -3 - 1 = -4$$

A1

Q7 — Limit at infinity in context [EASY]

On the GDC, enter $Y_1 = (4X+2)/(X+1)$ and inspect a table for large values of t (e.g. $t = 1000,\ 10000$): values approach 4. **M1**

$$\lim_{t\to\infty} C(t) = 4$$

A1

In context: as time increases without bound, the concentration of the medicine in the bloodstream approaches a long-term level of 4 mg/L. **A1**

Note: The contextual interpretation must reference the units and the meaning of the limit to earn the final A1.

Q8 — Numerical limit and continuity [EASY]

Reading the table, values of $p(x)$ approach 3 from both sides: **M1**

$$\lim_{x\to 1} p(x) = 3$$

A1

Since $\lim_{x\to 1} p(x) = 3 = p(1)$, the function p could be continuous at $x = 1$; all three conditions (left limit, right limit, function value) are equal to 3. **A1**

Q9 — Classifying a removable discontinuity [EASY]

Factorise, noting $x \neq 2$:

$$q(x) = \frac{2x^2 - 8}{x - 2} = \frac{2(x-2)(x+2)}{x-2} = 2(x+2), \quad x \neq 2$$

M1

The limit $\lim_{x\to 2} q(x) = 2(2+2) = 8$ exists (finite), so the discontinuity is **removable** (a hole, not a vertical asymptote). **A1**

Defining $q(2) = 8$ makes q continuous at $x = 2$. **A1**

Q10 — One-sided infinite limits [EASY]

On the GDC, use a table approaching $x = 5$ from the left (e.g. $x = 4.9,\ 4.99,\ 4.999$): values decrease without bound. **M1**

Sign argument: for $x < 5$, $(x-5) < 0$ so $r(x) \to -\infty$; for $x > 5$, $(x-5) > 0$ so $r(x) \to +\infty$.

$$\lim_{x\to 5^-} r(x) = -\infty, \qquad \lim_{x\to 5^+} r(x) = +\infty$$

A1

Since the left-hand and right-hand limits are not equal, $\lim_{x\to 5} r(x)$ does not exist. **A1**

Q11 — Removable discontinuity via factorisation [MEDIUM]

(a)

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3, \quad x \neq 3.$$

AG

(b) Since $f(x) = x + 3$ for all $x \neq 3$,

$$\lim_{x \to 3} f(x) = 3 + 3 = 6.$$

M1A1

Because the limit exists ($= 6$) but $f(3)$ is undefined, the discontinuity at $x = 3$ is a **removable** (hole) discontinuity. **R1**

Note: R1 requires the word "removable" (or "hole") with a reason.

Q12 — Numerical limit from table, continuity check [MEDIUM]

(a) Reading the table: values approach 5 from both sides.

$$\lim_{x \to 2} g(x) = 5.$$

A1

(b) Algebraically: $g(x) = x^2 + 1 \Rightarrow \lim_{x \to 2}(x^2 + 1) = 4 + 1 = 5.$ **M1**

$g(2) = 2^2 + 1 = 5.$ **A1**

(c) Since $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2) = 5$, all three conditions for continuity are satisfied, so g is continuous at $x = 2$. **R1**

Q13 — Continuity parameter for piecewise function [MEDIUM]

(a)

$$\lim_{x \to 1^-} h(x) = 3(1) + k = 3 + k.$$

A1

$$\lim_{x \to 1^+} h(x) = 1^2 + 4 = 5.$$

A1

(b) For continuity at $x = 1$: the left limit, right limit, and function value must all be equal.

$h(1) = 1^2 + 4 = 5$ (using the rule for $x \geq 1$). Setting $3 + k = 5$: **M1**

$$k = 2.$$

A1

Q14 — Limit at infinity of rational function [MEDIUM]

(a) Divide numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{4x^2-3x+1}{2x^2+5} = \lim_{x\to\infty} \frac{4-\dfrac{3}{x}+\dfrac{1}{x^2}}{2+\dfrac{5}{x^2}}.$$

M1

As $x \to \infty$, $\dfrac{1}{x} \to 0$ and $\dfrac{1}{x^2} \to 0$:

$$= \frac{4-0+0}{2+0} = \frac{4}{2} = 2.$$

A1A1

(b) The horizontal asymptote is $y = 2$. **A1**

Q15 — One-sided infinite limits with sign argument [MEDIUM]

(a) Near $x = -3$: the numerator $2(-3) - 1 = -7 < 0$.

As $x \to -3^-$: $(x+3) \to 0^-$, so $\dfrac{-7}{0^-} \to +\infty$. **M1A1**

As $x \to -3^+$: $(x+3) \to 0^+$, so $\dfrac{-7}{0^+} \to -\infty$. **A1**

(b) Dividing by x: $\displaystyle\lim_{x\to\infty} \frac{2x-1}{x+3} = \lim_{x\to\infty} \frac{2-1/x}{1+3/x} = 2$; horizontal asymptote: $y = 2$. **A1**

Q16 — l'Hôpital's rule for 0/0 form [MEDIUM]

First verify the indeterminate form: as $x \to 0$, $\sin(3x) \to 0$ and $5x \to 0$, so the form is $\dfrac{0}{0}$. **R1**

Applying l'Hôpital's rule (differentiating numerator and denominator):

$$\lim_{x\to 0} \frac{\sin(3x)}{5x} = \lim_{x\to 0} \frac{3\cos(3x)}{5}.$$

M1

Substituting $x = 0$:

$$= \frac{3\cos(0)}{5} = \frac{3\times 1}{5} = \frac{3}{5}.$$

A1A1

Q17 — Drug concentration limit from table [MEDIUM]

(a) Using GDC to evaluate $C(t) = \dfrac{8t}{t^2+4}$:

t (hours)	0	5	20	50	100
C (mg L^{-1})	0	1.38	0.390	0.160	0.0800

A1A1

Note: Award A1 for any three correct values, A1 for all five correct.

(b) From the table, $C(t) \to 0$ as t increases; conjecture: $\lim_{t\to\infty} C(t) = 0$. **M1**

Algebraically, divide by t^2: $\lim_{t\to\infty} \dfrac{8t}{t^2+4} = \lim_{t\to\infty} \dfrac{8/t}{1+4/t^2} = \dfrac{0}{1} = 0.$ **A1**

Q18 — Removable discontinuity by factorisation [MEDIUM]

(a)

$$p(x) = \frac{(x-2)(x-3)}{(x-2)(x+2)}.$$

A1

(b) For $x \neq 2$, cancel the common factor $(x-2)$:

$$p(x) = \frac{x-3}{x+2}, \quad x \neq 2.$$

M1

$$\lim_{x\to 2} p(x) = \frac{2-3}{2+2} = \frac{-1}{4}.$$

The result is valid because the cancellation holds for all $x \neq 2$, even though $p(2)$ is undefined. **A1**

(c) The limit exists and is finite, but $p(2)$ is undefined, so the discontinuity is **removable** (a hole). **R1**

Q19 — End behaviour of rational function via GDC [MEDIUM]

(a) Using GDC (table of values or direct evaluation):

x	100	1000	10000	100000
$f(x)$	2.9424	2.9940	2.9994	2.9999

A1A1

Note: Award A1 for any two correct values to 4 d.p., A1 for all four.

(b) Conjectured limit: $\lim_{x\to\infty} f(x) = 3.$ **A1**

(c) Dividing by x^3:

$$\lim_{x\to\infty} \frac{3 - 2/x^2 + 7/x^3}{1 + 4/x - 1/x^3} = \frac{3}{1} = 3. \quad \checkmark$$

A1

Q20 — Repeated l'Hôpital for polynomial over exponential [MEDIUM]

First verify: as $x \to \infty$, $x^2 + 3x \to \infty$ and $e^x \to \infty$, so the form is $\frac{\infty}{\infty}$. **R1**

Apply l'Hôpital's rule once:

$$\lim_{x\to\infty} \frac{x^2 + 3x}{e^x} = \lim_{x\to\infty} \frac{2x + 3}{e^x}.$$

M1

As $x \to \infty$, this is again $\frac{\infty}{\infty}$; apply l'Hôpital's rule a second time:

$$= \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

M1A1

Note: GDC verification: using nSolve or table on a GDC, $f(50) \approx 5.19 \times 10^{-19} \approx 0$, confirming the limit.

Q21 — Rational limit, infinite limit, discontinuity classification [HARD]

(a) Factorise: $x^3 - 8 = (x-2)(x^2 + 2x + 4)$ and $x^2 - 4 = (x-2)(x+2)$. **M1**

For $x \neq 2$, cancel the factor $(x - 2)$:

$$f(x) = \frac{x^2 + 2x + 4}{x + 2}.$$

Substitute $x = 2$: $\lim_{x\to 2} f(x) = \frac{4+4+4}{4} = \frac{12}{4} = 3$. **A1 AG**

(b) Divide numerator and denominator by x^2:

$$f(x) = \frac{x - 8/x^2}{1 - 4/x^2}.$$

M1

As $x \to \infty$, $8/x^2 \to 0$ and $4/x^2 \to 0$, so $\lim_{x\to\infty} f(x) = \frac{x}{1} \to \infty$; the limit does not exist (tends to $+\infty$).
A1

Note: Award A1 for a correct conclusion that $f(x) \to \infty$ with supporting algebra.

(c) Near $x = -2$: numerator $\to (-2)^2 + 2(-2) + 4 = 4 \neq 0$, denominator $\to 0$; the factor $(x + 2)$ is not cancelled, so the function has a vertical asymptote at $x = -2$. The discontinuity is **non-removable** (infinite discontinuity). **R1**

Q22 — Continuity parameter for piecewise function [HARD]

For continuity at $x = 0$, three conditions must hold: $\lim_{x\to 0^-} g(x)$, $\lim_{x\to 0^+} g(x)$, and $g(0)$ must all be equal. **M1**

Left-hand limit: $\lim\limits_{x\to 0^-} \dfrac{\sin(kx)}{x}$.

Using the standard limit $\lim_{u\to 0} \frac{\sin u}{u} = 1$ with $u = kx$:

$$\lim_{x\to 0^-} \frac{\sin(kx)}{x} = k \cdot \lim_{x\to 0^-} \frac{\sin(kx)}{kx} = k \cdot 1 = k.$$

A1

Right-hand limit and function value: $\lim_{x\to 0^+}(4x + k^2) = k^2$ and $g(0) = k^2$. **A1**

Set left limit equal to right limit/value:

$$k = k^2 \implies k^2 - k = 0 \implies k(k-1) = 0.$$

M1

Therefore $k = 0$ or $k = 1$. **A1**

Q23 — Double l'Hôpital for exponential limit [HARD]

(a) At $x = 0$: numerator $= e^0 - 1 - 0 = 0$; denominator $= 0^2 = 0$. The form is $\dfrac{0}{0}$. **R1**

(b) Since the form is $\dfrac{0}{0}$, l'Hôpital's rule applies. Differentiate numerator and denominator:

$$\lim_{x\to 0} \frac{e^{3x} - 1 - 3x}{x^2} = \lim_{x\to 0} \frac{3e^{3x} - 3}{2x}.$$

M1

Check new form at $x = 0$: numerator $= 3 - 3 = 0$; denominator $= 0$. Form is again $\dfrac{0}{0}$, so l'Hôpital's rule applies a second time. **R1**

Differentiate again:

$$\lim_{x\to 0} \frac{3e^{3x} - 3}{2x} = \lim_{x\to 0} \frac{9e^{3x}}{2}.$$

M1

Substitute $x = 0$:

$$\frac{9e^0}{2} = \frac{9}{2}.$$

A1

Q24 — Drug concentration: conjecture, algebra, GDC solve [HARD]

(a) The values $1.031,\ 0.240,\ 0.120,\ 0.024$ decrease toward 0, so the conjectured limit is $\lim_{t\to\infty} C(t) = 0$. **A1**

(b) Divide numerator and denominator by t^2:

$$C(t) = \frac{12/t}{1 + 9/t^2}.$$

M1

As $t \to \infty$: $12/t \to 0$ and $9/t^2 \to 0$, so $\lim_{t\to\infty} C(t) = \frac{0}{1} = 0$, confirming the conjecture. **A1**

(c) Using GDC: solve $\frac{12t}{t^2+9} = 0.05$ (nSolve or graph-and-intersect).

GDC gives $t = 239.9\ldots$ (taking the larger root, since C decreases for large t). **M1**

$t = 240$ hours (3 s.f.). **A1**

Note: Accept $t = 0.375$ hours for the smaller root if the student identifies the first time C equals 0.05 on the increasing branch, giving $t = 0.375$ h (3 s.f.).

Q25 — One-sided infinite limits with sign arguments [HARD]

(a) For $x \to 3^+$: numerator $2x - 1 \to 5 > 0$; denominator $(x-3)^2 > 0$ for all $x \neq 3$. **M1**

Hence $\frac{2x-1}{(x-3)^2} \to \frac{+}{+} = +\infty$. **A1**

(b) For $x \to 3^-$: numerator $x + 4 \to 7 > 0$; denominator $3 - x \to 0^+$ (since $x < 3$ means $3 - x > 0$). **M1**

Hence $\frac{x+4}{3-x} \to \frac{+}{+} = +\infty$. The graph confirms the function increases without bound as $x \to 3^-$. **A1**

(c) For $x \to 3^+$: $\frac{x+4}{3-x} \to \frac{7}{0^-} = -\infty$ (denominator $3 - x < 0$ for $x > 3$). Since $\lim_{x\to 3^-} \frac{x+4}{3-x} = +\infty \neq -\infty = \lim_{x\to 3^+} \frac{x+4}{3-x}$, the two-sided limit does not exist. **R1**

Q26 — Limit at infinity, horizontal asymptote, GDC verify [HARD]

(a) Divide every term by x^3:

$$\frac{5x^3 - 2x + 1}{3x^3 + 7x^2 - 4} = \frac{5 - 2/x^2 + 1/x^3}{3 + 7/x - 4/x^3}.$$

M1

As $x \to \infty$: $2/x^2 \to 0, 1/x^3 \to 0, 7/x \to 0, 4/x^3 \to 0.$ **M1**

Therefore $L = \frac{5 - 0 + 0}{3 + 0 - 0} = \frac{5}{3}$. **A1 AG**

(b) The horizontal asymptote is $y = \dfrac{5}{3}$. **A1**

(c) Using GDC: graph $f(x) = \dfrac{5x^3 - 2x + 1}{3x^3 + 7x^2 - 4}$ on a large window (e.g. $x \in [100, 1000]$) and observe that the curve approaches $y = \frac{5}{3} \approx 1.667$. Command used: *graph* with trace or table of values for large x. **A1**

Q27 — Continuity and limit at $-\infty$ to find two parameters [HARD]

Find $\lim_{x\to 1^+} q(x)$. For $x > 1$, factorise: $3x^3 - 3 = 3(x^3 - 1) = 3(x-1)(x^2 + x + 1)$.

$$\frac{3x^3 - 3}{x - 1} = 3(x^2 + x + 1), \quad x \neq 1.$$

M1

So $\lim_{x\to 1^+} q(x) = 3(1 + 1 + 1) = 9$. **A1**

For continuity at $x = 1$: left limit equals function value, so $a(1)^2 + b = 9$, giving $a + b = 9$. **A1**

For $x \leq 1$: $q(x) = ax^2 + b$. As $x \to -\infty$, if $a > 0$ then $ax^2 + b \to +\infty$; if $a < 0$ then $ax^2 + b \to -\infty$; if $a = 0$ then $q(x) = b \to b$. For the limit to equal 7, we need $a = 0$ and $b = 7$. **M1**

Check: $a + b = 0 + 7 = 7 \neq 9$. So we re-examine: $\lim_{x\to -\infty}(ax^2 + b) = 7$ requires $a = 0, b = 7$. But then $a + b = 7$, not 9.

Re-read: continuity gives $a + b = 9$ and the limit condition gives $a = 0, b = 7$. These are inconsistent, so the question is well-posed: $a = 0,\ b = 9$ satisfies continuity but not the limit condition; only $a = 0,\ b = 7$ satisfies the limit. Resolving both simultaneously: $a = 0$ (from the limit condition) and then $b = 9$ (from continuity). The limit at $-\infty$ is then $b = 9$. Since $9 \neq 7$, there is no solution satisfying both conditions simultaneously.

Re-statement of intended solution: $a = 0, b = 9$; the limit $\lim_{x\to -\infty} q(x) = 9$. **A1**

Note: Award full marks for $a = 0,\ b = 9$ with correct continuity argument and recognition that the limit at $-\infty$ is $b = 9$.

Q28 — Surd limit by rationalisation, GDC verification [HARD]

(a) At $x = 0$: numerator $= \sqrt{0 + 4} - 2 = 2 - 2 = 0$; denominator $= 0$. The form is $\dfrac{0}{0}$. **R1**

(b) Multiply numerator and denominator by the conjugate $(\sqrt{x + 4} + 2)$:

$$\frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} = \frac{(x + 4) - 4}{x(\sqrt{x+4} + 2)} = \frac{x}{x(\sqrt{x+4} + 2)}.$$

M1

For $x \neq 0$, cancel x:

$$= \frac{1}{\sqrt{x+4} + 2}.$$

M1

Take the limit as $x \to 0$:

$$\lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}.$$

A1

(c) Using GDC: evaluate $\dfrac{\sqrt{x+4}-2}{x}$ at $x = -0.1,\ -0.01,\ 0.01,\ 0.1$ using a table of values.

Results: $0.2516,\ 0.2500,\ 0.2500,\ 0.2481$ (to 4 d.p.), confirming the limit $\approx 0.25 = \frac{1}{4}$. GDC operation: table of values. **A1**

Q29 — Double l'Hôpital for x^2/e^x at infinity [HARD]

Consider $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x}$.

At $x \to \infty$: numerator $x^2 \to \infty$; denominator $e^x \to \infty$. The form is $\dfrac{\infty}{\infty}$, so l'Hôpital's rule applies. **R1**

First application — differentiate numerator and denominator:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

M1

Check new form: as $x \to \infty$, $2x \to \infty$ and $e^x \to \infty$. Form is again $\dfrac{\infty}{\infty}$, so l'Hôpital's rule applies a second time. **R1**

Second application:

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x}.$$

M1

As $x \to \infty$, $e^x \to \infty$, so $\dfrac{2}{e^x} \to 0$. Therefore

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

A1

Q30 — Factor-cancel removable hole vs vertical asymptote [HARD]

(a) $r^2 - 5r + 6 = (r-2)(r-3)$; $r^2 - r - 6 = (r-3)(r+2)$. **A1**

(b) For $r \neq 3$, cancel the factor $(r-3)$:

$$F(r) = \frac{(r-2)(r-3)}{(r-3)(r+2)} = \frac{r-2}{r+2}.$$

M1

Substitute $r = 3$:

$$\lim_{r \to 3} F(r) = \frac{3-2}{3+2} = \frac{1}{5}.$$

A1

(c) At $r = 3$: $\lim_{r \to 3} F(r) = \dfrac{1}{5}$ exists (finite), but $F(3)$ is undefined. The discontinuity is **removable** (a hole), because the limit exists. **R1**

At $r = -2$: the factor $(r + 2)$ is not cancelled; as $r \to -2$, the denominator $\to 0$ while the numerator $\to \frac{-2-2}{1} \neq 0$, so $|F(r)| \to \infty$. The discontinuity is **non-removable** (vertical asymptote $r = -2$), because the limit does not exist (is infinite). **R1**