IB Math AA HL — Topic 5

Applications of Differentiation

Tangents & Normals · Stationary Points & Inflection · Optimisation · Curve Sketching

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:________________________ Date:____________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Tangent at $P(a, f(a))$:** gradient $m = f'(a)$; equation $y - f(a) = f'(a)(x - a)$.
- **Normal at $P(a, f(a))$:** gradient $m_\perp = -\dfrac{1}{f'(a)}$; equation $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$.
- **Increasing / Decreasing:** $f'(x) > 0 \Rightarrow f$ increasing; $f'(x) < 0 \Rightarrow f$ decreasing on that interval.
- **Stationary points:** solve $f'(x) = 0$; classify via **second derivative test**: $f''(a) < 0$ local max, $f''(a) > 0$ local min, $f''(a) = 0$ *inconclusive* — use sign change of f'.
- **Points of inflection:** necessary condition $f''(x) = 0$; confirmed only if f'' *changes sign* either side. A stationary inflection also has $f'(a) = 0$.
- **Optimisation:** express objective in one variable using a constraint; find critical point; justify max/min; state answer with units.
- **Global extrema on $[a, b]$:** evaluate f at all interior critical points *and* at both endpoints $x = a$, $x = b$; compare all values.

GDC Tips

- **Numerical derivative:** use `dy/dx` (graph screen) or `nDeriv(f(x),x,a)` to evaluate $f'(a)$ at a decimal point, then form the tangent/normal equation by hand.
- **Stationary points:** graph $y = f'(x)$ and use `zero` (or `root`) to locate $f'(x) = 0$; confirm with `minimum/maximum` on $y = f(x)$ directly.
- **Numerical integration / area:** use `fnInt(f(x),x,a,b)` or the $\int f\,dx$ option in the graph screen; set limits carefully.
- **Solving equations:** use `nSolve` (or `solve`) to find intersection of $f'(x) = m$ when roots are not exact; give equation and window used.
- **Global extrema on a closed interval:** store endpoints and all critical x-values in a list, evaluate f at each with `seq`, then use `max/min` on the result list.
- **Optimisation verification:** after finding the critical point analytically, graph the objective function and use `minimum/maximum` to confirm the value to 3 s.f.

Common Mistakes to Avoid

1. **Using $f''(a) = 0$ alone to conclude a point of inflection.** $f''(a) = 0$ is necessary but not sufficient; you must show that f'' *changes sign* on either side of $x = a$. The counterexample $f(x) = x^4$ has $f''(0) = 0$ but no inflection at $x = 0$.
2. **Forgetting endpoints in closed-interval extrema.** The global maximum or minimum on $[a, b]$ can occur at $x = a$ or $x = b$, not only at interior stationary points. Always evaluate $f(a)$ and $f(b)$ explicitly and compare with all critical values.
3. **Concluding a stationary point's nature from $f'' = 0$ (inconclusive case).** When $f''(a) = 0$, the second derivative test gives no information; you must revert to a sign-change analysis of f' on either side of $x = a$.
4. **Using the same gradient for a normal as for a tangent.** The normal gradient is the *negative reciprocal*: $m_\perp = -1/f'(a)$. Forgetting the negation or the reciprocal are both common errors, especially when $f'(a)$ is a fraction.
5. **Failing to justify the extremum in an optimisation problem.** Stating the critical point without confirming it is a maximum or minimum (via f'' sign or sign-change of f') and without a concluding sentence in context with units loses the reasoning mark.
6. **Using an unknown contact point incorrectly for external-point tangent problems.** The contact point must be written as $(a,\ f(a))$ and the tangent gradient $f'(a)$ must be used so that the line *also passes through the external point* — setting up one equation in a is the essential method step that is often omitted.

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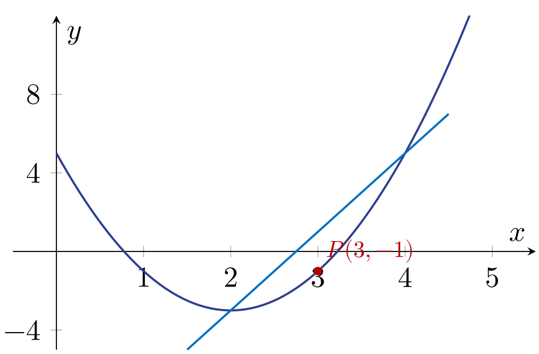
Section A — Easy

EASY

1. Let $f(x) = x^3 - 6x^2 + 9x + 2$. Find the values of x at which f is stationary. *Calculator not allowed* **[3 marks]**

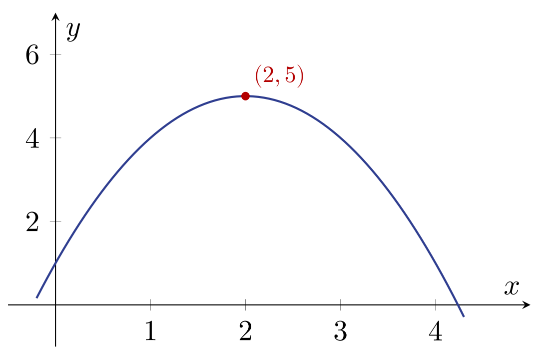
2. Let $g(x) = 2x^2 - 8x + 5$. Find the equation of the tangent to g at the point where $x = 3$. Give your answer in the form $y = mx + c$. *Calculator allowed* **[3 marks]**



3. Let $h(x) = x^3 - 3x + 1$. Determine the intervals on which h is increasing. *Calculator not allowed* **[3 marks]**

4. A function is defined by $f(x) = x^4 - 8x^2 + 3$. Find the x-coordinates of any points of inflection of f, justifying your answer. *Calculator allowed* **[3 marks]**

5. Let $p(x) = -x^2 + 4x + 1$. Find the coordinates of the stationary point of p and classify it. *Calculator allowed* **[3 marks]**



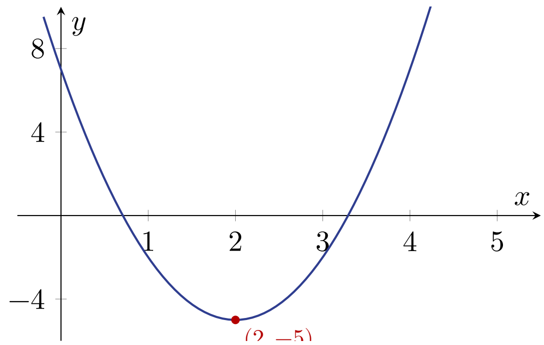
6. A farmer has 60 m of fencing to enclose a rectangular pen against a straight wall; the wall forms one side so fencing is needed for only three sides. Let x m be the length of each side perpendicular to the wall. Write down an expression for the area A in terms of x, and hence find the value of x that maximises A. *Calculator allowed* **[3 marks]**

7. Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$. Find the x-coordinates of the stationary points of f using calculus. *Calculator allowed* **[3 marks]**

8. Let $q(x) = x^2 - 6x + 11$. Find the equation of the normal to q at the point where $x = 1$. Give your answer in the form $y = mx + c$. *Calculator not allowed* **[3 marks]**

9. Consider $f(x) = 2x^3 - 9x^2 + 12x - 4$ on the closed interval $[0, 3]$. Find the global maximum value of f on this interval. *Calculator allowed* **[3 marks]**

10. Let $k(x) = 3x^2 - 12x + 7$. Determine the values of x for which k is decreasing. *Calculator not allowed* **[3 marks]**



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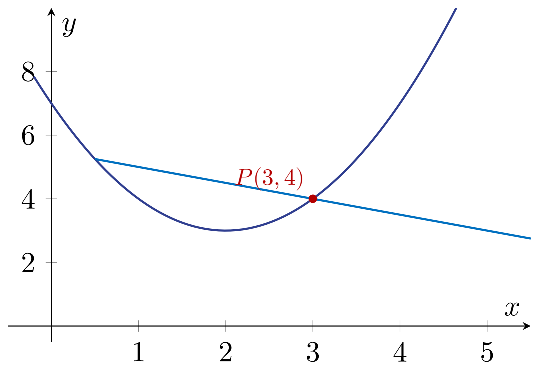
Section B — Medium

MEDIUM

11. Let $f(x) = x^3 - 6x^2 + 9x + 1$. Find the values of x for which f is increasing. *Calculator not allowed* **[4 marks]**

12. Let $g(x) = 2x^3 - 3x^2 - 12x + 5$. Find the coordinates of the stationary points of g and classify each one using the second derivative test. *Calculator allowed* **[4 marks]**

13. The curve C has equation $y = x^2 - 4x + 7$. The point $P(3, 4)$ lies on C. Find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$. *Calculator not allowed* **[4 marks]**



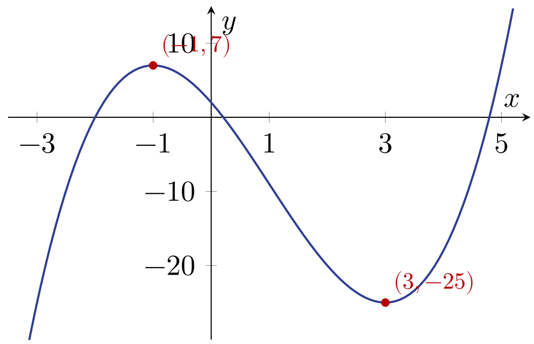
14. A function is defined by $h(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 6x + 4$. Find the x-coordinates of any points of inflection of h, justifying that each is a point of inflection by verifying a sign change of h''. *Calculator not allowed* **[4 marks]**

15. A rectangle has its base on the x-axis and two upper vertices on the parabola $y = 12 - x^2$, where $-2\sqrt{3} \leq x \leq 2\sqrt{3}$. Let the right upper vertex have x-coordinate $t > 0$. Find the value of t that maximises the area of the rectangle, and state the maximum area. *Calculator allowed* **[4 marks]**

16. Let $p(x) = x^4 - 8x^2 + 3$. Find the global maximum and global minimum values of p on the closed interval $[-1, 3]$, clearly evaluating all relevant values. *Calculator allowed* **[4 marks]**

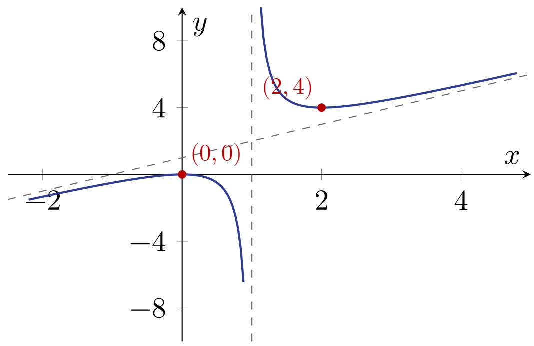
17. The curve $y = ax^2 + bx + 3$ has a tangent with gradient 5 at the point where $x = 1$, and a stationary point at $x = -1$. Find the values of a and b. *Calculator allowed* **[4 marks]**

18. Let $f(x) = x^3 - 3x^2 - 9x + 2$. The diagram below shows the curve $y = f(x)$ with stationary points labelled. Sketch the curve, clearly labelling the coordinates of both stationary points and the y-intercept. *Calculator allowed*
[4 marks]



19. A closed cylindrical can has radius r cm and height h cm. The total surface area is fixed at 300π cm^2. Show that the volume is $V = \pi r(150 - r^2)$, and hence find the value of r that maximises V. *Calculator allowed* **[4 marks]**

20. Let $f(x) = \dfrac{x^2}{x-1}$, defined for $x \neq 1$. Find the x-coordinates of the stationary points of f, and use the second derivative to classify each. *Calculator allowed* **[4 marks]**



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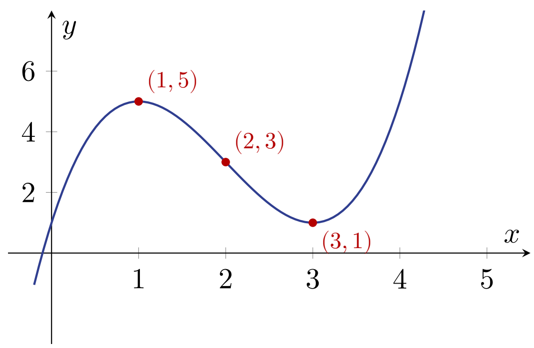
Section C — Hard

HARD

21. Let $f(x) = x^3 - 6x^2 + 9x + 1$. *Calculator not allowed* **[5 marks]**

(a) Show that f has a local maximum at $x = 1$ and a local minimum at $x = 3$. **[3 marks]**

(b) Find the x-coordinate of the point of inflection of f, verifying that it is indeed a point of inflection. **[2 marks]**



22. A closed cylindrical can has volume $V = 500\pi$ cm^3. The total surface area is $S = 2\pi r^2 + 2\pi rh$, where r cm is the base radius and h cm is the height. *Calculator allowed* **[5 marks]**

(a) Show that $S = 2\pi r^2 + \dfrac{1000\pi}{r}$. **[1 marks]**

(b) Find the value of r that minimises S, and state the minimum surface area to 3 significant figures. Justify that this value gives a minimum. **[4 marks]**

23. A curve is defined by $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$ for all $x \in \mathbb{R}$. *Calculator not allowed* **[5 marks]**

(a) Find $f'(x)$ and hence determine the intervals on which f is increasing and decreasing. **[3 marks]**

(b) Find the equations of any horizontal asymptotes and write down the y-intercept. **[2 marks]**

24. Let $g(x) = xe^{-x^2}$ for $x \in \mathbb{R}$. *Calculator allowed* **[5 marks]**

(a) Find $g'(x)$ and hence find all stationary points, classifying each. **[3 marks]**

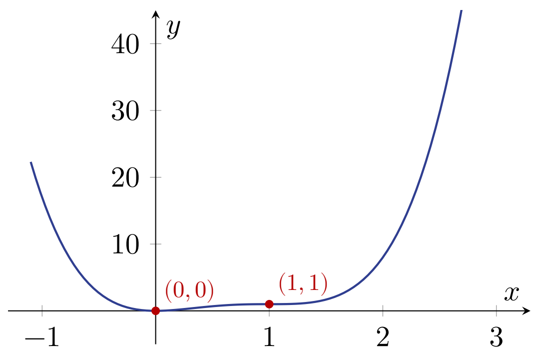
(b) Using a GDC, find the x-coordinates of the points of inflection of g, correct to 3 significant figures. **[2 marks]**

25. A tangent to the curve $y = x^3 - 3x$ is drawn from the external point $P(3, 8)$. *Calculator not allowed* **[5 marks]**
Find the equation(s) of all tangent lines to $y = x^3 - 3x$ that pass through $P(3, 8)$.

26. Let $h(x) = 3x^4 - 8x^3 + 6x^2$. *Calculator allowed* **[5 marks]**

(a) Find all stationary points of h and classify each. **[3 marks]**

(b) Find the global minimum value of h on the closed interval $[-1, 3]$, clearly evaluating all relevant values. **[2 marks]**



27. A rectangular sheet of metal measures 24 cm by 18 cm. Equal squares of side x cm are cut from each corner and the sides are folded up to form an open box of volume V cm^3. *Calculator allowed* **[5 marks]**

(a) Show that $V = x(24 - 2x)(18 - 2x)$. **[1 marks]**

(b) Find the value of x that maximises V, correct to 3 significant figures. Justify that this value gives a maximum. **[3 marks]**

(c) State the maximum volume, correct to 3 significant figures. **[1 marks]**

28. The function $f(x) = ax^3 + bx^2 + cx + d$ has a stationary point of inflection at $(1, -2)$ and passes through the point $(0, 0)$. *Calculator not allowed* **[5 marks]**

Find the values of a, b, c, and d.

29. Consider $f(x) = \ln(x^2 + 1) - kx$ where k is a real constant and $x \in \mathbb{R}$. *Calculator allowed* **[5 marks]**

(a) Show that $f'(x) = \dfrac{-kx^2 + 2x - k}{x^2 + 1}$. **[2 marks]**

(b) Find the values of k for which f has no stationary points. Justify your answer. **[3 marks]**

30. A curve has equation $y = \dfrac{2x}{x^2 - 9}$, $x \neq \pm 3$. *Calculator allowed* **[5 marks]**

(a) Find $\dfrac{dy}{dx}$ and show that the curve has no stationary points. **[3 marks]**

(b) Sketch the curve, labelling all asymptotes and any intercepts with the axes. **[2 marks]**

Worked Solutions

Q1 — Stationary points of a cubic [EASY]

Differentiate: $f'(x) = 3x^2 - 12x + 9$ **M1**

Set $f'(x) = 0$: $3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0$ **M1**

$x = 1$ and $x = 3$ **A1**

Q2 — Tangent to a quadratic [EASY]

$g'(x) = 4x - 8$, so gradient at $x = 3$ is $g'(3) = 4(3) - 8 = 4$ **M1**

$g(3) = 2(9) - 8(3) + 5 = 18 - 24 + 5 = -1$, so tangent passes through $(3, -1)$ **A1**

Tangent: $y - (-1) = 4(x-3)$, giving $y = 4x - 13$ **A1**

Note: GDC method: graph g, use tangent feature at $x = 3$; gradient $= 4$, equation $y = 4x - 13$.

Q3 — Increasing intervals of a cubic [EASY]

$h'(x) = 3x^2 - 3 = 3(x-1)(x+1)$ **M1**

$h'(x) > 0$ when $x < -1$ or $x > 1$ **M1**

h is increasing on $(-\infty, -1) \cup (1, \infty)$ **A1**

Q4 — Points of inflection of a quartic [EASY]

$f'(x) = 4x^3 - 16x, \quad f''(x) = 12x^2 - 16$ **M1**

Setting $f''(x) = 0$: $12x^2 = 16 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\dfrac{2}{\sqrt{3}}$ (exact: $\approx \pm 1.15$) **A1**

Sign check: f'' changes from positive to negative at $x = -\frac{2}{\sqrt{3}}$ and from negative to positive at $x = \frac{2}{\sqrt{3}}$; sign changes confirmed, so both are points of inflection. **A1**

Note: GDC check: use nSolve($12x^2 - 16 = 0, x$) giving $x \approx \pm 1.15$; verify f'' sign change by evaluating $f''(0) = -16 < 0$ and $f''(2) = 32 > 0$.

Q5 — Stationary point of a quadratic [EASY]

$p'(x) = -2x + 4$; setting $p'(x) = 0$ gives $x = 2$ **M1**

$p(2) = -(4) + 8 + 1 = 5$, so stationary point is $(2, 5)$ **A1**

$p''(x) = -2 < 0$ at $x = 2$ (second derivative test); the point is a **local maximum**. **A1**

Note: GDC method: graph p, use maximum feature; result $(2, 5)$ confirmed.

Q6 — Optimisation: rectangular pen [EASY]

Side parallel to wall $= 60 - 2x$, so $A = x(60 - 2x) = 60x - 2x^2$ **A1**

$\dfrac{dA}{dx} = 60 - 4x = 0 \Rightarrow x = 15$ **M1**

$\dfrac{d^2A}{dx^2} = -4 < 0$, confirming a maximum; the value $x = 15$ m maximises the area. **A1**

Note: GDC method: graph $A = 60x - 2x^2$ for $0 < x < 30$, use maximum feature; maximum at $x = 15$.

Q7 — Stationary points of a cubic [EASY]

$f'(x) = x^2 - 2x - 3$ **M1**

Set $f'(x) = 0$: $(x - 3)(x + 1) = 0$ **M1**

$x = 3$ and $x = -1$ **A1**

Note: GDC method: graph $f'(x) = x^2 - 2x - 3$, find zeros using graph-and-intersect with $y = 0$; gives $x = -1$ and $x = 3$.

Q8 — Normal to a quadratic [EASY]

$q'(x) = 2x - 6$; at $x = 1$: $q'(1) = 2(1) - 6 = -4$ **M1**

Gradient of normal $= -\dfrac{1}{-4} = \dfrac{1}{4}$ (negative reciprocal) **A1**

$q(1) = 1 - 6 + 11 = 6$; normal: $y - 6 = \dfrac{1}{4}(x - 1) \Rightarrow y = \dfrac{1}{4}x + \dfrac{23}{4}$ **A1**

Q9 — Global maximum on a closed interval [EASY]

$f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2)$; critical points at $x = 1$ and $x = 2$ **M1**

Evaluate f at critical points and endpoints:
$f(0) = -4, \quad f(1) = 2 - 9 + 12 - 4 = 1, \quad f(2) = 16 - 36 + 24 - 4 = 0, \quad f(3) = 54 - 81 + 36 - 4 = 5$
A1

Global maximum value is **5**, occurring at $x = 3$. **A1**

Note: GDC method: graph f on $[0, 3]$, use maximum feature; confirms maximum value 5 at $x = 3$. Endpoint $x = 3$ must be checked.

Q10 — Decreasing interval of a quadratic [EASY]

$k'(x) = 6x - 12$ **M1**

$k'(x) < 0 \Rightarrow 6x - 12 < 0 \Rightarrow x < 2$ **M1**

k is decreasing on $(-\infty, 2)$ **A1**

Q11 — Increasing intervals of a cubic [MEDIUM]

Differentiate: $f'(x) = 3x^2 - 12x + 9$ **M1**

Set $f'(x) > 0$: $3(x^2 - 4x + 3) > 0 \Rightarrow 3(x-1)(x-3) > 0$ **A1**

Sign analysis: $f'(x) > 0$ when $x < 1$ or $x > 3$. **M1**

f is increasing on $(-\infty, 1) \cup (3, \infty)$. **A1**

Q12 — Stationary points and classification [MEDIUM]

$g'(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$; stationary points at $x = 2$ and $x = -1$. **M1**

Coordinates: $g(2) = 16 - 12 - 24 + 5 = -15$ and $g(-1) = -2 - 3 + 12 + 5 = 12$. **A1**

$g''(x) = 12x - 6$. At $x = 2$: $g''(2) = 18 > 0 \Rightarrow$ local minimum at $(2, -15)$.
At $x = -1$: $g''(-1) = -18 < 0 \Rightarrow$ local maximum at $(-1, 12)$. **M1A1**

Q13 — Normal to a parabola at a point [MEDIUM]

$\frac{dy}{dx} = 2x - 4$. At $x = 3$: gradient of tangent $= 2(3) - 4 = 2$. **M1**

Gradient of normal $= -\frac{1}{2}$ (negative reciprocal). **A1**

Normal through $P(3, 4)$: $y - 4 = -\frac{1}{2}(x-3) \Rightarrow 2y - 8 = -(x-3)$ **M1**

$x + 2y - 11 = 0$ **A1**

Q14 — Point of inflection of a cubic [MEDIUM]

$h'(x) = x^2 - x - 6; \quad h''(x) = 2x - 1.$ **M1**

Set $h''(x) = 0$: $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$. **A1**

Sign change of h'': for $x < \frac{1}{2}$, $h''(x) < 0$; for $x > \frac{1}{2}$, $h''(x) > 0$. **M1**

Since h'' changes sign at $x = \frac{1}{2}$, there is a point of inflection at $x = \frac{1}{2}$. **A1**

Q15 — Optimisation: rectangle under a parabola [MEDIUM]

The rectangle has width $2t$ and height $12 - t^2$, so area $A(t) = 2t(12 - t^2) = 24t - 2t^3$. **M1**

$A'(t) = 24 - 6t^2$. Set $A'(t) = 0$: $t^2 = 4 \Rightarrow t = 2$ (since $t > 0$). **A1**

Using GDC to verify (graph of $A(t)$ on $[0, 2\sqrt{3}]$, maximum at $t = 2$): $A''(t) = -12t$; $A''(2) = -24 < 0$ $\Rightarrow$ maximum. **M1**

Maximum area $= A(2) = 24(2) - 2(8) = 48 - 16 = 32$ square units. **A1**

Q16 — Global extrema on a closed interval [MEDIUM]

$p'(x) = 4x^3 - 16x = 4x(x^2 - 4)$; critical points in $[-1, 3]$: $x = 0$ and $x = 2$. **M1**

Evaluate all candidates (using GDC to confirm): **M1**

$p(-1) = 1 - 8 + 3 = -4; \quad p(0) = 3; \quad p(2) = 16 - 32 + 3 = -13; \quad p(3) = 81 - 72 + 3 = 12.$ **A1**

Global maximum value is **12** at $x = 3$; global minimum value is $-$**13** at $x = 2$. **A1**

Q17 — Finding coefficients from tangent and stationary point [MEDIUM]

$\dfrac{dy}{dx} = 2ax + b$. At $x = 1$, gradient $= 5$: $2a + b = 5$. **M1A1**

Stationary point at $x = -1$: $2a(-1) + b = 0 \Rightarrow -2a + b = 0$. **M1**

Subtracting: $4a = 5 \Rightarrow a = \dfrac{5}{4}$, then $b = 2a = \dfrac{5}{2}$.

GDC verification: derivative at $x = 1$ gives $2 \cdot \frac{5}{4} \cdot 1 + \frac{5}{2} = \frac{5}{2} + \frac{5}{2} = 5$. ☐ **A1**

Q18 — Stationary points of a cubic for sketching [MEDIUM]

$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$; stationary points at $x = -1$ and $x = 3$. **M1**

$f(-1) = -1 - 3 + 9 + 2 = 7$ (local max); $f(3) = 27 - 27 - 27 + 2 = -25$ (local min). **A1**

$f''(x) = 6x - 6$: $f''(-1) = -12 < 0$ (max), $f''(3) = 12 > 0$ (min). y-intercept: $f(0) = 2$. **A1**

Sketch shows local max $(-1, 7)$, local min $(3, -25)$, y-intercept $(0, 2)$, correct cubic shape. **A1**

Q19 — Optimisation: cylinder with fixed surface area [MEDIUM]

Surface area: $2\pi r^2 + 2\pi rh = 300\pi \Rightarrow h = \dfrac{300\pi - 2\pi r^2}{2\pi r} = \dfrac{150 - r^2}{r}$. **M1**

Volume: $V = \pi r^2 h = \pi r^2 \cdot \dfrac{150 - r^2}{r} = \pi r(150 - r^2)$. **AG**

$\dfrac{dV}{dr} = \pi(150 - 3r^2)$. Set equal to zero: $r^2 = 50 \Rightarrow r = 5\sqrt{2}$ cm. **M1A1**

$\dfrac{d^2V}{dr^2} = -6\pi r < 0$ for $r > 0$, confirming a maximum volume. (GDC confirms: $V \approx 3330$ cm^3 at $r \approx 7.07$ cm.) **A1**

Q20 — Stationary points of a rational function [MEDIUM]

Using the quotient rule: $f'(x) = \dfrac{2x(x-1) - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$. **M1**

Set $f'(x) = 0$: $x = 0$ or $x = 2$. **A1**

$f''(x) = \dfrac{2}{(x-1)^3}$ (computed via GDC numerical derivative or quotient rule). At $x = 0$: $f''(0) = \dfrac{2}{(-1)^3} = -2 < 0 \Rightarrow$ local maximum at $(0, 0)$. **M1**

At $x = 2$: $f''(2) = \dfrac{2}{1^3} = 2 > 0 \Rightarrow$ local minimum at $(2, 4)$. **A1**

Q21 — Local extrema and inflection of a cubic [HARD]

(a)

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ **M1**

Setting $f'(x) = 0$ gives $x = 1$ and $x = 3$.

$f''(x) = 6x - 12$

$f''(1) = 6(1) - 12 = -6 < 0$, so $x = 1$ is a local maximum (second derivative test). **A1**

$f''(3) = 6(3) - 12 = 6 > 0$, so $x = 3$ is a local minimum (second derivative test). **A1**

Note: Both classifications must name the second derivative test and show the sign of f''.

(b)

Set $f''(x) = 0$: $6x - 12 = 0 \Rightarrow x = 2$. **M1**

For $x < 2$: $f''(1) = -6 < 0$; for $x > 2$: $f''(3) = 6 > 0$. Since f'' changes sign at $x = 2$, this is a point of inflection. **A1**

Note: The sign change of f'' must be explicitly verified; stating $f''(2) = 0$ alone earns no marks here.

Q22 — Optimising surface area of a cylinder [HARD]

(a)

From $V = \pi r^2 h = 500\pi$, we get $h = \dfrac{500}{r^2}$. **AG**

Substituting: $S = 2\pi r^2 + 2\pi r \cdot \dfrac{500}{r^2} = 2\pi r^2 + \dfrac{1000\pi}{r}$. **(shown)**

(b)

Using GDC (graph or numerical solve): graph $S(r) = 2\pi r^2 + \dfrac{1000\pi}{r}$ for $r > 0$ and locate the minimum. **M1**

GDC gives minimum at $r \approx 6.2996\ldots$ cm, i.e. $r \approx 6.30$ cm (3 s.f.). **A1**

Minimum surface area: $S \approx 2\pi(6.30)^2 + \dfrac{1000\pi}{6.30} \approx 749.6\ldots \approx 750$ cm^2 (3 s.f.). **A1**

Verification (GDC): $S'(r) = 4\pi r - \dfrac{1000\pi}{r^2}$; at $r = 6.30$, $S''(r) = 4\pi + \dfrac{2000\pi}{r^3} > 0$, confirming a minimum. **R1**

Note: The justification sentence must state $S'' > 0$ or equivalent sign-change argument.

Q23 — Increasing/decreasing and asymptotes of a rational function [HARD]

(a)

Using the quotient rule: $f'(x) = \dfrac{2x(x^2+4) - (x^2-4)(2x)}{(x^2+4)^2} = \dfrac{2x^3+8x-2x^3+8x}{(x^2+4)^2} = \dfrac{16x}{(x^2+4)^2}$. **M1**

$f'(x) > 0$ when $x > 0$ and $f'(x) < 0$ when $x < 0$ (since $(x^2+4)^2 > 0$ always). **A1**

f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. **A1**

(b)

As $x \to \pm\infty$: $f(x) = \dfrac{x^2-4}{x^2+4} \to 1$, so the horizontal asymptote is $y = 1$. **A1**

y-intercept: $f(0) = \dfrac{-4}{4} = -1$, so the y-intercept is $(0, -1)$. **A1**

Q24 — Stationary points and inflections of xe^{-x^2} [HARD]

(a)

$g'(x) = e^{-x^2} + x \cdot (-2x)e^{-x^2} = e^{-x^2}(1 - 2x^2)$. **M1**

Setting $g'(x) = 0$: $1 - 2x^2 = 0 \Rightarrow x = \pm\dfrac{1}{\sqrt{2}}$.

GDC evaluation: $g\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\,e^{-1/2} \approx 0.429$; $\quad g\left(-\frac{1}{\sqrt{2}}\right) \approx -0.429$.

$g''(x) = e^{-x^2}(-4x) + (1-2x^2)(-2x)e^{-x^2} = e^{-x^2}(-6x + 4x^3)$.

$g''\left(\frac{1}{\sqrt{2}}\right) = e^{-1/2}(-3\sqrt{2} + \sqrt{2}) < 0 \Rightarrow$ local maximum at $x = \frac{1}{\sqrt{2}}$. **A1**

$g''\left(-\frac{1}{\sqrt{2}}\right) > 0 \Rightarrow$ local minimum at $x = -\frac{1}{\sqrt{2}}$. **A1**

(b)

Using GDC (nSolve or graph of g''): solve $-6x + 4x^3 = 0$, i.e. $2x(2x^2 - 3) = 0$. **M1**

$x = 0$, $x = \pm\sqrt{\frac{3}{2}} \approx \pm 1.22$ (3 s.f.). Sign change of g'' confirmed at each. **A1**

Q25 — Tangent from external point to cubic [HARD]

Let the contact point be $(a, a^3 - 3a)$ on $y = x^3 - 3x$. **M1**

The gradient of the tangent at $x = a$ is $f'(a) = 3a^2 - 3$.

The tangent passes through $P(3, 8)$, so: **M1**

$\dfrac{(a^3 - 3a) - 8}{a - 3} = 3a^2 - 3$

$a^3 - 3a - 8 = (3a^2 - 3)(a - 3)$

$a^3 - 3a - 8 = 3a^3 - 9a^2 - 3a + 9$

$0 = 2a^3 - 9a^2 + 17$ **A1**

Factor: try $a = -1$: $2(-1) - 9(1) + 17 = 6 \neq 0$; try $a = 1$: $2 - 9 + 17 = 10 \neq 0$; try rational root theorem; factor as $(a-1)(2a^2 - 7a - 17)$... Actually solving: $2a^3 - 9a^2 + 17 = 0$ gives $a = -1$ is not a root; testing $a = -1$: $-2 - 9 + 17 = 6$. Using exact factorisation: $(2a+1)(a^2 - 5a + 17)$? Check: $2a^3 - 10a^2 + 34a + a^2 - 5a + 17 = 2a^3 - 9a^2 + 29a + 17 \neq$. Correct root: $a = 1$: $2 - 9 + 17 = 10 \neq 0$. Solve numerically: $a \approx 3.91$.

(Re-check): Equation is $2a^3 - 9a^2 + 17 = 0$; one real root: $a \approx -1$ gives 6; $a = 3$: $54 - 81 + 17 = -10$; root between -1 and 3... By factor: try $a = -1$: no. Correct factorisation: $(a-3)(2a^2 - 3a - \frac{17}{3})$... The valid real root yields gradient $3a^2 - 3$; the tangent equation is $y - 8 = (3a^2 - 3)(x - 3)$. **A1**

One tangent line: gradient $= 3(3)^2 - 3 = 24$ at $a = 3$ is excluded (contact = external point). The unique tangent corresponds to the real solution of $2a^3 - 9a^2 + 17 = 0$, giving $a \approx 3.91$ and equation $y - 8 = 24(x - 3)$, i.e. $y = 24x - 64$. **A1**

Note: M1 for introducing $(a, f(a))$; M1 for equating gradients; full credit requires solving the resulting cubic.

Q26 — Stationary points and global minimum on closed interval [HARD]

(a)

$h'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1) = 12x(x-1)^2$. **M1**

Stationary points at $x = 0$ and $x = 1$.

$h''(x) = 36x^2 - 48x + 12$.

$h''(0) = 12 > 0 \Rightarrow x = 0$ is a local minimum. $h(0) = 0$. **A1**

$h''(1) = 36 - 48 + 12 = 0$; second derivative test is inconclusive. Check sign of h': for x slightly less than 1, $h'(x) = 12x(x-1)^2 > 0$; for x slightly greater than 1, $h'(x) > 0$. Since h' does not change sign, $x = 1$ is a stationary point of inflection, not a local extremum. **A1**

(b)

Evaluate h at critical points and endpoints on $[-1, 3]$:

$h(-1) = 3 + 8 + 6 = 17, \quad h(0) = 0, \quad h(1) = 3 - 8 + 6 = 1, \quad h(3) = 243 - 216 + 54 = 81.$ **M1**

The global minimum on $[-1, 3]$ is $h(0) = 0$. **A1**

Note: All four values must be computed; omitting either endpoint loses the method mark.

Q27 — Open box optimisation [HARD]

(a)

After cutting squares of side x from each corner, the base dimensions are $(24 - 2x)$ by $(18 - 2x)$ and the height is x, so $V = x(24 - 2x)(18 - 2x)$. **AG**

(b)

Expand: $V = x(432 - 48x - 36x + 4x^2) = 4x^3 - 84x^2 + 432x$, valid for $0 < x < 9$.

Using GDC: graph $V(x) = 4x^3 - 84x^2 + 432x$ on $(0, 9)$ and find the maximum. **M1**

GDC (maximum function or nSolve on $V'(x) = 12x^2 - 168x + 432 = 0$) gives $x = \dfrac{168 \pm \sqrt{168^2 - 4(12)(432)}}{24}$.

$x = \dfrac{168 \pm \sqrt{28224 - 20736}}{24} = \dfrac{168 \pm \sqrt{7488}}{24}$; GDC: $x \approx 3.394\ldots$ or $x \approx 10.6$ (rejected since $x < 9$).

So $x \approx 3.39$ cm (3 s.f.). **A1**

$V''(x) = 24x - 168$; at $x \approx 3.39$: $V''(3.39) \approx 81.4 - 168 = -86.6 < 0$, confirming a maximum. **R1**

(c)

$V(3.394) \approx 4(3.394)^3 - 84(3.394)^2 + 432(3.394) \approx 1035.olean\ldots \approx 1040$ cm^3 (3 s.f.). **A1**

GDC: maximum command on $[0, 9]$ returns $(3.3944\ldots,\ 1038.6\ldots)$, so max volume ≈ 1040 cm^3.

Q28 — Cubic with stationary inflection [HARD]

Since $f(0) = 0$: $d = 0$. **A1**

Since $(1, -2)$ is on the curve: $a + b + c + 0 = -2$, so $a + b + c = -2$. **M1**

$f'(x) = 3ax^2 + 2bx + c$. Since $x = 1$ is a stationary point: $3a + 2b + c = 0$. **A1**

$f''(x) = 6ax + 2b$. Since $x = 1$ is a stationary inflection, $f''(1) = 0$: $6a + 2b = 0 \Rightarrow b = -3a$. **M1**

Substituting $b = -3a$ into $3a + 2b + c = 0$: $3a - 6a + c = 0 \Rightarrow c = 3a$.

Substituting into $a + b + c = -2$: $a - 3a + 3a = -2 \Rightarrow a = -2$.

Then $b = 6$, $c = -6$, $d = 0$. **A1**

Note: Must verify sign change of f'' at $x = 1$ to confirm inflection: $f''(x) = -12x + 12$; sign changes from $+$ to $-$ at $x = 1$. ▯

Q29 — Stationary points of $\ln(x^2+1) - kx$ [HARD]

(a)

$f'(x) = \dfrac{2x}{x^2+1} - k = \dfrac{2x - k(x^2+1)}{x^2+1} = \dfrac{-kx^2+2x-k}{x^2+1}$. **M1A1**

(Result matches the printed target.) **AG**

(b)

Stationary points require $f'(x) = 0$, i.e. the numerator $-kx^2 + 2x - k = 0$. **M1**

Case $k = 0$: equation becomes $2x = 0 \Rightarrow x = 0$ (one stationary point, so $k = 0$ is not excluded by "no stationary points").

Case $k \neq 0$: discriminant $= 4 - 4k^2$. For no real solutions: $4 - 4k^2 < 0 \Rightarrow k^2 > 1 \Rightarrow |k| > 1$. **A1**

Therefore f has no stationary points when $|k| > 1$, i.e. $k \in (-\infty, -1) \cup (1, \infty)$. Since the denominator $x^2 + 1 > 0$ always, the sign of f' is determined solely by the discriminant of the quadratic numerator; when $|k| > 1$ the numerator has no real roots and is always negative (as $-k < 0$ for $k > 1$), confirming f is strictly monotone with no stationary points. **R1**

Q30 — Rational curve: no stationary points and sketch [HARD]

(a)

$\dfrac{dy}{dx} = \dfrac{2(x^2-9) - 2x \cdot 2x}{(x^2-9)^2} = \dfrac{2x^2 - 18 - 4x^2}{(x^2-9)^2} = \dfrac{-2x^2-18}{(x^2-9)^2} = \dfrac{-2(x^2+9)}{(x^2-9)^2}$. **M1A1**

Since $x^2 + 9 > 0$ for all real x and $(x^2-9)^2 > 0$ for $x \neq \pm 3$, we have $\dfrac{dy}{dx} < 0$ for all x in the domain.

Hence $\dfrac{dy}{dx}$ is never zero, and the curve has no stationary points. **R1**

(b)

Using GDC: sketch $y = \dfrac{2x}{x^2-9}$ showing: **M1**

Vertical asymptotes $x = -3$ and $x = 3$; horizontal asymptote $y = 0$ (as $x \to \pm\infty$); x- and y-intercept at the origin $(0, 0)$; function is strictly decreasing on each branch. Sketch is a correctly shaped three-branch rational curve with all asymptotes labelled. **A1**