

IB Math AA HL — Topic 5 Applications of Differentiation

Tangents & Normals · Stationary Points & Inflection · Optimisation · Curve Sketching



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Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name: _____ Date: _____
Score: _____ / 120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.



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Key Formulae

- Tangent at $P(a, f(a))$: gradient $m = f'(a)$; equation $y - f(a) = f'(a)(x - a)$.
- Normal at $P(a, f(a))$: gradient $m_{\perp} = -\frac{1}{f'(a)}$; equation $y - f(a) = -\frac{1}{f'(a)}(x - a)$.
- Increasing / Decreasing: $f'(x) > 0 \Rightarrow f$ increasing; $f'(x) < 0 \Rightarrow f$ decreasing on that interval.
- Stationary points: solve $f'(x) = 0$; classify via second derivative test: $f''(a) < 0$ local max, $f''(a) > 0$ local min, $f''(a) = 0$ inconclusive — use sign change of f' .
- Points of inflection: necessary condition $f''(x) = 0$; confirmed only if f'' changes sign either side. A stationary inflection also has $f'(a) = 0$.
- Optimisation: express objective in one variable using a constraint; find critical point; justify max/min; state answer with units.
- Global extrema on $[a, b]$: evaluate f at all interior critical points and at both endpoints $x = a, x = b$; compare all values.

GDC Tips

- Numerical derivative: use dy/dx (graph screen) or $\text{nDeriv}(f(x), x, a)$ to evaluate $f'(a)$ at a decimal point, then form the tangent/normal equation by hand.
- Stationary points: graph $y = f'(x)$ and use zero (or root) to locate $f'(x) = 0$; confirm with minimum/maximum on $y = f(x)$ directly.
- Numerical integration / area: use $\text{fnInt}(f(x), x, a, b)$ or the $\int f dx$ option in the graph screen; set limits carefully.
- Solving equations: use nSolve (or solve) to find intersection of $f'(x) = m$ when roots are not exact; give equation and window used.
- Global extrema on a closed interval: store endpoints and all critical x -values in a list, evaluate f at each with seq , then use max/min on the result list.
- Optimisation verification: after finding the critical point analytically, graph the objective function and use minimum/maximum to confirm the value to 3 s.f.

Common Mistakes to Avoid

1. **Using $f''(a) = 0$ alone to conclude a point of inflection.** $f''(a) = 0$ is necessary but not sufficient; you must show that f'' changes sign on either side of $x = a$. The counterexample $f(x) = x^4$ has $f''(0) = 0$ but no inflection at $x = 0$.
2. **Forgetting endpoints in closed-interval extrema.** The global maximum or minimum on $[a, b]$ can occur at $x = a$ or $x = b$, not only at interior stationary points. Always evaluate $f(a)$ and $f(b)$ explicitly and compare with all critical values.
3. **Concluding a stationary point's nature from $f'' = 0$ (inconclusive case).** When $f''(a) = 0$, the second derivative test gives no information; you must revert to a sign-change analysis of f' on either side of $x = a$.
4. **Using the same gradient for a normal as for a tangent.** The normal gradient is the negative reciprocal: $m_{\perp} = -1/f'(a)$. Forgetting the negation or the reciprocal are both common errors, especially when $f'(a)$ is a fraction.
5. **Failing to justify the extremum in an optimisation problem.** Stating the critical point without confirming it is a maximum or minimum (via f'' sign or sign-change of f') and without a concluding sentence in context with units loses the reasoning mark.
6. **Using an unknown contact point incorrectly for external-point tangent problems.** The contact point must be written as $(a, f(a))$ and the tangent gradient $f'(a)$ must be used so that the line also passes through the external point — setting up one equation in a is the essential method step that is often omitted.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)



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1. **EASY** **No Calculator**

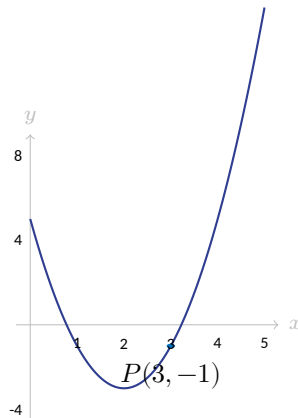
[3 marks]

Let $f(x) = x^3 - 6x^2 + 9x + 2$. Find the values of x at which f is stationary.

2. **EASY** **Calculator**

[3 marks]

Let $g(x) = 2x^2 - 8x + 5$. Find the equation of the tangent to g at the point where $x = 3$. Give your answer in the form $y = mx + c$.



3. **EASY** No Calculator

[3 marks]

Let $h(x) = x^3 - 3x + 1$. Determine the intervals on which h is increasing.

4. **EASY** Calculator

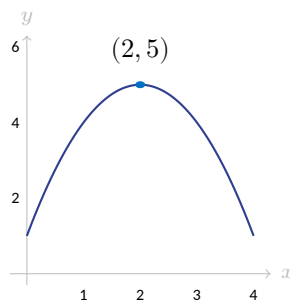
[3 marks]

A function is defined by $f(x) = x^4 - 8x^2 + 3$. Find the x -coordinates of any points of inflection of f , justifying your answer.

5. **EASY** Calculator

[3 marks]

Let $p(x) = -x^2 + 4x + 1$. Find the coordinates of the stationary point of p and classify it.



6. **EASY** **Calculator**

[3 marks]

A farmer has 60 m of fencing to enclose a rectangular pen against a straight wall; the wall forms one side so fencing is needed for only three sides. Let x m be the length of each side perpendicular to the wall. Write down an expression for the area A in terms of x , and hence find the value of x that maximises A .

7. **EASY** **Calculator**

[3 marks]

Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$. Find the x -coordinates of the stationary points of f using calculus.

8. **EASY** **No Calculator**

[3 marks]

Let $q(x) = x^2 - 6x + 11$. Find the equation of the normal to q at the point where $x = 1$. Give your answer in the form $y = mx + c$.

9. **EASY** **Calculator**

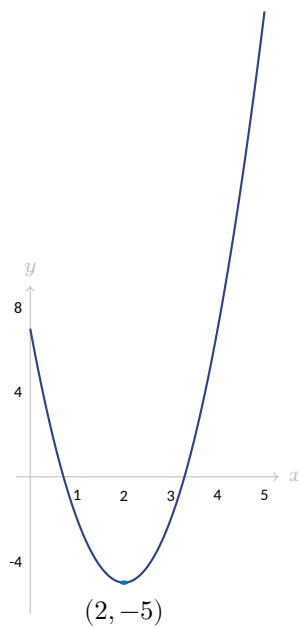
[3 marks]

Consider $f(x) = 2x^3 - 9x^2 + 12x - 4$ on the closed interval $[0, 3]$. Find the global maximum value of f on this interval.

10. **EASY** No Calculator

[3 marks]

Let $k(x) = 3x^2 - 12x + 7$. Determine the values of x for which k is decreasing.



Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)



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11. MEDIUM No Calculator [4 marks]

Let $f(x) = x^3 - 6x^2 + 9x + 1$. Find the values of x for which f is increasing.

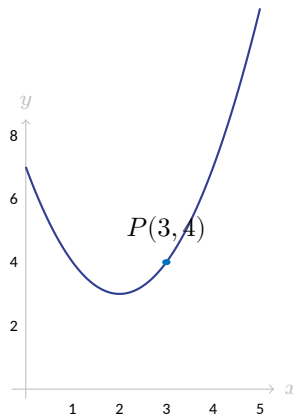
12. MEDIUM Calculator [4 marks]

Let $g(x) = 2x^3 - 3x^2 - 12x + 5$. Find the coordinates of the stationary points of g and classify each one using the second derivative test.

13. MEDIUM No Calculator

[4 marks]

The curve C has equation $y = x^2 - 4x + 7$. The point $P(3, 4)$ lies on C . Find the equation of the normal to C at P , giving your answer in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$.



14. MEDIUM No Calculator

[4 marks]

A function is defined by $h(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x + 4$. Find the x -coordinates of any points of inflection of h , justifying that each is a point of inflection by verifying a sign change of h'' .

15. MEDIUM Calculator

[4 marks]

A rectangle has its base on the x -axis and two upper vertices on the parabola $y = 12 - x^2$, where $-2\sqrt{3} \leq x \leq 2\sqrt{3}$. Let the right upper vertex have x -coordinate $t > 0$. Find the value of t that maximises the area of the rectangle, and state the maximum area.

16. MEDIUM Calculator

[4 marks]

Let $p(x) = x^4 - 8x^2 + 3$. Find the global maximum and global minimum values of p on the closed interval $[-1, 3]$, clearly evaluating all relevant values.

17. MEDIUM Calculator

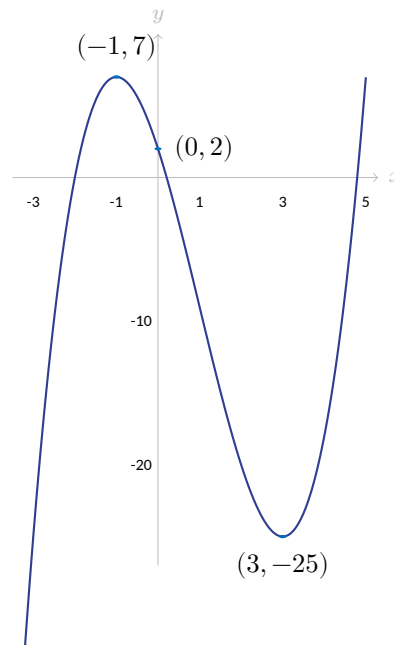
[4 marks]

The curve $y = ax^2 + bx + 3$ has a tangent with gradient 5 at the point where $x = 1$, and a stationary point at $x = -1$. Find the values of a and b .

18. MEDIUM Calculator

[4 marks]

Let $f(x) = x^3 - 3x^2 - 9x + 2$. The diagram below shows the curve $y = f(x)$ with stationary points labelled. Sketch the curve, clearly labelling the coordinates of both stationary points and the y -intercept.



19. MEDIUM Calculator

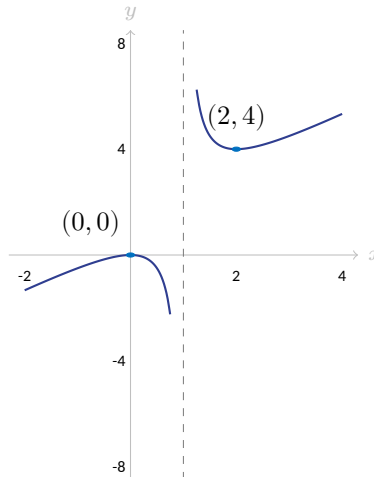
[4 marks]

A closed cylindrical can has radius r cm and height h cm. The total surface area is fixed at 300π cm². Show that the volume is $V = \pi r(150 - r^2)$, and hence find the value of r that maximises V .

20. **MEDIUM** Calculator

[4 marks]

Let $f(x) = \frac{x^2}{x-1}$, defined for $x \neq 1$. Find the x -coordinates of the stationary points of f , and use the second derivative to classify each.



Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)



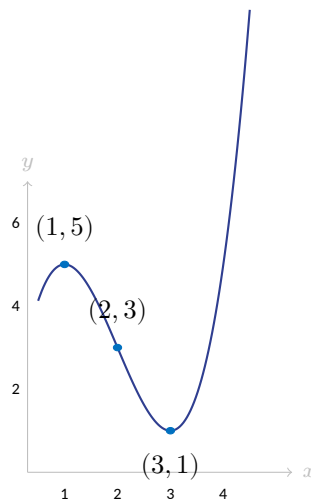
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21. **HARD** No Calculator

[5 marks]

Let $f(x) = x^3 - 6x^2 + 9x + 1$.

- (a) Show that f has a local maximum at $x = 1$ and a local minimum at $x = 3$. [3]
- (b) Find the x -coordinate of the point of inflection of f , verifying that it is indeed a point of inflection. [2]



22. **HARD** Calculator

[5 marks]

A closed cylindrical can has volume $V = 500\pi \text{ cm}^3$. The total surface area is $S = 2\pi r^2 + 2\pi rh$, where r cm is the base radius and h cm is the height.

(a) Show that $S = 2\pi r^2 + \frac{1000\pi}{r}$. [1]

(b) Find the value of r that minimises S , and state the minimum surface area to 3 significant figures. Justify that this value gives a minimum. [4]

23. **HARD** No Calculator

[5 marks]

A curve is defined by $f(x) = \frac{x^2 - 4}{x^2 + 4}$ for all $x \in \mathbb{R}$.

(a) Find $f'(x)$ and hence determine the intervals on which f is increasing and decreasing. [3]

(b) Find the equations of any horizontal asymptotes and write down the y -intercept. [2]

24. **HARD** **Calculator**

[5 marks]

Let $g(x) = xe^{-x^2}$ for $x \in \mathbb{R}$.

(a) Find $g'(x)$ and hence find all stationary points, classifying each. [3]

(b) Using a GDC, find the x -coordinates of the points of inflection of g , correct to 3 significant figures. [2]

25. **HARD** **No Calculator**

[5 marks]

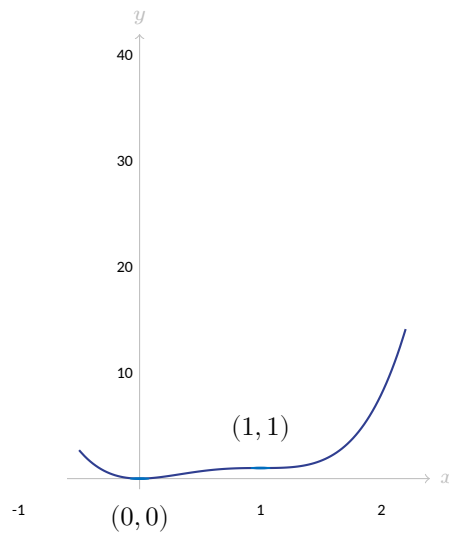
A tangent to the curve $y = x^3 - 3x$ is drawn from the external point $P(3, 8)$. Find the equation(s) of all tangent lines to $y = x^3 - 3x$ that pass through $P(3, 8)$.

26. **HARD** **Calculator**

[5 marks]

Let $h(x) = 3x^4 - 8x^3 + 6x^2$.

- (a) Find all stationary points of h and classify each. [3]
- (b) Find the global minimum value of h on the closed interval $[-1, 3]$, clearly evaluating all relevant values. [2]



27. **HARD** **Calculator**

[5 marks]

A rectangular sheet of metal measures 24 cm by 18 cm. Equal squares of side x cm are cut from each corner and the sides are folded up to form an open box of volume V cm³.

(a) Show that $V = x(24 - 2x)(18 - 2x)$. [1]

(b) Find the value of x that maximises V , correct to 3 significant figures. Justify that this value gives a maximum. [3]

(c) State the maximum volume, correct to 3 significant figures. [1]

28. **HARD** **No Calculator**

[5 marks]

The function $f(x) = ax^3 + bx^2 + cx + d$ has a stationary point of inflection at $(1, -2)$ and passes through the point $(0, 0)$. Find the values of a , b , c , and d .

29. **HARD** Calculator

[5 marks]

Consider $f(x) = \ln(x^2 + 1) - kx$ where k is a real constant and $x \in \mathbb{R}$.

(a) Show that $f'(x) = \frac{-kx^2 + 2x - k}{x^2 + 1}$. [2]

(b) Find the values of k for which f has no stationary points. Justify your answer. [3]

30. **HARD** Calculator

[5 marks]

A curve has equation $y = \frac{2x}{x^2 - 9}$, $x \neq \pm 3$.

(a) Find $\frac{dy}{dx}$ and show that the curve has no stationary points. [3]

(b) Sketch the curve, labelling all asymptotes and any intercepts with the axes. [2]

Mark Schemes

Q1 — Stationary Points of a Cubic EASY

Differentiate: $f'(x) = 3x^2 - 12x + 9$ M1
 Set $f'(x) = 0$: $3(x^2 - 4x + 3) = 3(x - 1)(x - 3) = 0$ M1
 $x = 1$ and $x = 3$ A1

Q2 — Tangent to a Quadratic EASY

$g'(x) = 4x - 8$, so gradient at $x = 3$ is $g'(3) = 4(3) - 8 = 4$ M1
 $g(3) = 2(9) - 8(3) + 5 = 18 - 24 + 5 = -1$, so tangent passes through $(3, -1)$ A1
 Tangent: $y - (-1) = 4(x - 3)$, giving $y = 4x - 13$ A1
 Note: GDC method: graph g , use tangent feature at $x = 3$; gradient = 4, equation $y = 4x - 13$.

Q3 — Increasing Intervals of a Cubic EASY

$h'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$ M1
 $h'(x) > 0$ when $x < -1$ or $x > 1$ M1
 h is increasing on $(-\infty, -1) \cup (1, \infty)$ A1

Q4 — Points of Inflection of a Quartic EASY

$f'(x) = 4x^3 - 16x$, $f''(x) = 12x^2 - 16$ M1
 Setting $f''(x) = 0$: $12x^2 = 16 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm \frac{2}{\sqrt{3}}$, i.e. $x = \pm 2/\sqrt{3}$ (exact: $\approx \pm 1.15$, i.e. $x \approx \pm 1.15$) A1
 Sign check: f'' changes from positive to negative at $x = -\frac{2}{\sqrt{3}}$ and from negative to positive at $x = \frac{2}{\sqrt{3}}$; A1
 sign changes confirmed, so both are points of inflection. A1
 Note: GDC check: use $nSolve(12x^2 - 16 = 0, x)$ giving $x \approx \pm 1.15$; verify f'' sign change by evaluating $f''(0) = -16 < 0$ and $f''(2) = 32 > 0$.

Q5 — Stationary Point of a Quadratic EASY

$p'(x) = -2x + 4$; setting $p'(x) = 0$ gives $x = 2$ M1
 $p(2) = -(4) + 8 + 1 = 5$, so stationary point is $(2, 5)$ A1
 $p''(x) = -2 < 0$ at $x = 2$ (second derivative test); the point is a local maximum. A1
 Note: GDC method: graph p , use maximum feature; result $(2, 5)$ confirmed.

Q6 — Optimisation: Rectangular Pen EASY

Side parallel to wall = $60 - 2x$, so $A = x(60 - 2x) = 60x - 2x^2$ **A1**

$$\frac{dA}{dx} = 60 - 4x = 0 \Rightarrow x = 15 \quad \text{M1}$$

$$\frac{d^2A}{dx^2} = -4 < 0, \text{ confirming a maximum; the value } x = 15 \text{ m maximises the area.} \quad \text{A1}$$

Note: GDC method: graph $A = 60x - 2x^2$ for $0 < x < 30$, use maximum feature; maximum at $x = 15$.

Q7 — Stationary Points of a Cubic EASY

$$f'(x) = x^2 - 2x - 3 \quad \text{M1}$$

$$\text{Set } f'(x) = 0: (x - 3)(x + 1) = 0 \quad \text{M1}$$

$$x = 3 \text{ and } x = -1 \quad \text{A1}$$

Note: GDC method: graph $f'(x) = x^2 - 2x - 3$, find zeros using graph-and-intersect with $y = 0$; gives $x = -1$ and $x = 3$.

Q8 — Normal to a Quadratic EASY

$$q'(x) = 2x - 6; \text{ at } x = 1: q'(1) = 2(1) - 6 = -4 \quad \text{M1}$$

$$\text{Gradient of normal} = -\frac{1}{-4} = \frac{1}{4} \text{ (negative reciprocal)} \quad \text{A1}$$

$$q(1) = 1 - 6 + 11 = 6; \text{ normal: } y - 6 = \frac{1}{4}(x - 1) \Rightarrow y = \frac{1}{4}x + \frac{23}{4}, \text{ i.e. } y = 1/4x + 23/4 \quad \text{A1}$$

Q9 — Global Maximum on a Closed Interval EASY

$$f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2); \text{ critical points at } x = 1 \text{ and } x = 2 \quad \text{M1}$$

$$\text{Evaluate } f \text{ at critical points and endpoints: } f(0) = -4, f(1) = 2 - 9 + 12 - 4 = 1, f(2) = 16 - 36 + 24 - 4 = 0, f(3) = 54 - 81 + 36 - 4 = 5 \quad \text{A1}$$

$$\text{Global maximum value is } 5, \text{ occurring at } x = 3. \quad \text{A1}$$

Note: GDC method: graph f on $[0, 3]$, use maximum feature; confirms maximum value 5 at $x = 3$. End-point $x = 3$ must be checked.

Q10 — Decreasing Interval of a Quadratic EASY

$$k'(x) = 6x - 12 \quad \text{M1}$$

$$k'(x) < 0 \Rightarrow 6x - 12 < 0 \Rightarrow x < 2 \quad \text{M1}$$

$$k \text{ is decreasing on } (-\infty, 2) \quad \text{A1}$$

Q11 — Increasing Intervals of a Cubic MEDIUM

Differentiate: $f'(x) = 3x^2 - 12x + 9$ **M1**
 Set $f'(x) > 0$: $3(x^2 - 4x + 3) > 0 \Rightarrow 3(x - 1)(x - 3) > 0$ **A1**
 Sign analysis: $f'(x) > 0$ when $x < 1$ or $x > 3$. **M1**
 f is increasing on $(-\infty, 1) \cup (3, \infty)$. **A1**

Q12 — Stationary Points and Classification MEDIUM

$g'(x) = 6x^2 - 6x - 12 = 6(x - 2)(x + 1)$; stationary points at $x = 2$ and $x = -1$. **M1**
 Coordinates: $g(2) = 16 - 12 - 24 + 5 = -15$ and $g(-1) = -2 - 3 + 12 + 5 = 12$. **A1**
 $g''(x) = 12x - 6$. At $x = 2$: $g''(2) = 18 > 0 \Rightarrow$ local minimum at $(2, -15)$.
 At $x = -1$: $g''(-1) = -18 < 0 \Rightarrow$ local maximum at $(-1, 12)$. **M1A1**

Q13 — Normal to a Parabola at a Point MEDIUM

$\frac{dy}{dx} = 2x - 4$. At $x = 3$: gradient of tangent $= 2(3) - 4 = 2$. **M1**
 Gradient of normal $= -\frac{1}{2}$ (negative reciprocal). **A1**
 Normal through $P(3, 4)$: $y - 4 = -\frac{1}{2}(x - 3) \Rightarrow 2y - 8 = -(x - 3)$ **M1**
 $x + 2y - 11 = 0$, i.e. $x + 2y - 11 = 0$ **A1**

Q14 — Point of Inflection of a Cubic MEDIUM

$h'(x) = x^2 - x - 6$, $h''(x) = 2x - 1$. **M1**
 Set $h''(x) = 0$: $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$. **A1**
 Sign change of h'' : for $x < \frac{1}{2}$, $h''(x) < 0$; for $x > \frac{1}{2}$, $h''(x) > 0$. **M1**
 Since h'' changes sign at $x = \frac{1}{2}$, there is a point of inflection at $x = \frac{1}{2}$, i.e. $x = 1/2$. **A1**

Q15 — Optimisation: Rectangle Under a Parabola MEDIUM

The rectangle has width $2t$ and height $12 - t^2$, so area $A(t) = 2t(12 - t^2) = 24t - 2t^3$. **M1**
 $A'(t) = 24 - 6t^2$. Set $A'(t) = 0$: $t^2 = 4 \Rightarrow t = 2$ (since $t > 0$). **A1**
 Using GDC to verify (graph of $A(t)$ on $[0, 2\sqrt{3}]$, maximum at $t = 2$): $A''(t) = -12t$; $A''(2) = -24 < 0 \Rightarrow$ maximum. **M1**
 Maximum area $= A(2) = 24(2) - 2(8) = 48 - 16 = 32$ square units, i.e. $t = 2$ gives maximum area 32. **A1**

Q16 — Global Extrema on a Closed Interval MEDIUM

$p'(x) = 4x^3 - 16x = 4x(x^2 - 4)$; critical points in $[-1, 3]$: $x = 0$ and $x = 2$. **M1**
Evaluate all candidates (using GDC to confirm): **M1**
 $p(-1) = 1 - 8 + 3 = -4$; $p(0) = 3$; $p(2) = 16 - 32 + 3 = -13$; $p(3) = 81 - 72 + 3 = 12$. **A1**
Global maximum value is 12 at $x = 3$; global minimum value is -13 at $x = 2$. **A1**

Q17 — Finding Coefficients from Tangent and Stationary Point MEDIUM

$\frac{dy}{dx} = 2ax + b$. At $x = 1$, gradient = 5: $2a + b = 5$. **M1A1**
Stationary point at $x = -1$: $2a(-1) + b = 0 \Rightarrow -2a + b = 0$. **M1**
Subtracting: $4a = 5 \Rightarrow a = \frac{5}{4}$, then $b = 2a = \frac{5}{2}$, i.e. $a = 5/4$ and $b = 5/2$.
GDC verification: derivative at $x = 1$ gives $2 \cdot \frac{5}{4} \cdot 1 + \frac{5}{2} = \frac{5}{2} + \frac{5}{2} = 5$. ✓ **A1**

Q18 — Stationary Points of a Cubic for Sketching MEDIUM

$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$; stationary points at $x = -1$ and $x = 3$. **M1**
 $f(-1) = -1 - 3 + 9 + 2 = 7$ (local max); $f(3) = 27 - 27 - 27 + 2 = -25$ (local min), giving stationary points $(-1, 7)$ and $(3, -25)$. **A1**
 $f''(x) = 6x - 6$: $f''(-1) = -12 < 0$ (max), $f''(3) = 12 > 0$ (min). y -intercept: $f(0) = 2$. **A1**
Sketch shows local max $(-1, 7)$, local min $(3, -25)$, y -intercept $(0, 2)$, correct cubic shape. **A1**

Q19 — Optimisation: Cylinder with Fixed Surface Area MEDIUM

Surface area: $2\pi r^2 + 2\pi rh = 300\pi \Rightarrow h = \frac{300\pi - 2\pi r^2}{2\pi r} = \frac{150 - r^2}{r}$. **M1**
Volume: $V = \pi r^2 h = \pi r^2 \cdot \frac{150 - r^2}{r} = \pi r(150 - r^2)$. **AG**
 $\frac{dV}{dr} = \pi(150 - 3r^2)$. Set equal to zero: $r^2 = 50 \Rightarrow r = 5\sqrt{2}$ cm. **M1A1**
 $\frac{d^2V}{dr^2} = -6\pi r < 0$ for $r > 0$, confirming a maximum volume. (GDC confirms: $V \approx 2220 \text{ cm}^3$ at $r \approx 7.07$ cm.) **A1**

Q20 — Stationary Points of a Rational Function MEDIUM

Using the quotient rule: $f'(x) = \frac{2x(x - 1) - x^2}{(x - 1)^2} = \frac{x^2 - 2x}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$. **M1**
Set $f'(x) = 0$: $x = 0$ or $x = 2$. **A1**

$$f''(x) = \frac{2}{(x-1)^3} \text{ (computed via GDC numerical derivative or quotient rule). At } x = 0: f''(0) = \frac{2}{(-1)^3} = -2 < 0 \Rightarrow \text{local maximum at } (0, 0). \quad \text{M1}$$

$$\text{At } x = 2: f''(2) = \frac{2}{1^3} = 2 > 0 \Rightarrow \text{local minimum at } (2, 4). \quad \text{A1}$$

Q21 — Local Extrema and Inflection of a Cubic **HARD**

(a)

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) \quad \text{M1}$$

Setting $f'(x) = 0$ gives $x = 1$ and $x = 3$.

$$f''(x) = 6x - 12$$

$$f''(1) = 6(1) - 12 = -6 < 0, \text{ so } x = 1 \text{ is a local maximum (second derivative test).} \quad \text{A1}$$

$$f''(3) = 6(3) - 12 = 6 > 0, \text{ so } x = 3 \text{ is a local minimum (second derivative test).} \quad \text{A1}$$

Note: Both classifications must name the second derivative test and show the sign of f'' .

(b)

$$\text{Set } f''(x) = 0: 6x - 12 = 0 \Rightarrow x = 2. \quad \text{M1}$$

For $x < 2$: $f''(1) = -6 < 0$; for $x > 2$: $f''(3) = 6 > 0$. Since f'' changes sign at $x = 2$, this is a point of inflection. A1

Note: The sign change of f'' must be explicitly verified; stating $f''(2) = 0$ alone earns no marks here.

Q22 — Optimising Surface Area of a Cylinder **HARD**

(a)

$$\text{From } V = \pi r^2 h = 500\pi, \text{ we get } h = \frac{500}{r^2}. \quad \text{AG}$$

$$\text{Substituting: } S = 2\pi r^2 + 2\pi r \cdot \frac{500}{r^2} = 2\pi r^2 + \frac{1000\pi}{r}. \text{ (shown)}$$

(b)

Using GDC (graph or numerical solve): graph $S(r) = 2\pi r^2 + \frac{1000\pi}{r}$ for $r > 0$ and locate the minimum.

M1

GDC gives minimum at $r \approx 6.2996 \dots$ cm, i.e. $r \approx 6.30$ cm (3 s.f.). A1

Minimum surface area: $S \approx 2\pi(6.30)^2 + \frac{1000\pi}{6.30} \approx 748.0 \dots \approx 748 \text{ cm}^2$ (3 s.f.). A1

Verification (GDC): $S'(r) = 4\pi r - \frac{1000\pi}{r^2}$; at $r = 6.30$, $S''(r) = 4\pi + \frac{2000\pi}{r^3} > 0$, confirming a minimum.

R1

Note: The justification sentence must state $S'' > 0$ or equivalent sign-change argument.

Q23 — Increasing/Decreasing and Asymptotes of a Rational Function **HARD**

(a)

Using the quotient rule: $f'(x) = \frac{2x(x^2 + 4) - (x^2 - 4)(2x)}{(x^2 + 4)^2} = \frac{2x^3 + 8x - 2x^3 + 8x}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2}$.

M1

$f'(x) > 0$ when $x > 0$ and $f'(x) < 0$ when $x < 0$ (since $(x^2 + 4)^2 > 0$ always).

A1

f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

A1

(b)

As $x \rightarrow \pm\infty$: $f(x) = \frac{x^2 - 4}{x^2 + 4} \rightarrow 1$, so the horizontal asymptote is $y = 1$.

A1

y -intercept: $f(0) = \frac{-4}{4} = -1$, so the y -intercept is $(0, -1)$.

A1

Q24 — Stationary Points and Inflections of xe^{-x^2} **HARD**

(a)

$g'(x) = e^{-x^2} + x \cdot (-2x)e^{-x^2} = e^{-x^2}(1 - 2x^2)$.

M1

Setting $g'(x) = 0$: $1 - 2x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$.

GDC evaluation: $g\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-1/2} \approx 0.429$; $g\left(-\frac{1}{\sqrt{2}}\right) \approx -0.429$.

$g''(x) = e^{-x^2}(-4x) + (1 - 2x^2)(-2x)e^{-x^2} = e^{-x^2}(-6x + 4x^3)$.

$g''\left(\frac{1}{\sqrt{2}}\right) = e^{-1/2}(-3\sqrt{2} + 2) < 0 \Rightarrow$ local maximum at $x = \frac{1}{\sqrt{2}}$.

A1

$g''\left(-\frac{1}{\sqrt{2}}\right) > 0 \Rightarrow$ local minimum at $x = -\frac{1}{\sqrt{2}}$.

A1

(b)

Using GDC (nSolve or graph of g''): solve $-6x + 4x^3 = 0$, i.e. $2x(2x^2 - 3) = 0$.

M1

$x = 0$, $x = \pm\sqrt{\frac{3}{2}} \approx \pm 1.22$, i.e. $x \approx \pm 1.22$ (3 s.f.). Sign change of g'' confirmed at each.

A1

Q25 — Tangent from External Point to Cubic **HARD**

Let the contact point be $(a, a^3 - 3a)$ on $y = x^3 - 3x$.

M1

The gradient of the tangent at $x = a$ is $f'(a) = 3a^2 - 3$.

The tangent passes through $P(3, 8)$, so:

$\frac{(a^3 - 3a) - 8}{a - 3} = 3a^2 - 3$

M1

$a^3 - 3a - 8 = (3a^2 - 3)(a - 3)$

$a^3 - 3a - 8 = 3a^3 - 9a^2 - 3a + 9$

$0 = 2a^3 - 9a^2 + 17$

A1

This cubic in a has **three** real roots (use GDC: graph $y = 2a^3 - 9a^2 + 17$ and locate all sign changes, or nSolve repeatedly on different starting intervals):

$$a \approx -1.22, \quad a \approx 1.76, \quad a \approx 3.96 \text{ (3 s.f.)}$$

M1A1

Each root gives a genuine point of tangency, so there are **three tangent lines** from $P(3, 8)$ to the curve:

at $a \approx -1.22$: $m = 3a^2 - 3 \approx 1.46$, giving $y \approx 1.46x + 3.62$

at $a \approx 1.76$: $m \approx 6.31$, giving $y \approx 6.31x - 10.9$

at $a \approx 3.96$: $m \approx 44.0$, giving $y \approx 44.0x - 124$

A1

Note: M1 for introducing $(a, f(a))$; M1 for equating gradients; a cubic can have up to three real roots, and each must be checked — full credit requires finding all three tangent lines, not just one.

Q26 — Stationary Points and Global Minimum on Closed Interval **HARD**

(a)

$$h'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1) = 12x(x - 1)^2.$$

M1

Stationary points at $x = 0$ and $x = 1$.

$$h''(x) = 36x^2 - 48x + 12.$$

$$h''(0) = 12 > 0 \Rightarrow x = 0 \text{ is a local minimum. } h(0) = 0, \text{ i.e. stationary point } (0, 0).$$

A1

$h''(1) = 36 - 48 + 12 = 0$; second derivative test is inconclusive. Check sign of h' : for x slightly less than 1, $h'(x) = 12x(x - 1)^2 > 0$; for x slightly greater than 1, $h'(x) > 0$. Since h' does not change sign, $x = 1$ is a stationary point of inflection, not a local extremum.

A1

(b)

Evaluate h at critical points and endpoints on $[-1, 3]$:

$$h(-1) = 3 + 8 + 6 = 17, \quad h(0) = 0, \quad h(1) = 3 - 8 + 6 = 1, \quad h(3) = 243 - 216 + 54 = 81.$$

M1

The global minimum on $[-1, 3]$ is $h(0) = 0$.

A1

Note: All four values must be computed; omitting either endpoint loses the method mark.

Q27 — Open Box Optimisation **HARD**

(a)

After cutting squares of side x from each corner, the base dimensions are $(24 - 2x)$ by $(18 - 2x)$ and the height is x , so $V = x(24 - 2x)(18 - 2x)$.

AG

(b)

$$\text{Expand: } V = x(432 - 48x - 36x + 4x^2) = 4x^3 - 84x^2 + 432x, \text{ valid for } 0 < x < 9.$$

Using GDC: graph $V(x) = 4x^3 - 84x^2 + 432x$ on $(0, 9)$ and find the maximum.

M1

$$\text{GDC (maximum function or nSolve on } V'(x) = 12x^2 - 168x + 432 = 0) \text{ gives } x = \frac{168 \pm \sqrt{168^2 - 4(12)(432)}}{24}.$$

$$x = \frac{168 \pm \sqrt{28224 - 20736}}{24} = \frac{168 \pm \sqrt{7488}}{24}; \text{ GDC: } x \approx 3.394 \dots \text{ or } x \approx 10.6 \text{ (rejected since } x < 9).$$

$$\text{So } x \approx 3.39 \text{ cm (3 s.f.)}$$

A1

$$V''(x) = 24x - 168; \text{ at } x \approx 3.39: V''(3.39) \approx 81.4 - 168 = -86.6 < 0, \text{ confirming a maximum.}$$

R1

(c)

$$V(3.394) \approx 4(3.394)^3 - 84(3.394)^2 + 432(3.394) \approx 655 \text{ cm}^3 \text{ (3 s.f.)}$$

A1

GDC: maximum command on $[0, 9]$ returns $(3.3944 \dots, 654.98 \dots)$, so max volume $\approx 655 \text{ cm}^3$.

Q28 — Cubic with Stationary Inflection HARD

Since $f(0) = 0$: $d = 0$.

A1

Since $(1, -2)$ is on the curve: $a + b + c + 0 = -2$, so $a + b + c = -2$.

M1

$f'(x) = 3ax^2 + 2bx + c$. Since $x = 1$ is a stationary point: $3a + 2b + c = 0$.

A1

$f''(x) = 6ax + 2b$. Since $x = 1$ is a stationary inflection, $f''(1) = 0$: $6a + 2b = 0 \Rightarrow b = -3a$.

M1

Substituting $b = -3a$ into $3a + 2b + c = 0$: $3a - 6a + c = 0 \Rightarrow c = 3a$.

Substituting into $a + b + c = -2$: $a - 3a + 3a = -2 \Rightarrow a = -2$.

Then $b = 6$, $c = -6$, $d = 0$, i.e. $a = -2$, $b = 6$, $c = -6$, $d = 0$.

A1

Note: Must verify sign change of f'' at $x = 1$ to confirm inflection: $f''(x) = -12x + 12$; sign changes from $+$ to $-$ at $x = 1$. ✓

Q29 — Stationary Points of $\ln(x^2 + 1) - kx$ HARD

(a)

$$f'(x) = \frac{2x}{x^2 + 1} - k = \frac{2x - k(x^2 + 1)}{x^2 + 1} = \frac{-kx^2 + 2x - k}{x^2 + 1}$$

M1A1

(Result matches the printed target.)

AG

(b)

Stationary points require $f'(x) = 0$, i.e. the numerator $-kx^2 + 2x - k = 0$.

M1

Case $k = 0$: equation becomes $2x = 0 \Rightarrow x = 0$ (one stationary point, so $k = 0$ is not excluded by "no stationary points").

Case $k \neq 0$: discriminant $= 4 - 4k^2$. For no real solutions: $4 - 4k^2 < 0 \Rightarrow k^2 > 1 \Rightarrow |k| > 1$, i.e. $|k| > 1$.

A1

Therefore f has no stationary points when $|k| > 1$, i.e. $k \in (-\infty, -1) \cup (1, \infty)$. Since the denominator $x^2 + 1 > 0$ always, the sign of f' is determined solely by the discriminant of the quadratic numerator; when $|k| > 1$ the numerator has no real roots and is always negative (as $-k < 0$ for $k > 1$), confirming f is strictly monotone with no stationary points.

R1

Q30 — Rational Curve: No Stationary Points and Sketch HARD

(a)

$$\frac{dy}{dx} = \frac{2(x^2 - 9) - 2x \cdot 2x}{(x^2 - 9)^2} = \frac{2x^2 - 18 - 4x^2}{(x^2 - 9)^2} = \frac{-2x^2 - 18}{(x^2 - 9)^2} = \frac{-2(x^2 + 9)}{(x^2 - 9)^2}$$

M1A1

Since $x^2 + 9 > 0$ for all real x and $(x^2 - 9)^2 > 0$ for $x \neq \pm 3$, we have $\frac{dy}{dx} < 0$ for all x in the domain.

Hence $\frac{dy}{dx}$ is never zero, and the curve has no stationary points.

R1

(b)

Using GDC: sketch $y = \frac{2x}{x^2 - 9}$ showing:

M1

Vertical asymptotes $x = -3$ and $x = 3$; horizontal asymptote $y = 0$ (as $x \rightarrow \pm\infty$); x - and y -intercept at the origin $(0, 0)$; function is strictly decreasing on each branch. Sketch is a correctly shaped three-branch rational curve with all asymptotes labelled.

A1