IB Math AA HL — Topic 5

Differentiation Toolkit

First Principles · Chain, Product & Quotient Rules · Implicit & Parametric · Related Rates

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:____________________________ Date:______________

Score:_________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **First Principles:** $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$; limit notation is kept until all h factors cancel.
- **Power / Sum Rules:** $\dfrac{d}{dx}[x^n] = nx^{n-1}$; $\dfrac{d}{dx}[f \pm g] = f' \pm g'$.
- **Chain, Product, Quotient Rules:** $\dfrac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$; $\dfrac{d}{dx}[uv] = u'v + uv'$; $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{u'v - uv'}{v^2}$.
- **Standard Derivatives:** $\dfrac{d}{dx}e^x = e^x$; $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$; $\dfrac{d}{dx}a^x = a^x \ln a$; $\dfrac{d}{dx}\sin x = \cos x$; $\dfrac{d}{dx}\cos x = -\sin x$; $\dfrac{d}{dx}\tan x = \sec^2 x$.
- **Implicit Differentiation:** $\dfrac{d}{dx}[y^n] = ny^{n-1}\dfrac{dy}{dx}$; differentiate both sides with respect to x, then collect $\dfrac{dy}{dx}$ terms.
- **Parametric Differentiation:** $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$; $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left[\dfrac{dy}{dx}\right] \div \dfrac{dx}{dt}$.
- **Related Rates (Chain Rule):** Write the connecting equation first, e.g. $\dfrac{dV}{dt} = \dfrac{dV}{dr}\cdot\dfrac{dr}{dt}$, before substituting any values; every rate carries its units.

GDC Tips

- **Numerical Derivative at a Point:** Use `dy/dx` (graph screen) or `nDeriv(f(x),x,a)` (TI) / `diff(f(x),x) | x=a` (Casio CAS) to verify an exact gradient answer.
- **Graphing to Find Gradient:** Plot $y = f(x)$, trace to the required x-value, and select `dy/dx` from the Calc menu; record the result to 3 significant figures.
- **Implicit Curves:** Enter the relation as two explicit branches (solve for y first, or use the GDC's implicit-plot mode) to visualise the curve before differentiating by hand.
- **Intersection for Related-Rates Check:** Use `graph-and-intersect` to find when two rates are equal; input both rate expressions as separate functions of the variable and locate the crossing point.
- **Parametric Plotting:** In parametric mode enter $x(t)$ and $y(t)$; use `dy/dx` at the required t-value to confirm the parametric gradient, and zoom in to verify tangent direction.
- **Numerical Solve (`nSolve` / `Solve`):** Use `nSolve(f'(x)=0,x,a)` to locate stationary points when the derivative is too complex to solve by hand; state the GDC command and initial guess in your working.

Common Mistakes to Avoid

1. **Cancelling h before expanding in first principles.** Students substitute $h = 0$ into the unsimplified quotient $\dfrac{f(x+h)-f(x)}{h}$, obtaining $\frac{0}{0}$. You must expand $f(x+h)$, cancel every remaining h factor algebraically, and only then take $\lim_{h\to 0}$.

2. **Forgetting the chain-rule factor inside implicit differentiation.** Writing $\dfrac{d}{dx}[y^2] = 2y$ instead of $2y\,\dfrac{dy}{dx}$ is the most common implicit error; every y-term differentiated with respect to x must carry $\dfrac{dy}{dx}$ as a factor.

3. **Wrong sign order in the quotient rule.** Writing $\dfrac{u'v+uv'}{v^2}$ (product-rule sign) instead of $\dfrac{u'v-uv'}{v^2}$; always form *low d(high) minus high d(low)* in the numerator.

4. **Dividing by dx/dt only once for the parametric second derivative.** Students compute $\dfrac{d}{dt}\left[\dfrac{dy}{dx}\right]$ and call it $\dfrac{d^2y}{dx^2}$; the correct formula requires a second division by $\dfrac{dx}{dt}$: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left[\dfrac{dy}{dx}\right] \div \dfrac{dx}{dt}$.

5. **Substituting values before writing the related-rates chain equation.** Inserting numerical values of r, V, etc. into the formula *before* establishing $\dfrac{dV}{dt} = \dfrac{dV}{dr}\cdot\dfrac{dr}{dt}$ loses the method mark; the connecting equation with units must appear first.

6. **Dropping the chain-rule factor when differentiating $e^{g(x)}$ or $\ln(g(x))$.** Writing $\dfrac{d}{dx}[e^{3x}] = e^{3x}$ or $\dfrac{d}{dx}[\ln(x^2+1)] = \dfrac{1}{x^2+1}$ without the inner derivative; the correct results are $3e^{3x}$ and $\dfrac{2x}{x^2+1}$ respectively.

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Section A — Easy

EASY

1. Let $f(x) = 3x^2 - 5x + 4$. Find $f'(x)$ from first principles using the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Calculator not allowed **[3 marks]**

2. Differentiate $g(x) = 4x^3 - \dfrac{6}{x^2} + \sqrt{x}$ with respect to x. *Calculator not allowed* **[3 marks]**

3. Let $y = (3x^2 + 1)^5$. Find $\dfrac{dy}{dx}$. *Calculator allowed* **[3 marks]**

4. A curve is defined by $x^2 + y^2 = 25$. Find $\dfrac{dy}{dx}$ using implicit differentiation. *Calculator not allowed* **[3 marks]**

5. A curve is defined parametrically by $x = t^2 + 1$ and $y = 2t^3 - 3t$, where $t \in \mathbb{R}$. Find $\dfrac{dy}{dx}$ in terms of t. *Calculator allowed* **[3 marks]**

6. Let $h(x) = x^2 e^{3x}$. Find $h'(x)$ using the product rule. *Calculator allowed* **[3 marks]**

7. The table below shows values of two differentiable functions f and g and their derivatives at $x = 2$.

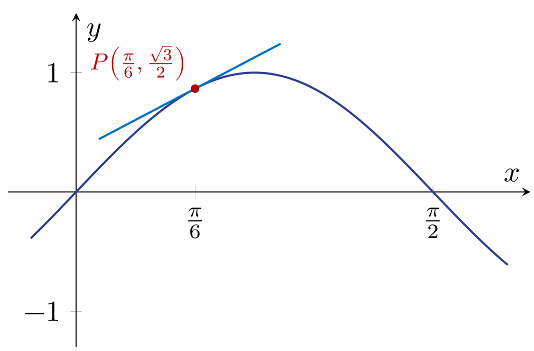
x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	3	-1	5	4

Let $p(x) = f(x) \cdot g(x)$. Find $p'(2)$. *Calculator allowed* **[3 marks]**

8. A spherical balloon is being inflated. At the instant when the radius is $r = 4$ cm, the radius is increasing at a rate of 0.5 cm s^{-1}. The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Find the rate of change of volume at this instant, giving your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

9. Let $f(x) = \sin(2x)$. The diagram below shows the curve $y = f(x)$ and the tangent to the curve at the point $P\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$.

Find the gradient of the tangent at P.



Calculator allowed **[3 marks]**

10. Let $f(x) = e^{2x}$. Find the second derivative $f''(x)$ and write down the value of $f''(0)$. *Calculator allowed* **[3 marks]**

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Section B — Medium

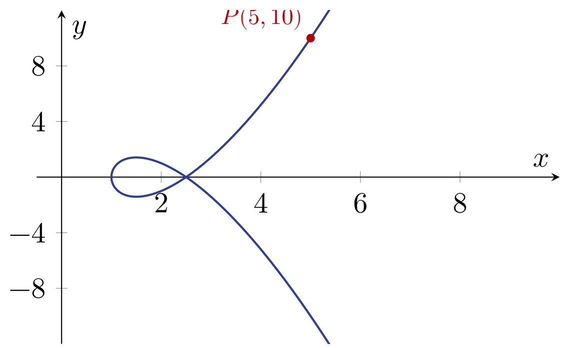
MEDIUM

11. Let $f(x) = 3x^2 - 5x$. Using differentiation from first principles, find $f'(x)$. *Calculator not allowed* **[4 marks]**

12. A curve is defined by $y = x^3 e^{2x}$. Find $\dfrac{dy}{dx}$. *Calculator not allowed* **[4 marks]**

13. The curve C has equation $x^2 + 3xy + y^2 = 7$. Find the gradient of C at the point $(1, 1)$. *Calculator not allowed* **[4 marks]**

14. A curve is defined parametrically by $x = t^2 + 1$ and $y = 2t^3 - 3t$, where $t \in \mathbb{R}$. Find the gradient of the curve at the point where $t = 2$. *Calculator allowed* **[4 marks]**



15. A spherical balloon is being inflated. At the instant when the radius is $r = 4$ cm, the radius is increasing at 0.3 cm s^{-1}. The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$.
Find the rate at which the volume is increasing at this instant, giving your answer in cm^3 s^{-1} correct to 3 significant figures. *Calculator allowed* **[4 marks]**

16. The functions f and g are twice differentiable. Selected values are given in the table below.

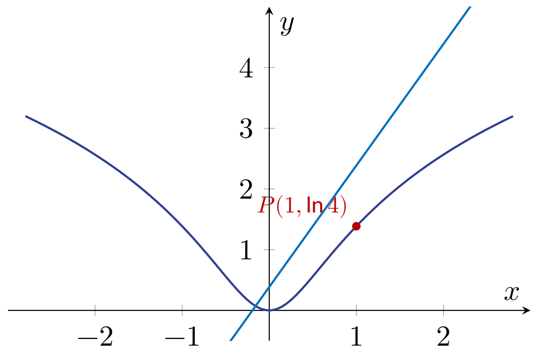
x	1	2	3	4
$f(x)$	3	5	2	6
$f'(x)$	-1	4	-3	7
$g(x)$	2	3	1	4
$g'(x)$	5	-2	6	-1

Let $h(x) = f(x)\,g(x)$. Find $h'(2)$. *Calculator allowed* **[4 marks]**

17. A ladder of length 5 m leans against a vertical wall. The foot of the ladder slides away from the wall along horizontal ground. At a certain instant, the foot of the ladder is 3 m from the wall and moving at 0.4 m s^{-1}.
Let x be the distance of the foot of the ladder from the wall and y be the height of the top of the ladder on the wall, both in metres.
Find the rate at which the top of the ladder is sliding **down** the wall at this instant, giving your answer in m s^{-1} correct to 3 significant figures. *Calculator allowed* **[4 marks]**

18. Let $f(x) = \dfrac{\sin x}{e^x}$. Find $f''(x)$. *Calculator not allowed* **[4 marks]**

19. A curve is defined by $y = \ln(3x^2 + 1)$.



Find the equation of the tangent to the curve at the point P where $x = 1$, giving your answer in the form $y = mx + c$. *Calculator allowed* **[4 marks]**

20. Water drains from a conical tank (vertex downwards) such that when the depth of water is h cm the volume is $V = \dfrac{\pi h^3}{12}$ cm^3. At the instant when $h = 6$ cm, water is draining at 9π cm^3 min^{-1}.
Find the rate at which the depth is decreasing at this instant, in cm min^{-1}. *Calculator allowed* **[4 marks]**

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Section C — Hard

HARD

21. A curve is defined implicitly by

$$x^3 + 2xy^2 - y^3 = 8.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2 + 2y^2}{3y^2 - 4xy}$. [3]

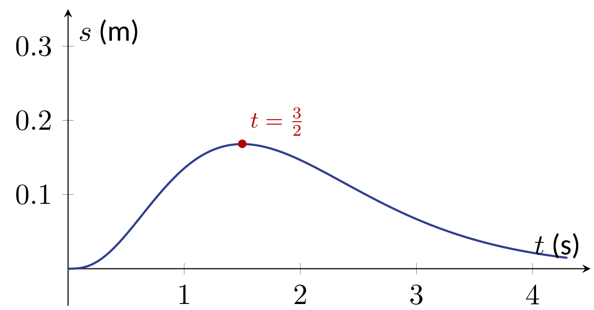
(b) Hence find the gradient of the curve at the point $(2, 0)$. [1]

(c) Write down the equation of the tangent to the curve at $(2, 0)$. [1] *Calculator not allowed* **[5 marks]**

22. A particle moves along a straight line. Its displacement from the origin at time $t \geq 0$ seconds is given by

$$s(t) = t^3 e^{-2t} \quad \text{(metres)}.$$

(a) Find $s'(t)$. [2]
(b) Find the exact value of t at which the particle's velocity is zero for $t > 0$. [2]
(c) Justify, using $s''(t)$, whether the velocity is increasing or decreasing at this instant. [1] *Calculator not allowed* **[5 marks]**



23. A curve is defined parametrically by

$$x = 3\cos t - \cos 3t, \qquad y = 3\sin t - \sin 3t, \qquad 0 < t < \pi.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{3\cos t - 3\cos 3t}{-3\sin t + 3\sin 3t}$. [2]

(b) Find the gradient of the curve at $t = \dfrac{\pi}{2}$. Give your answer as an exact value. [2]

(c) Write down the coordinates of the point on the curve at $t = \dfrac{\pi}{2}$. [1] *Calculator not allowed* **[5 marks]**

24. A spherical balloon is being inflated so that its radius r cm increases at a constant rate of 0.4 cm s^{-1}. At a particular instant, $r = 7.5$ cm.

(a) Write down the chain-rule equation linking $\dfrac{dV}{dt}$ and $\dfrac{dr}{dt}$, where $V = \frac{4}{3}\pi r^3$. [1]

(b) Find the rate of increase of the surface area $A = 4\pi r^2$ at this instant. Give your answer in cm^2 s^{-1} to 3 significant figures. [2]

(c) At the same instant, find the rate of increase of V to 3 significant figures. [2] *Calculator allowed* **[5 marks]**

25. Let $f(x) = \dfrac{x^2 \ln x}{e^x}$ for $x > 0$.

(a) Find $f'(x)$. [3]

(b) Using a GDC, find the x-coordinate of the stationary point of f for $x > 0$, giving your answer to 3 significant figures. [2] *Calculator allowed* **[5 marks]**

26. The function $f(x) = \sqrt{4 + x^2}$ is to be differentiated from first principles.

(a) Using the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, show that

$$f'(x) = \lim_{h \to 0} \frac{h}{h\left(\sqrt{4 + (x+h)^2} + \sqrt{4 + x^2}\right)}.$$

[3]

(b) Hence show that $f'(x) = \dfrac{x}{\sqrt{4 + x^2}}$. [2] *Calculator not allowed* **[5 marks]**

27. The table below gives values of two differentiable functions f and g, and their derivatives, at $x = 2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	5	-3	1	4

Let $h(x) = \dfrac{f(x)\,g(x)}{e^{g(x)}}$.

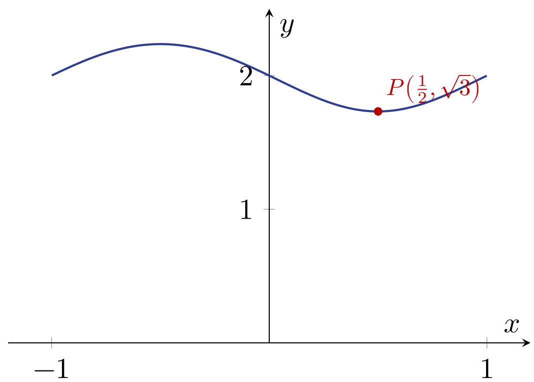
(a) Find $h'(x)$ in terms of f, g and their derivatives. [3]

(b) Hence find the exact value of $h'(2)$. [2] *Calculator allowed* **[5 marks]**

28. A curve C is defined implicitly by $y^2 + \sin(\pi x) = 4$, where $x \in (-1, 1)$ and $y > 0$.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y. [2]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of x and y. [3]



Calculator not allowed **[5 marks]**

29. A conical funnel (vertex pointing downward) has a fixed half-angle of 30°. Water drains out of the vertex at 6 cm^3 s^{-1} while being poured in at 10 cm^3 s^{-1}. When the depth of water is $h = 12$ cm:

(a) show that the volume of water is $V = \dfrac{\pi h^3}{9}$, [1]

(b) find $\dfrac{dh}{dt}$ at this instant, giving your answer in cm s^{-1} to 3 significant figures. [4] *Calculator allowed* **[5 marks]**

30. Consider the function $f(x) = x^2 \sin x$.
(a) Find $f'(x)$ and $f''(x)$. [3]
(b) Using a GDC, find the x-coordinate of the inflection point of f in the interval $(1, 4)$, giving your answer to 3 significant figures. [1]
(c) Justify that this point is indeed an inflection by verifying the sign change of $f''(x)$ on either side of it. [1]
Calculator allowed **[5 marks]**

Worked Solutions

Q1 — First principles for a quadratic [EASY]

Expanding $f(x+h) = 3(x+h)^2 - 5(x+h) + 4 = 3x^2 + 6xh + 3h^2 - 5x - 5h + 4$ **M1**

$$\frac{f(x+h) - f(x)}{h} = \frac{6xh + 3h^2 - 5h}{h} = 6x + 3h - 5$$

A1

Taking the limit: $f'(x) = \lim_{h \to 0}(6x + 3h - 5) = 6x - 5$ **A1**

Note: Award M0 if limit notation is absent before the h terms cancel.

Q2 — Power rule differentiation [EASY]

Rewrite: $g(x) = 4x^3 - 6x^{-2} + x^{1/2}$ **M1**

Differentiating each term:

$$g'(x) = 12x^2 + 12x^{-3} + \tfrac{1}{2}x^{-1/2}$$

A1

Writing in equivalent form: $g'(x) = 12x^2 + \dfrac{12}{x^3} + \dfrac{1}{2\sqrt{x}}$ **A1**

Note: Award A1 A0 if exactly one term is incorrect. Accept any equivalent unsimplified form.

Q3 — Chain rule on a composite power [EASY]

Identifying the outer and inner functions, applying the chain rule: **M1**

$$\frac{dy}{dx} = 5(3x^2 + 1)^4 \cdot 6x$$

A1

$$\frac{dy}{dx} = 30x(3x^2 + 1)^4$$

A1

Q4 — Implicit differentiation of a circle [EASY]

Differentiating both sides with respect to x: **M1**

$$2x + 2y\,\frac{dy}{dx} = 0$$

A1

$$\frac{dy}{dx} = -\frac{x}{y}$$

A1

Note: Award M1 for correctly differentiating y^2 *as* $2y\,\frac{dy}{dx}$*, showing the chain-rule step.*

Q5 — Parametric differentiation [EASY]

Finding the two component derivatives: $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 6t^2 - 3$ **M1**

Applying $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$: **A1**

$$\frac{dy}{dx} = \frac{6t^2 - 3}{2t}$$

A1

Note: Award A0 for inverting the ratio $dx/dt \div dy/dt$.

Q6 — Product rule with exponential [EASY]

Identifying $u = x^2$, $v = e^{3x}$, so $u' = 2x$, $v' = 3e^{3x}$ **M1**

Applying the product rule $h'(x) = u'v + uv'$:

$$h'(x) = 2x\,e^{3x} + x^2 \cdot 3e^{3x}$$

A1

$$h'(x) = e^{3x}(2x + 3x^2)$$

A1

Q7 — Product rule from a table [EASY]

Applying the product rule: $p'(x) = f'(x)\,g(x) + f(x)\,g'(x)$ **M1**

Substituting values from the table at $x = 2$:

$$p'(2) = (-1)(5) + (3)(4)$$

A1

$$p'(2) = -5 + 12 = 7$$

A1

Q8 — Related rates for a sphere [EASY]

Writing the chain-rule connecting equation before substituting:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

M1

Substituting $r = 4$ cm and $\frac{dr}{dt} = 0.5$ cm s^{-1}:

$$\frac{dV}{dt} = 4\pi(4)^2 \times 0.5 = 32\pi$$

A1

Using GDC to evaluate: $32\pi = 100.530\ldots \approx 101$ cm^3 s^{-1} **A1**

Note: The connecting equation $\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$ *must appear before values are substituted; award M0 if values are inserted into* $\frac{dV}{dr}$ *without the chain-rule step.*

Q9 — Gradient of tangent to a sine curve [EASY]

Differentiating: $f'(x) = 2\cos(2x)$ **M1**

Substituting $x = \frac{\pi}{6}$:

$$f'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right)$$

A1

$$= 2 \times \frac{1}{2} = 1$$

A1

Note: GDC numerical derivative at $x = \pi/6$ *gives* $1.00000\ldots$, *confirming the exact answer of* 1.

Q10 — Second derivative of an exponential [EASY]

First derivative: $f'(x) = 2e^{2x}$ **M1**

Second derivative: $f''(x) = 4e^{2x}$ **A1**

Substituting $x = 0$: $f''(0) = 4e^0 = 4$ **A1**

Q11 — Differentiation from first principles for $3x^2 - 5x$ [MEDIUM]

Using the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$: **M1**

$f(x+h) = 3(x+h)^2 - 5(x+h) = 3x^2 + 6xh + 3h^2 - 5x - 5h$

$\dfrac{f(x+h) - f(x)}{h} = \dfrac{6xh + 3h^2 - 5h}{h} = 6x + 3h - 5$ **A1**

Taking the limit as $h \to 0$ (do not substitute $h = 0$ into the unsimplified quotient): **M1**

$f'(x) = \lim_{h \to 0}(6x + 3h - 5) = 6x - 5$ **A1**

Note: Award M0 if $h = 0$ is substituted before cancellation.

Q12 — Product rule with $x^3 e^{2x}$ [MEDIUM]

Let $u = x^3$ and $v = e^{2x}$, so $u' = 3x^2$ and $v' = 2e^{2x}$. **M1**

Applying the product rule $\dfrac{dy}{dx} = u'v + uv'$: **M1**

$\dfrac{dy}{dx} = 3x^2 e^{2x} + x^3 \cdot 2e^{2x}$ **A1**

$= e^{2x}(3x^2 + 2x^3)$ **A1**

Note: Accept unsimplified form $3x^2 e^{2x} + 2x^3 e^{2x}$ for full marks.

Q13 — Implicit differentiation of $x^2 + 3xy + y^2 = 7$ [MEDIUM]

Differentiating both sides with respect to x: **M1**

$2x + 3\left(y + x\dfrac{dy}{dx}\right) + 2y\dfrac{dy}{dx} = 0$ **A1**

Note: Award A1 for correct application of chain rule on y^2 giving $2y\,\frac{dy}{dx}$, and product rule on $3xy$.

Collecting $\dfrac{dy}{dx}$ terms: $(3x + 2y)\dfrac{dy}{dx} = -2x - 3y$ **M1**

At $(1, 1)$: $(3 + 2)\dfrac{dy}{dx} = -2 - 3 \implies \dfrac{dy}{dx} = -1$ **A1**

Q14 — Parametric gradient at $t = 2$ [MEDIUM]

$\dfrac{dx}{dt} = 2t, \quad \dfrac{dy}{dt} = 6t^2 - 3$ **A1**

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{6t^2 - 3}{2t}$ **M1**

At $t = 2$: $\dfrac{dy}{dx} = \dfrac{6(4) - 3}{2(2)} = \dfrac{21}{4}$ **A1**

Using GDC to verify: at $t = 2$ the point is $(5, 10)$; numerical derivative of the parametric curve at $t = 2$ confirms gradient $= 5.25$. **A1**

Note: The point $P(5, 10)$ is shown on the diagram.

Q15 — Related rates, spherical balloon [MEDIUM]

Write the chain-rule link before substituting values: **M1**

$\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$

$\dfrac{dV}{dr} = 4\pi r^2$ **A1**

At $r = 4$ cm, $\dfrac{dr}{dt} = 0.3$ cm s^{-1}:

$\dfrac{dV}{dt} = 4\pi(4)^2 \times 0.3 = 19.2\pi$ **M1**

GDC evaluate: $19.2\pi = 60.318\ldots \approx 60.3$ cm^3 s^{-1} **A1**

Note: Units cm^3 s^{-1} required for final A1.

Q16 — Product rule from a table of values [MEDIUM]

The product rule gives $h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$. **M1**

Identifying the required values from the table:

$f'(2) = 4, \quad g(2) = 3, \quad f(2) = 5, \quad g'(2) = -2$ **A1**

$h'(2) = (4)(3) + (5)(-2)$ **M1**

$h'(2) = 12 - 10 = 2$ **A1**

Q17 — Related rates, sliding ladder [MEDIUM]

By Pythagoras: $x^2 + y^2 = 25$. Write the chain-rule equation before substituting: **M1**

Differentiating with respect to t: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ **A1**

At $x = 3$: $y = \sqrt{25 - 9} = 4$ m.

$\dfrac{dy}{dt} = -\dfrac{x}{y} \cdot \dfrac{dx}{dt} = -\dfrac{3}{4} \times 0.4$ **M1**

$\dfrac{dy}{dt} = -0.3$ m s^{-1}; the top is sliding **down** at 0.300 m s^{-1}. **A1**

Note: The negative sign confirms downward motion; accept 0.3 m s^{-1} downwards.

Q18 — Second derivative of $\sin x/e^x$ [MEDIUM]

Write $f(x) = e^{-x}\sin x$. Using the product rule with $u = e^{-x}$, $v = \sin x$:

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x - \sin x)$ **M1A1**

Differentiating again using the product rule on $e^{-x}(\cos x - \sin x)$: **M1**

$f''(x) = -e^{-x}(\cos x - \sin x) + e^{-x}(-\sin x - \cos x)$

$= e^{-x}(-\cos x + \sin x - \sin x - \cos x) = -2e^{-x}\cos x$ **A1**

Note: Award M1A1 for correct first derivative, M1A1 for correct second derivative.

Q19 — Tangent to $\ln(3x^2 + 1)$ at $x = 1$ [MEDIUM]

$\dfrac{dy}{dx} = \dfrac{6x}{3x^2 + 1}$ **M1A1**

At $x = 1$: gradient $m = \dfrac{6}{4} = \dfrac{3}{2}$. **A1**

Point on curve: $y = \ln(3 + 1) = \ln 4$.

Tangent: $y - \ln 4 = \dfrac{3}{2}(x - 1)$

GDC evaluate $\ln 4 = 1.386\ldots$; *tangent equation:*

$y = \dfrac{3}{2}x - \dfrac{3}{2} + \ln 4$ **A1**

Note: Accept exact form $y = \frac{3}{2}x + \ln 4 - \frac{3}{2}$ or decimal $y \approx \frac{3}{2}x - 0.114$.

Q20 — Related rates, conical tank draining [MEDIUM]

Write the chain-rule link before substituting values: **M1**

$\dfrac{dV}{dt} = \dfrac{dV}{dh} \cdot \dfrac{dh}{dt}$

$\dfrac{dV}{dh} = \dfrac{\pi h^2}{4}$ **A1**

At $h = 6$ cm: $\dfrac{dV}{dh} = \dfrac{\pi(36)}{4} = 9\pi$ cm^2.

Water draining means $\dfrac{dV}{dt} = -9\pi$ cm^3 min^{-1}:

$-9\pi = 9\pi \cdot \dfrac{dh}{dt} \implies \dfrac{dh}{dt} = -1$ cm min^{-1} **M1**

The depth is decreasing at 1 cm min^{-1}. **A1**

Note: Negative sign confirms depth is decreasing; accept 1 cm min^{-1} (decreasing).

Q21 — Implicit differentiation, tangent line [HARD]

(a) Differentiating $x^3 + 2xy^2 - y^3 = 8$ implicitly with respect to x:

$$3x^2 + 2y^2 + 4xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

M1

(chain-rule terms $4xy\frac{dy}{dx}$ and $-3y^2\frac{dy}{dx}$ visible) **A1**

Collecting $\frac{dy}{dx}$ terms:

$$\frac{dy}{dx}(4xy - 3y^2) = -(3x^2 + 2y^2) \implies \frac{dy}{dx} = \frac{3x^2 + 2y^2}{3y^2 - 4xy}$$

AG

(b) At $(2, 0)$:

$$\frac{dy}{dx} = \frac{3(4) + 0}{0 - 0}$$

Substituting $x = 2, y = 0$: numerator $= 12$, denominator $= 0 - 4(2)(0) = 0$.

Note: The denominator is zero; we check that $(2, 0)$ lies on the curve: $8 + 0 - 0 = 8$ ✓. The tangent is vertical. **A1**

(c) Since the gradient is undefined (vertical tangent), the equation of the tangent is:

$$x = 2$$

A1

Note: Award the final A1 only if the student identifies the vertical tangent; do not accept $y =$ (anything).

Q22 — Product rule, velocity zero, second derivative test [HARD]

(a) Using the product rule on $s(t) = t^3e^{-2t}$:

$$s'(t) = 3t^2e^{-2t} + t^3(-2e^{-2t}) = e^{-2t}(3t^2 - 2t^3)$$

M1A1

(b) Setting $s'(t) = 0$ for $t > 0$: since $e^{-2t} > 0$,

$$3t^2 - 2t^3 = 0 \implies t^2(3 - 2t) = 0 \implies t = \tfrac{3}{2}$$

M1A1

(c) Finding $s''(t)$:

$$s''(t) = e^{-2t}(6t - 6t^2) + (3t^2 - 2t^3)(-2e^{-2t}) = e^{-2t}(6t - 6t^2 - 6t^2 + 4t^3)$$

At $t = \frac{3}{2}$:

$$s''\left(\tfrac{3}{2}\right) = e^{-3}\left(9 - \tfrac{27}{2} - \tfrac{27}{2} + \tfrac{27}{2}\right) = e^{-3}\left(9 - \tfrac{27}{2}\right) = e^{-3}\left(-\tfrac{9}{2}\right) < 0$$

Since $s''\left(\frac{3}{2}\right) < 0$, the velocity is **decreasing** at this instant. **R1**

Note: The R1 requires a sentence in context; accept "the particle is decelerating" with justification from $s'' < 0$.

Q23 — Parametric differentiation, exact gradient [HARD]

(a) Differentiating:

$$\frac{dx}{dt} = -3\sin t + 3\sin 3t, \qquad \frac{dy}{dt} = 3\cos t - 3\cos 3t$$

M1

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t - 3\cos 3t}{-3\sin t + 3\sin 3t}$$

AG

(b) At $t = \dfrac{\pi}{2}$: $\cos\frac{\pi}{2} = 0$, $\cos\frac{3\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, $\sin\frac{3\pi}{2} = -1$:

$$\frac{dy}{dx} = \frac{3(0) - 3(0)}{-3(1) + 3(-1)} = \frac{0}{-6} = 0$$

M1A1

(c) At $t = \dfrac{\pi}{2}$:

$$x = 3\cos\tfrac{\pi}{2} - \cos\tfrac{3\pi}{2} = 0 - 0 = 0, \quad y = 3\sin\tfrac{\pi}{2} - \sin\tfrac{3\pi}{2} = 3 - (-1) = 4$$

The coordinates are $(0, 4)$. **A1**

Q24 — Related rates, balloon, surface area and volume [HARD]

(a) Chain-rule equation written explicitly:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

A1

(b) For surface area $A = 4\pi r^2$:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

M1

At $r = 7.5$, $\frac{dr}{dt} = 0.4$:

$$\frac{dA}{dt} = 8\pi(7.5)(0.4) = 24\pi \approx 75.4\ \text{cm}^2\ \text{s}^{-1}$$

A1

(c) From part (a):

$$\frac{dV}{dt} = 4\pi(7.5)^2(0.4) = 4\pi(56.25)(0.4) = 90\pi$$

M1

$$\frac{dV}{dt} \approx 283\ \text{cm}^3\ \text{s}^{-1}$$

A1

GDC note: evaluate 90π numerically to give $282.743\ldots \approx 283$ cm^3 s^{-1}.

Q25 — Quotient/product/ln derivative, GDC stationary point [HARD]

(a) Write $f(x) = \frac{x^2 \ln x}{e^x}$. Using the quotient rule (low $= e^x$, high $= x^2 \ln x$):

$$\frac{d}{dx}(x^2 \ln x) = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$$

M1

$$f'(x) = \frac{(2x \ln x + x)e^x - x^2 \ln x \cdot e^x}{e^{2x}} = \frac{e^x(x + 2x \ln x - x^2 \ln x)}{e^{2x}}$$

A1

$$f'(x) = \frac{x + 2x \ln x - x^2 \ln x}{e^x} = \frac{x(1 + (2 - x) \ln x)}{e^x}$$

A1

(b) Using GDC: enter `nSolve(`$f'(x) = 0,\ x,\ x > 0$`)` or graph $y = f'(x)$ and use *graph-and-intersect* with the x-axis.

Unrounded result: $x = 3.5131\ldots$

$$x \approx 3.51$$

M1A1

Q26 — First principles for $\sqrt{4+x^2}$ [HARD]

(a) Form the difference quotient:

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{4 + (x+h)^2} - \sqrt{4 + x^2}}{h}$$

M1

Multiply numerator and denominator by the conjugate $\sqrt{4 + (x+h)^2} + \sqrt{4 + x^2}$:

$$= \frac{[4 + (x+h)^2] - [4 + x^2]}{h\left(\sqrt{4 + (x+h)^2} + \sqrt{4 + x^2}\right)}$$

M1

Expanding: $(x+h)^2 - x^2 = 2xh + h^2$, so numerator $= 2xh + h^2 = h(2x+h)$.

$$= \frac{h(2x+h)}{h\left(\sqrt{4 + (x+h)^2} + \sqrt{4 + x^2}\right)}$$

which, after cancelling one factor of h (valid for $h \neq 0$), gives the printed form. **AG (A1 for the cancellation step)**

(b) Taking the limit as $h \to 0$:

$$f'(x) = \lim_{h \to 0} \frac{2x + h}{\sqrt{4 + (x+h)^2} + \sqrt{4 + x^2}} = \frac{2x}{2\sqrt{4 + x^2}} = \frac{x}{\sqrt{4 + x^2}}$$

M1A1

Note: Award M1 for substituting $h = 0$ into the simplified expression only (not into the original unsimplified quotient); the limit notation must be present until the h terms are gone. **AG**

Q27 — Table-based derivative, quotient with composite [HARD]

(a) Write $h(x) = f(x)g(x) \cdot e^{-g(x)}$. Apply the product rule to $f(x)g(x)$ and chain rule on $e^{-g(x)}$:

$$h'(x) = [f'(x)g(x) + f(x)g'(x)]e^{-g(x)} + f(x)g(x) \cdot (-g'(x)e^{-g(x)})$$

M1A1

$$h'(x) = e^{-g(x)}\Big[f'(x)g(x) + f(x)g'(x) - f(x)g(x)g'(x)\Big]$$

A1

(b) Substituting values at $x = 2$: $f = 5$, $f' = -3$, $g = 1$, $g' = 4$:

$$h'(2) = e^{-1}[(-3)(1) + (5)(4) - (5)(1)(4)] = e^{-1}[-3 + 20 - 20] = \frac{-3}{e}$$

M1A1

GDC note: evaluate $-3/e \approx -1.10$ as numerical confirmation; exact answer $-3e^{-1}$ required.

Q28 — Implicit second derivative with trig [HARD]

(a) Differentiating $y^2 + \sin(\pi x) = 4$ implicitly:

$$2y\frac{dy}{dx} + \pi\cos(\pi x) = 0$$

M1

$$\frac{dy}{dx} = -\frac{\pi\cos(\pi x)}{2y}$$

A1

(b) Differentiating $\frac{dy}{dx} = -\frac{\pi\cos(\pi x)}{2y}$ with respect to x using the quotient rule:

$$\frac{d^2y}{dx^2} = -\frac{(-\pi^2\sin(\pi x))(2y) - \pi\cos(\pi x)\cdot 2\frac{dy}{dx}}{4y^2}$$

M1A1

Substituting $\frac{dy}{dx} = -\frac{\pi\cos(\pi x)}{2y}$:

$$= -\frac{-2\pi^2 y\sin(\pi x) + 2\pi\cos(\pi x)\cdot\frac{\pi\cos(\pi x)}{2y}}{4y^2} = -\frac{-2\pi^2 y\sin(\pi x) + \frac{\pi^2\cos^2(\pi x)}{y}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{\pi^2(2y^2 \sin(\pi x) - \cos^2(\pi x))}{4y^3}$$

A1

Note: Accept equivalent unsimplified forms; penalise only if chain-rule y-derivative terms are absent.

Q29 — Related rates, conical funnel, net flow [HARD]

(a) For a cone with half-angle $30°$: $r = h \tan 30° = \dfrac{h}{\sqrt{3}}$.

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi \cdot \frac{h^2}{3} \cdot h = \frac{\pi h^3}{9}$$

AG (A1)

(b) The chain-rule equation linking rates (written before substituting values):

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \frac{\pi h^2}{3} \cdot \frac{dh}{dt}$$

M1

The net rate of change of volume is $10 - 6 = 4$ cm^3 s^{-1}: **A1**

$$4 = \frac{\pi(12)^2}{3} \cdot \frac{dh}{dt} = 48\pi \cdot \frac{dh}{dt}$$

M1

$$\frac{dh}{dt} = \frac{4}{48\pi} = \frac{1}{12\pi} \approx 0.0265 \text{ cm s}^{-1}$$

A1

GDC note: evaluate $1/(12\pi)$ numerically: $0.026525\ldots \approx 0.0265$ cm s^{-1}.

Q30 — Higher derivatives, inflection, sign-change justification [HARD]

(a) $f(x) = x^2 \sin x$. Applying the product rule:

$$f'(x) = 2x \sin x + x^2 \cos x$$

M1A1

Differentiating again (product rule on each term):

$$f''(x) = 2\sin x + 2x\cos x + 2x\cos x - x^2 \sin x = 2\sin x + 4x\cos x - x^2 \sin x$$

A1

(b) Using GDC: graph $y = f''(x)$ and use *graph-and-intersect* with the x-axis on $(1, 4)$.
Unrounded: $x = 2.2889\ldots$

$$x \approx 2.29$$

A1

(c) Evaluating f'' on either side: $f''(2.2) \approx 0.28 > 0$ and $f''(2.4) \approx -0.37 < 0$. Since f'' changes sign from positive to negative at $x \approx 2.29$, this confirms the point is an inflection point. **R1**

Note: The R1 requires explicit numerical evidence of the sign change and a conclusion in context.