IB Math AA HL — Topic 5

Differentiation Toolkit

First Principles · Chain, Product & Quotient Rules · Implicit & Parametric · Related Rates

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **First Principles:** $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$; limit notation is kept until all h factors cancel.
- **Power / Sum Rules:** $\dfrac{d}{dx}[x^n] = nx^{n-1}$; $\dfrac{d}{dx}[f \pm g] = f' \pm g'$.
- **Chain, Product, Quotient Rules:** $\dfrac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$; $\dfrac{d}{dx}[uv] = u'v + uv'$; $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{u'v - uv'}{v^2}$.
- **Standard Derivatives:** $\dfrac{d}{dx}e^x = e^x$; $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$; $\dfrac{d}{dx}a^x = a^x \ln a$; $\dfrac{d}{dx}\sin x = \cos x$; $\dfrac{d}{dx}\cos x = -\sin x$; $\dfrac{d}{dx}\tan x = \sec^2 x$.
- **Implicit Differentiation:** $\dfrac{d}{dx}[y^n] = ny^{n-1}\dfrac{dy}{dx}$; differentiate both sides with respect to x, then collect $\dfrac{dy}{dx}$ terms.
- **Parametric Differentiation:** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$; $\dfrac{d^2y}{dx^2} = \left[\dfrac{d}{dt}\dfrac{dy}{dx}\right] \div \dfrac{dx}{dt}$.
- **Related Rates (Chain Rule):** Write the connecting equation first, e.g. $\dfrac{dV}{dt} = \dfrac{dV}{dr}\cdot\dfrac{dr}{dt}$, before substituting any values; every rate carries its units.

GDC Tips

- **Numerical Derivative at a Point:** Use dy/dx (graph screen) or `nDeriv(f(x),x,a)` (TI) / `diff(f(x),x)|x=a` (Casio CAS) to verify an exact gradient answer.
- **Graphing to Find Gradient:** Plot $y = f(x)$, trace to the required x-value, and select dy/dx from the Calc menu; record the result to 3 significant figures.
- **Implicit Curves:** Enter the relation as two explicit branches (solve for y first, or use the GDC's implicit-plot mode) to visualise the curve before differentiating by hand.
- **Intersection for Related-Rates Check:** Use graph-and-intersect to find when two rates are equal; input both rate expressions as separate functions of the variable and locate the crossing point.
- **Parametric Plotting:** In parametric mode enter $x(t)$ and $y(t)$; use dy/dx at the required t-value to confirm the parametric gradient, and zoom in to verify tangent direction.
- **Numerical Solve (nSolve / Solve):** Use `nSolve(f'(x)=0,x,a)` to locate stationary points when the derivative is too complex to solve by hand; state the GDC command and initial guess in your working.

Common Mistakes to Avoid

1. **Cancelling h before expanding in first principles.** Students substitute $h = 0$ into the unsimplified quotient $\dfrac{f(x+h)-f(x)}{h}$, obtaining $\frac{0}{0}$. You must expand $f(x+h)$, cancel every remaining h factor algebraically, and only then take $\lim_{h\to 0}$.

2. **Forgetting the chain-rule factor inside implicit differentiation.** Writing $\dfrac{d}{dx}[y^2] = 2y$ instead of $2y\dfrac{dy}{dx}$ is the most common implicit error; every y-term differentiated with respect to x must carry $\dfrac{dy}{dx}$ as a factor.

3. **Wrong sign order in the quotient rule.** Writing $\dfrac{u'v+uv'}{v^2}$ (product-rule sign) instead of $\dfrac{u'v-uv'}{v^2}$; always form low d(high) minus high d(low) in the numerator.

4. **Dividing by dx/dt only once for the parametric second derivative.** Students compute $\dfrac{d}{dt}\left[\dfrac{dy}{dx}\right]$ and call it $\dfrac{d^2y}{dx^2}$; the correct formula requires a second division by dx/dt: $\dfrac{d^2y}{dx^2} = \left[\dfrac{d}{dt}\dfrac{dy}{dx}\right] \div \dfrac{dx}{dt}$.

5. **Substituting values before writing the related-rates chain equation.** Inserting numerical values of r, V, etc. into the formula before establishing $\dfrac{dV}{dt} = \dfrac{dV}{dr}\cdot\dfrac{dr}{dt}$ loses the method mark; the connecting equation with units must appear first.

6. **Dropping the chain-rule factor when differentiating $e^{g(x)}$ or $\ln(g(x))$.** Writing $\dfrac{d}{dx}[e^{3x}] = e^{3x}$ or $\dfrac{d}{dx}[\ln(x^2+1)] = \dfrac{1}{x^2+1}$ without the inner derivative; the correct results are $3e^{3x}$ and $\dfrac{2x}{x^2+1}$ respectively.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY No Calculator **[3 marks]**

Let $f(x) = 3x^2 - 5x + 4$. Find $f'(x)$ from first principles using the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2. EASY No Calculator **[3 marks]**

Differentiate $g(x) = 4x^3 - \dfrac{6}{x^2} + \sqrt{x}$ with respect to x.

3. EASY Calculator **[3 marks]**

Let $y = (3x^2 + 1)^5$. Find $\dfrac{dy}{dx}$.

4. EASY No Calculator **[3 marks]**

A curve is defined by $x^2 + y^2 = 25$. Find $\dfrac{dy}{dx}$ using implicit differentiation.

5. EASY Calculator **[3 marks]**

A curve is defined parametrically by $x = t^2 + 1$ and $y = 2t^3 - 3t$, where $t \in \mathbb{R}$. Find $\dfrac{dy}{dx}$ in terms of t.

6. EASY Calculator **[3 marks]**

Let $h(x) = x^2 e^{3x}$. Find $h'(x)$ using the product rule.

7. EASY Calculator **[3 marks]**

The table below shows values of two differentiable functions f and g and their derivatives at $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	3	-1	5	4

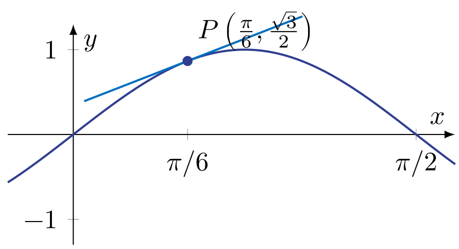
Let $p(x) = f(x) \cdot g(x)$. Find $p'(2)$.

8. EASY Calculator **[3 marks]**

A spherical balloon is being inflated. At the instant when the radius is $r = 4$ cm, the radius is increasing at a rate of 0.5 cm s^{-1}. The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Find the rate of change of volume at this instant, giving your answer to 3 significant figures.

9. EASY Calculator **[3 marks]**

Let $f(x) = \sin(2x)$. The diagram below shows the curve $y = f(x)$ and the tangent to the curve at the point $P\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$.



Find the gradient of the tangent at P.

10. EASY Calculator **[3 marks]**

Let $f(x) = e^{2x}$. Find the second derivative $f''(x)$ and write down the value of $f''(0)$.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

Let $f(x) = 3x^2 - 5x$. Using differentiation from first principles, find $f'(x)$.

12. MEDIUM No Calculator **[4 marks]**

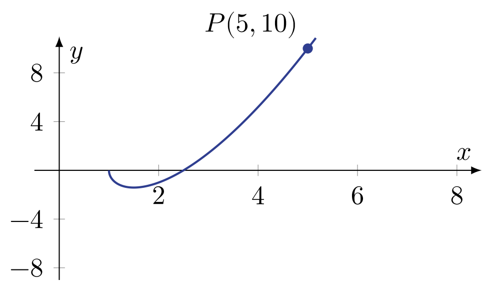
A curve is defined by $y = x^3 e^{2x}$. Find $\dfrac{dy}{dx}$.

13. MEDIUM No Calculator **[4 marks]**

The curve C has equation $x^2 + 3xy + y^2 = 5$. Find the gradient of C at the point $(1, 1)$.

14. MEDIUM Calculator **[4 marks]**

A curve is defined parametrically by $x = t^2 + 1$ and $y = 2t^3 - 3t$, where $t \in \mathbb{R}$. Find the gradient of the curve at the point where $t = 2$.



15. MEDIUM Calculator **[4 marks]**

A spherical balloon is being inflated. At the instant when the radius is $r = 4$ cm, the radius is increasing at 0.3 cm s^{-1}. The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$.

Find the rate at which the volume is increasing at this instant, giving your answer in cm^{3}s^{-1} correct to 3 significant figures.

16. MEDIUM Calculator **[4 marks]**

The functions f and g are twice differentiable. Selected values are given in the table below.

x	1	2	3	4
$f(x)$	3	5	2	6
$f'(x)$	-1	4	-3	7
$g(x)$	2	3	1	4
$g'(x)$	5	-2	6	-1

Let $h(x) = f(x)\,g(x)$. Find $h'(2)$.

17. MEDIUM Calculator **[4 marks]**

A ladder of length 5 m leans against a vertical wall. The foot of the ladder slides away from the wall along horizontal ground. At a certain instant, the foot of the ladder is 3 m from the wall and moving at 0.4 m s^{-1}. Let x be the distance of the foot of the ladder from the wall and y be the height of the top of the ladder on the wall, both in metres.

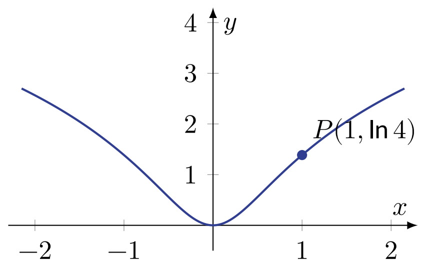
Find the rate at which the top of the ladder is sliding down the wall at this instant, giving your answer in m s^{-1} correct to 3 significant figures.

18. MEDIUM No Calculator **[4 marks]**

Let $f(x) = \dfrac{\sin x}{e^x}$. Find $f''(x)$.

19. MEDIUM Calculator **[4 marks]**

A curve is defined by $y = \ln(3x^2 + 1)$.



Find the equation of the tangent to the curve at the point P where $x = 1$, giving your answer in the form $y = mx + c$.

20. MEDIUM Calculator **[4 marks]**

Water drains from a conical tank (vertex downwards) such that when the depth of water is h cm the volume is $V = \dfrac{\pi h^3}{12}$ cm^3. At the instant when $h = 6$ cm, water is draining at 9π cm^3 min^{-1}.

Find the rate at which the depth is decreasing at this instant, in cm min^{-1}.

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **No Calculator** **[5 marks]**

A curve is defined implicitly by

$$x^3 + 2xy^2 - y^3 = 8.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2 + 2y^2}{3y^2 - 4xy}$. [3]

(b) Hence find the gradient of the curve at the point $(2, 0)$. [1]

(c) Write down the equation of the tangent to the curve at $(2, 0)$. [1]

22. **HARD** **No Calculator** **[5 marks]**

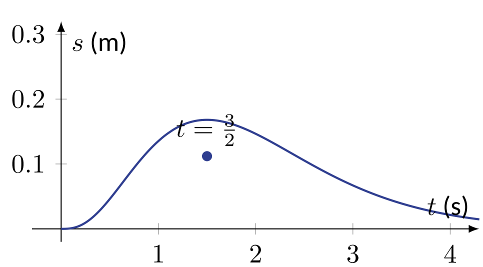
A particle moves along a straight line. Its displacement from the origin at time $t \geq 0$ seconds is given by

$$s(t) = t^3 e^{-2t} \quad \text{(metres)}.$$

(a) Find $s'(t)$. [2]

(b) Find the exact value of t at which the particle's velocity is zero for $t > 0$. [2]

(c) Justify, using $s''(t)$, whether the velocity is increasing or decreasing at this instant. [1]



23. HARD No Calculator **[5 marks]**

A curve is defined parametrically by

$$x = 3\cos t - \cos 3t, \qquad y = 3\sin t - \sin 3t, \qquad 0 < t < \pi.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{3\cos t - 3\cos 3t}{-3\sin t + 3\sin 3t}$. [2]

(b) Find the gradient of the curve at $t = \dfrac{\pi}{2}$. Give your answer as an exact value. [2]

(c) Write down the coordinates of the point on the curve at $t = \dfrac{\pi}{2}$. [1]

24. HARD Calculator **[5 marks]**

A spherical balloon is being inflated so that its radius r cm increases at a constant rate of 0.4 cm s^{-1}. At a particular instant, $r = 7.5$ cm.

(a) Write down the chain-rule equation linking $\dfrac{dV}{dt}$ and $\dfrac{dr}{dt}$, where $V = \frac{4}{3}\pi r^3$. [1]

(b) Find the rate of increase of the surface area $A = 4\pi r^2$ at this instant. Give your answer in cm^{2}s^{-1} to 3 significant figures. [2]

(c) At the same instant, find the rate of increase of V to 3 significant figures. [2]

25. HARD Calculator **[5 marks]**

Let $f(x) = \dfrac{\ln x}{e^x}$ for $x > 0$.

(a) Find $f'(x)$. [3]

(b) Using a GDC, find the x-coordinate of the stationary point of f for $x > 0$, giving your answer to 3 significant figures. [2]

26. HARD No Calculator **[5 marks]**

The function $f(x) = \sqrt{4 + x^2}$ is to be differentiated from first principles.

(a) Using the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, show that

$$f'(x) = \lim_{h \to 0} \frac{h}{h\left(\sqrt{4 + (x+h)^2} + \sqrt{4 + x^2}\right)}.$$

[3]

(b) Hence show that $f'(x) = \dfrac{x}{\sqrt{4 + x^2}}$. [2]

27. **HARD** **Calculator** **[5 marks]**

The table below gives values of two differentiable functions f and g, and their derivatives, at $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	5	-3	1	4

Let $h(x) = \dfrac{f(x)\,g(x)}{e^{g(x)}}$.

(a) Find $h'(x)$ in terms of f, g and their derivatives. [3]

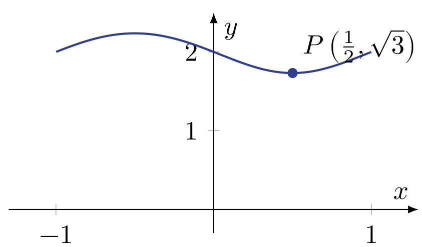
(b) Hence find the exact value of $h'(2)$. [2]

28. **HARD** **No Calculator** **[5 marks]**

A curve C is defined implicitly by $y^2 + \sin(\pi x) = 4$, where $x \in (-1, 1)$ and $y > 0$.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y. [2]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y. [3]



29. HARD Calculator **[5 marks]**

A conical funnel (vertex pointing downward) has a fixed half-angle of $30°$. Water drains out of the vertex at 6 cm^{3}s^{-1} while being poured in at 10 cm^{3}s^{-1}. When the depth of water is $h = 12$ cm:

(a) show that the volume of water is $V = \dfrac{\pi h^3}{9}$, [1]

(b) find $\dfrac{dh}{dt}$ at this instant, giving your answer in cm s^{-1} to 3 significant figures. [4]

30. HARD Calculator **[5 marks]**

Consider the function $f(x) = x^2 \cos x$.

(a) Find $f'(x)$ and $f''(x)$. [3]

(b) Using a GDC, find the x-coordinate of the inflection point of f in the interval $(1, 4)$, giving your answer to 3 significant figures. [1]

(c) Justify that this point is indeed an inflection by verifying the sign change of $f''(x)$ on either side of it. [1]

Mark Schemes

Q1 — First Principles for a Quadratic EASY

Expanding $f(x+h) = 3(x+h)^2 - 5(x+h) + 4 = 3x^2 + 6xh + 3h^2 - 5x - 5h + 4$ **M1**

$$\frac{f(x+h) - f(x)}{h} = \frac{6xh + 3h^2 - 5h}{h} = 6x + 3h - 5$$

A1

Taking the limit: $f'(x) = \lim_{h \to 0}(6x + 3h - 5) = 6x - 5$ **A1**

Note: Award M0 if limit notation is absent before the h terms cancel. Final answer $f'(x) = 6x - 5$.

Q2 — Power Rule Differentiation EASY

Rewrite: $g(x) = 4x^3 - 6x^{-2} + x^{1/2}$ **M1**

Differentiating each term: $g'(x) = 12x^2 + 12x^{-3} + \frac{1}{2}x^{-1/2}$ **A1**

Writing in equivalent form: $g'(x) = 12x^2 + \dfrac{12}{x^3} + \dfrac{1}{2\sqrt{x}}$ **A1**

Note: Award A1 A0 if exactly one term is incorrect. Accept any equivalent unsimplified form, e.g. plain notation $g'(x) = 12x^2 + 12/x^3 + 1/(2\sqrt{x})$.

Q3 — Chain Rule on a Composite Power EASY

Identifying the outer and inner functions, applying the chain rule: **M1**

$$\frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot 6x$$

A1

$$\frac{dy}{dx} = 30x(3x^2 + 1)^4$$

A1

Note: equivalently written $\dfrac{dy}{dx} = 30x(3x^2 + 1)^4$.

Q4 — Implicit Differentiation of a Circle EASY

Differentiating both sides with respect to x: **M1**

$$2x + 2y\frac{dy}{dx} = 0$$

A1

$$\frac{dy}{dx} = -\frac{x}{y}$$

A1

Note: Award M1 for correctly differentiating y^2 as $2y\dfrac{dy}{dx}$, showing the chain-rule step. In plain notation, $dy/dx = -x/y$.

Q5 — Parametric Differentiation EASY

Finding the two component derivatives: $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 6t^2 - 3$ **M1**

Applying $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$: **A1**

$$\frac{dy}{dx} = \frac{6t^2 - 3}{2t}$$

A1

Note: Award A0 for inverting the ratio $dx/dt \div dy/dt$. In plain notation, $dy/dx = (6t^2 - 3)/(2t)$, where the numerator $6t^2 - 3$ comes from dy/dt.

Q6 — Product Rule with Exponential EASY

Identifying $u = x^2$, $v = e^{3x}$, so $u' = 2x$, $v' = 3e^{3x}$ **M1**

Applying the product rule $h'(x) = u'v + uv'$:

$$h'(x) = 2x\,e^{3x} + x^2 \cdot 3e^{3x}$$

A1

$$h'(x) = e^{3x}(2x + 3x^2)$$

A1

Note: equivalently written $h'(x) = e^{3x}(2x + 3x^2)$.

Q7 — Product Rule from a Table EASY

Applying the product rule: $p'(x) = f'(x)\,g(x) + f(x)\,g'(x)$ **M1**

Substituting values from the table at $x = 2$:

$$p'(2) = (-1)(5) + (3)(4)$$

A1

$$p'(2) = -5 + 12 = 7$$

A1

Q8 — Related Rates for a Sphere EASY

Writing the chain-rule connecting equation before substituting:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

M1

Substituting $r = 4$ cm and $\frac{dr}{dt} = 0.5$ cm s^{-1}:

$$\frac{dV}{dt} = 4\pi(4)^2 \times 0.5 = 32\pi$$

A1

Using GDC to evaluate: $32\pi = 100.530\ldots \approx 101$ cm^{3}s^{-1} **A1**

Note: The connecting equation $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$ must appear before values are substituted; award M0 if values are inserted into $\frac{dV}{dr}$ without the chain-rule step.

Q9 — Gradient of Tangent to a Sine Curve EASY

Differentiating: $f'(x) = 2\cos(2x)$ **M1**

Substituting $x = \frac{\pi}{6}$:

$$f'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right)$$

A1

$$= 2 \times \frac{1}{2} = 1$$

A1

Note: GDC numerical derivative at $x = \pi/6$ gives $1.00000\ldots$, confirming the exact answer of 1.

Q10 — Second Derivative of an Exponential EASY

First derivative: $f'(x) = 2e^{2x}$ **M1**

Second derivative: $f''(x) = 4e^{2x}$ **A1**

Substituting $x = 0$: $f''(0) = 4e^0 = 4$ **A1**

Q11 — First Principles for a Second Quadratic MEDIUM

Using the definition $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$: **M1**

$f(x+h) = 3(x+h)^2 - 5(x+h) = 3x^2 + 6xh + 3h^2 - 5x - 5h$

$$\frac{f(x+h) - f(x)}{h} = \frac{6xh + 3h^2 - 5h}{h} = 6x + 3h - 5$$

A1

Taking the limit as $h \to 0$ (do not substitute $h = 0$ into the unsimplified quotient): **M1**

$f'(x) = \lim_{h\to 0}(6x + 3h - 5) = 6x - 5$ **A1**

Note: Award M0 if $h = 0$ is substituted before cancellation. Final answer $f'(x) = 6x - 5$.

Q12 — Product Rule with a Cubic Exponential MEDIUM

Let $u = x^3$ and $v = e^{2x}$, so $u' = 3x^2$ and $v' = 2e^{2x}$. **M1**

Applying the product rule $\dfrac{dy}{dx} = u'v + uv'$: **M1**

$\dfrac{dy}{dx} = 3x^2e^{2x} + x^3 \cdot 2e^{2x}$ **A1**

$= e^{2x}(3x^2 + 2x^3)$ **A1**

Note: Accept unsimplified form $3x^2e^{2x} + 2x^3e^{2x}$ for full marks, or equivalently $e^{2x}(3x^2 + 2x^3)$.

Q13 — Implicit Differentiation of a Second Curve MEDIUM

Differentiating both sides with respect to x: **M1**

$2x + 3\left(y + x\dfrac{dy}{dx}\right) + 2y\dfrac{dy}{dx} = 0$ **A1**

Note: Award A1 for correct application of chain rule on y^2 giving $2y\dfrac{dy}{dx}$, and product rule on $3xy$.

Collecting $\frac{dy}{dx}$ terms: $(3x+2y)\frac{dy}{dx} = -2x - 3y$ **M1**

At $(1,1)$: $(3+2)\frac{dy}{dx} = -2-3 \implies \frac{dy}{dx} = -1$ **A1**

Note: Verify $(1,1)$ lies on the curve first: $1+3+1=5$ ✓.

Q14 — Parametric Gradient Revisited MEDIUM

$\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 6t^2 - 3$ **A1**

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2-3}{2t}$ **M1**

At $t=2$: $\frac{dy}{dx} = \frac{6(4)-3}{2(2)} = \frac{21}{4}$ **A1**

Using GDC to verify: at $t=2$ the point is $(5,10)$; numerical derivative of the parametric curve at $t=2$ confirms gradient $= 5.25$. **A1**

Note: The point $P(5,10)$ is shown on the diagram. In plain notation, $dy/dx = 21/4 = 5.25$.

Q15 — Related Rates for a Spherical Balloon MEDIUM

Write the chain-rule link before substituting values: **M1**

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$\frac{dV}{dr} = 4\pi r^2$ **A1**

At $r=4$ cm, $\frac{dr}{dt} = 0.3$ cm s^{-1}: $\frac{dV}{dt} = 4\pi(4)^2 \times 0.3 = 19.2\pi$ **M1**

GDC evaluate: $19.2\pi = 60.318\ldots \approx 60.3$ cm^{3}s^{-1} **A1**

Note: Units cm^{3}s^{-1} required for final A1.

Q16 — Product Rule from a Table of Values MEDIUM

The product rule gives $h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$. **M1**

Identifying the required values from the table: $f'(2)=4$, $g(2)=3$, $f(2)=5$, $g'(2)=-2$ **A1**

$h'(2) = (4)(3) + (5)(-2)$ **M1**

$h'(2) = 12 - 10 = 2$ **A1**

Q17 — Related Rates for a Sliding Ladder MEDIUM

By Pythagoras: $x^2 + y^2 = 25$. Write the chain-rule equation before substituting: **M1**

Differentiating with respect to t: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ **A1**

At $x = 3$: $y = \sqrt{25-9} = 4$ m. $\dfrac{dy}{dt} = -\dfrac{x}{y} \cdot \dfrac{dx}{dt} = -\dfrac{3}{4} \times 0.4$ **M1**

$\dfrac{dy}{dt} = -0.3\ \text{m s}^{-1}$; the top is sliding down at $0.300\ \text{m s}^{-1}$. **A1**

Note: The negative sign confirms downward motion; accept $0.3\ \text{m s}^{-1}$ downwards.

Q18 — Second Derivative of a Sine-Exponential Quotient MEDIUM

Write $f(x) = e^{-x}\sin x$. Using the product rule with $u = e^{-x}$, $v = \sin x$: $f'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x - \sin x)$ **M1A1**

Differentiating again using the product rule on $e^{-x}(\cos x - \sin x)$: **M1**

$f''(x) = -e^{-x}(\cos x - \sin x) + e^{-x}(-\sin x - \cos x)$
$= e^{-x}(-\cos x + \sin x - \sin x - \cos x) = -2e^{-x}\cos x$ **A1**

Note: Award M1A1 for correct first derivative, M1A1 for correct second derivative. In plain notation this is written $-2e^{-x}cos(x)$ or, with a space, $-2e^{-x}cos(x)$.

Q19 — Tangent to a Logarithmic Curve MEDIUM

$\dfrac{dy}{dx} = \dfrac{6x}{3x^2+1}$ **M1A1**

At $x = 1$: gradient $m = \dfrac{6}{4} = \dfrac{3}{2}$. **A1**

Point on curve: $y = \ln(3+1) = \ln 4$. Tangent: $y - \ln 4 = \dfrac{3}{2}(x-1)$ GDC evaluate $\ln 4 = 1.386\ldots$; tangent equation: $y = \dfrac{3}{2}x - \dfrac{3}{2} + \ln 4$ **A1**

Note: Accept exact form $y = \frac{3}{2}x + \ln 4 - \frac{3}{2}$ or decimal $y \approx \frac{3}{2}x - 0.114$. Gradient in plain notation: $m = 3/2$; $\ln 4$ may also be written ln4.

Q20 — Related Rates for a Draining Conical Tank MEDIUM

Write the chain-rule link before substituting values: **M1**

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt}$

$\frac{dV}{dh} = \frac{\pi h^2}{4}$ **A1**

At $h = 6$ cm: $\frac{dV}{dh} = \frac{\pi(36)}{4} = 9\pi$ cm^2. Water draining means $\frac{dV}{dt} = -9\pi$ cm^{3}min^{-1}: $-9\pi = 9\pi \cdot \frac{dh}{dt} \implies$
$\frac{dh}{dt} = -1$ cm min^{-1} **M1**

The depth is decreasing at 1 cm min^{-1}. **A1**

Note: Negative sign confirms depth is decreasing; accept a rate of 1 cm per minute (decreasing).

Q21 — Implicit Differentiation and Tangent Line HARD

(a) Differentiating $x^3 + 2xy^2 - y^3 = 8$ implicitly with respect to x:

$$3x^2 + 2y^2 + 4xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

M1

(chain-rule terms $4xy\frac{dy}{dx}$ and $-3y^2\frac{dy}{dx}$ visible) **A1**

Collecting $\frac{dy}{dx}$ terms:

$$\frac{dy}{dx}(4xy - 3y^2) = -(3x^2 + 2y^2) \implies \frac{dy}{dx} = \frac{3x^2 + 2y^2}{3y^2 - 4xy}$$

AG

(b) At $(2, 0)$: numerator $= 3(4) + 0 = 12$, denominator $= 0 - 4(2)(0) = 0$.
Note: The denominator is zero; we check that $(2, 0)$ lies on the curve: $8 + 0 - 0 = 8$ ✓. The tangent is vertical. **A1**

(c) Since the gradient is undefined (vertical tangent), the equation of the tangent is:

$$x = 2$$

A1

Note: Award the final A1 only if the student identifies the vertical tangent; do not accept $y =$(anything).

Q22 — Product Rule and Second Derivative Test HARD

(a) Using the product rule on $s(t) = t^3e^{-2t}$:

$$s'(t) = 3t^2e^{-2t} + t^3(-2e^{-2t}) = e^{-2t}(3t^2 - 2t^3)$$

M1A1

Note: equivalently written $s'(t) = e^{-2t}(3t^2 - 2t^3)$.

(b) Setting $s'(t) = 0$ for $t > 0$: since $e^{-2t} > 0$,

$$3t^2 - 2t^3 = 0 \implies t^2(3 - 2t) = 0 \implies t = \tfrac{3}{2}$$

M1A1

Note: in plain notation, $t = 3/2$.

(c) Finding $s''(t)$:

$$s''(t) = e^{-2t}(6t - 6t^2) + (3t^2 - 2t^3)(-2e^{-2t}) = e^{-2t}(6t - 6t^2 - 6t^2 + 4t^3)$$

At $t = \frac{3}{2}$:

$$s''\left(\tfrac{3}{2}\right) = e^{-3}\left(9 - \tfrac{27}{2} - \tfrac{27}{2} + \tfrac{27}{2}\right) = e^{-3}\left(9 - \tfrac{27}{2}\right) = e^{-3}\left(-\tfrac{9}{2}\right) < 0$$

Since $s''(\frac{3}{2}) < 0$, the velocity is decreasing at this instant. **R1**

Note: The R1 requires a sentence in context; accept "the particle is decelerating" with justification from $s'' < 0$.

Q23 — Parametric Differentiation for an Exact Gradient HARD

(a) Differentiating:

$$\frac{dx}{dt} = -3\sin t + 3\sin 3t, \qquad \frac{dy}{dt} = 3\cos t - 3\cos 3t$$

M1

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t - 3\cos 3t}{-3\sin t + 3\sin 3t}$$

AG

(b) At $t = \dfrac{\pi}{2}$: $\cos\frac{\pi}{2} = 0$, $\cos\frac{3\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, $\sin\frac{3\pi}{2} = -1$:

$$\frac{dy}{dx} = \frac{3(0) - 3(0)}{-3(1) + 3(-1)} = \frac{0}{-6} = 0$$

M1A1

(c) At $t = \dfrac{\pi}{2}$: $x = 3\cos\frac{\pi}{2} - \cos\frac{3\pi}{2} = 0 - 0 = 0$, $y = 3\sin\frac{\pi}{2} - \sin\frac{3\pi}{2} = 3 - (-1) = 4$.

The coordinates are $(0, 4)$, i.e. $(0, 4)$. **A1**

Q24 — Related Rates for a Balloon's Surface Area and Volume HARD

(a) Chain-rule equation written explicitly:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

A1

(b) For surface area $A = 4\pi r^2$:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

M1

At $r = 7.5$, $\frac{dr}{dt} = 0.4$:

$$\frac{dA}{dt} = 8\pi(7.5)(0.4) = 24\pi \approx 75.4 \text{ cm}^2\text{s}^{-1}$$

A1

(c) From part (a):

$$\frac{dV}{dt} = 4\pi(7.5)^2(0.4) = 4\pi(56.25)(0.4) = 90\pi$$

M1

$$\frac{dV}{dt} \approx 283 \text{ cm}^3\text{s}^{-1}$$

A1

GDC note: evaluate 90π numerically to give $282.743\ldots \approx 283$ cm^{3}s^{-1}.

Q25 — Quotient Rule with Logarithm and Exponential and GDC Stationary Point HARD

(a) Write $f(x) = \frac{\ln x}{e^x}$. Using the quotient rule (low $= e^x$, high $= \ln x$): $\frac{d}{dx}(\ln x) = \frac{1}{x}$ **M1**

$$f'(x) = \frac{\left(\frac{1}{x}\right)e^x - \ln x \cdot e^x}{e^{2x}} = \frac{e^x\left(\frac{1}{x} - \ln x\right)}{e^{2x}}$$

A1

$$f'(x) = \frac{\frac{1}{x} - \ln x}{e^x} = \frac{1 - x\ln x}{x\,e^x}$$

A1

(b) Using GDC: enter `nSolve(f'(x)=0,x,x>0)` or graph $y = f'(x)$ and use graph-and-intersect with the x-axis.
Unrounded result: $x = 1.76322\ldots$

$$x \approx 1.76$$

M1A1

Note: Since $g(x) = \ln x - \frac{1}{x}$ is strictly increasing for $x > 0$ (as $g'(x) = \frac{1}{x} + \frac{1}{x^2} > 0$), this stationary point is unique. The second derivative test confirms it is a maximum.

Q26 — First Principles for a Square-Root Function HARD

(a) Form the difference quotient:

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{4+(x+h)^2}-\sqrt{4+x^2}}{h}$$

M1

Multiply numerator and denominator by the conjugate $\sqrt{4+(x+h)^2}+\sqrt{4+x^2}$:

$$=\frac{[4+(x+h)^2]-[4+x^2]}{h\left(\sqrt{4+(x+h)^2}+\sqrt{4+x^2}\right)}$$

M1

Expanding: $(x+h)^2-x^2=2xh+h^2$, so numerator $=2xh+h^2=h(2x+h)$.

$$=\frac{h(2x+h)}{h\left(\sqrt{4+(x+h)^2}+\sqrt{4+x^2}\right)}$$

which, after cancelling one factor of h (valid for $h\neq 0$), gives the printed form. **AG (A1 for the cancellation step)**

(b) Taking the limit as $h\to 0$:

$$f'(x)=\lim_{h\to 0}\frac{2x+h}{\sqrt{4+(x+h)^2}+\sqrt{4+x^2}}=\frac{2x}{2\sqrt{4+x^2}}=\frac{x}{\sqrt{4+x^2}}$$

M1A1

Note: Award M1 for substituting $h=0$ into the simplified expression only (not into the original unsimplified quotient); the limit notation must be present until the h terms are gone. In plain notation, $f'(x)=x/sqrt(4+x^2)$, or with unicode symbols $f'(x)=x/\sqrt{4+x^2}$ (x/√(4 + x²)). **AG**

Q27 — Table-Based Derivative and Quotient with Composite HARD

(a) Write $h(x)=f(x)g(x)\cdot e^{-g(x)}$. Apply the product rule to $f(x)g(x)$ and chain rule on $e^{-g(x)}$:

$$h'(x)=[f'(x)g(x)+f(x)g'(x)]e^{-g(x)}+f(x)g(x)\cdot(-g'(x)e^{-g(x)})$$

M1A1

$$h'(x)=e^{-g(x)}\left[f'(x)g(x)+f(x)g'(x)-f(x)g(x)g'(x)\right]$$

A1

(b) Substituting values at $x=2$: $f=5$, $f'=-3$, $g=1$, $g'=4$:

$$h'(2)=e^{-1}\left[(-3)(1)+(5)(4)-(5)(1)(4)\right]=e^{-1}[-3+20-20]=\frac{-3}{e}$$

M1A1

GDC note: evaluate $-3/e \approx -1.10$ as numerical confirmation; exact answer $-3e^{-1}$ required.

Q28 — Implicit Second Derivative with Trig HARD

(a) Differentiating $y^2 + \sin(\pi x) = 4$ implicitly:

$$2y\frac{dy}{dx} + \pi\cos(\pi x) = 0$$

M1

$$\frac{dy}{dx} = -\frac{\pi\cos(\pi x)}{2y}$$

A1

Note: equivalently written $dy/dx = -\pi\cos(\pi x)/(2y)$.

(b) Differentiating $\frac{dy}{dx} = -\frac{\pi\cos(\pi x)}{2y}$ with respect to x using the quotient rule:

$$\frac{d^2y}{dx^2} = -\frac{(-\pi^2\sin(\pi x))(2y) - \pi\cos(\pi x)\cdot 2\frac{dy}{dx}}{4y^2}$$

M1A1

Substituting $\frac{dy}{dx} = -\frac{\pi\cos(\pi x)}{2y}$:

$$= -\frac{-2\pi^2 y\sin(\pi x) + 2\pi\cos(\pi x)\cdot\frac{\pi\cos(\pi x)}{2y}}{4y^2} = -\frac{-2\pi^2 y\sin(\pi x) + \frac{\pi^2\cos^2(\pi x)}{y}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{\pi^2\left(2y^2\sin(\pi x) - \cos^2(\pi x)\right)}{4y^3}$$

A1

Note: Accept equivalent unsimplified forms; penalise only if chain-rule y-derivative terms are absent.

Q29 — Related Rates for a Conical Funnel with Net Flow HARD

(a) For a cone with half-angle 30°: $r = h\tan 30^\circ = \frac{h}{\sqrt{3}}$.

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi\cdot\frac{h^2}{3}\cdot h = \frac{\pi h^3}{9}$$

AG (A1)

Note: equivalently written $V = \pi h^3/9$.

(b) The chain-rule equation linking rates (written before substituting values):

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \frac{\pi h^2}{3} \cdot \frac{dh}{dt}$$

M1

The net rate of change of volume is $10 - 6 = 4$ cm^{3}s^{-1}: **A1**

$$4 = \frac{\pi(12)^2}{3} \cdot \frac{dh}{dt} = 48\pi \cdot \frac{dh}{dt}$$

M1

$$\frac{dh}{dt} = \frac{4}{48\pi} = \frac{1}{12\pi} \approx 0.0265 \text{ cm s}^{-1}$$

A1

GDC note: evaluate $1/(12\pi)$ *numerically:* $0.026525\ldots \approx 0.0265$ *cm s*$^{-1}$.

Q30 — Higher Derivatives and Inflection Sign-Change Justification HARD

(a) $f(x) = x^2 \cos x$. Applying the product rule:

$$f'(x) = 2x \cos x - x^2 \sin x$$

M1A1

Differentiating again (product rule on each term):

$$f''(x) = 2\cos x - 2x \sin x - 2x \sin x - x^2 \cos x = 2\cos x - 4x \sin x - x^2 \cos x$$

A1

(b) Using GDC: graph $y = f''(x)$ and use graph-and-intersect with the x-axis on $(1, 4)$.
Unrounded: $x = 2.68897\ldots$

$$x \approx 2.69$$

A1

(c) Evaluating f'' on either side: $f''(2.6) \approx -1.28 < 0$ and $f''(2.8) \approx 1.75 > 0$. Since f'' changes sign from negative to positive at $x \approx 2.69$, this confirms the point is an inflection point. **R1**

Note: The R1 requires explicit numerical evidence of the sign change and a conclusion in context.