IB Math AA HL — Topic 5

Integration Toolkit

Antiderivatives & Definite Integrals · Substitution · By Parts · Partial Fractions

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

- **Power rule:** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1;\quad \int \frac{1}{x}\,dx = \ln|x| + C$
- **Standard forms:** $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$; $\int \sin(kx)\,dx = -\frac{\cos(kx)}{k} + C$; $\int \cos(kx)\,dx = \frac{\sin(kx)}{k} + C$; $\int \sec^2(x)\,dx = \tan x + C$
- **Inverse trig:** $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C;\quad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$
- **Substitution:** $\int f(g(x))g'(x)\,dx = \int f(u)\,du$ with $u = g(x)$; for definite integrals the limits must be changed: $x = a \Rightarrow u = g(a)$, $x = b \Rightarrow u = g(b)$
- **Integration by parts:** $\int u\,dv = uv - \int v\,du$; choose u so that $\frac{du}{dx}$ simplifies (LIATE priority: Logarithm, Inverse trig, Algebraic, Trig, Exponential)
- **Partial fractions (two distinct linear factors):** $\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$; then $\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C$
- **Fundamental Theorem of Calculus:** $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x);\quad \int_a^b f(x)\,dx = F(b) - F(a)$ where $F' = f$

GDC Tips

- **Numerical definite integral:** Use `fnInt(f(x),x,a,b)` (TI) or the $\int$ template (Casio/TI-Nspire) to verify an exact answer or evaluate an awkward integral; always write the setup, e.g. `fnInt(x^2*e^x, x, 0, 2)`.
- **Graph-and-intersect for limits:** Plot both curves, use Intersect (or G-Solv $\rightarrow$ Intersection) to find the x-values of intersection before integrating to find an enclosed area.
- **Numerical solve (nSolve/solve):** Use `nSolve(F(x)=k, x)` to find a variable upper limit when a definite integral equals a given value.
- **Verify by-parts or substitution:** After finding an antiderivative $F(x)$ analytically, confirm with $d/dx(F(x))$ (numerical derivative at a test point) that it returns $f(x)$ to at least 6 significant figures.
- **Checking partial-fraction integrals:** Evaluate the original integrand and the decomposed form at several x-values using STO$\rightarrow$ to confirm they are identical before integrating.

Common Mistakes to Avoid

1. **Omitting the constant of integration.** Every indefinite integral requires $+C$; omitting it loses the accuracy mark and is penalised every time it occurs, even in multi-step working.

2. **Forgetting to change limits in a definite substitution.** When $u = g(x)$, the limits $x = a$ and $x = b$ must become $u = g(a)$ and $u = g(b)$ before evaluating; substituting x-limits into a u-expression gives a wrong numerical answer.

3. **Incorrect sign in integration by parts.** The formula is $uv - \int v\,du$; a missing minus sign, especially in repeated applications, changes every subsequent term and produces a wrong final answer.

4. **Wrong cover-up or sign error in partial fractions.** When finding constants A and B, substituting the root of the wrong factor, or mis-reading a sign in $ax + b = 0$, gives incorrect numerators and hence wrong logarithm coefficients.

5. **Dropping the chain-rule factor in the FTC with a function upper limit.** $\dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$; omitting $g'(x)$ earns no method mark for that step.

6. **Applying $\int x^n\,dx$ with $n = -1$.** The formula $\dfrac{x^{n+1}}{n+1}$ is undefined for $n = -1$; the correct antiderivative of x^{-1} is $\ln|x| + C$, and the absolute value signs must appear whenever the domain can include negative values.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. **EASY** **No Calculator** **[3 marks]**

Find $\int \left(3x^2 - \frac{2}{x} + e^{4x}\right) dx$.

2. **EASY** **Calculator** **[3 marks]**

Find $\int_0^2 (5x^4-3x+1)\, dx$, giving your answer as an exact value.

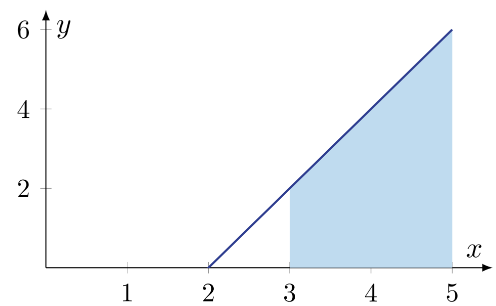
3. **EASY** **No Calculator** **[3 marks]**

Find $\int \frac{1}{\sqrt{9-x^2}}\, dx$.

4. EASY Calculator **[3 marks]**

The graph of $y = f'(x)$ is shown below, where $f'(x) = 2x - 4$. The shaded region is bounded by $y = f'(x)$, the x-axis, $x = 3$, and $x = 5$.

Find the exact value of $\int_3^5 (2x - 4)\,dx$.



5. EASY No Calculator **[3 marks]**

Use the substitution $u = 3x+1$ to find $\int (3x+1)^5\,dx$. Express your answer in terms of x.

6. EASY Calculator **[3 marks]**

Find $\int x\,e^{2x}\,dx$.

7. EASY Calculator **[3 marks]**

It is given that $\int_1^4 g(x)\,dx = 10$ and $\int_1^4 h(x)\,dx = 3$. Write down the value of $\int_1^4 [2g(x) - h(x)]\,dx$.

8. EASY Calculator **[3 marks]**

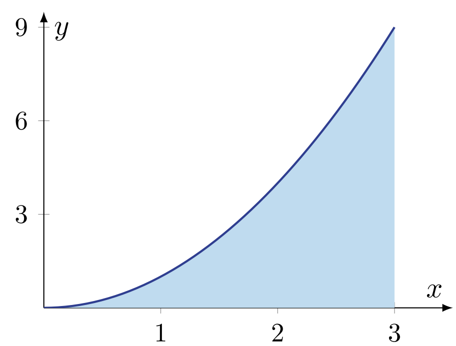
Let $F(x) = \int_0^{x^2} \cos t \, dt$. Find $F'(x)$.

9. EASY Calculator **[3 marks]**

The velocity of a particle at time t seconds ($t \geq 0$) is given by $v(t) = 6t^2 - 2t$ m s^{-1}. Given that the particle's displacement is $s = 4$ m when $t = 1$, find an expression for $s(t)$.

10. EASY Calculator **[3 marks]**

The region R is enclosed between the curve $y = x^2$, the x-axis, $x = 0$ and $x = 3$, as shown below.
Find the exact area of region R.



Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

Find $\displaystyle\int \frac{3x+1}{(x+2)(x-1)}\,dx$.

12. MEDIUM No Calculator **[4 marks]**

Find $\displaystyle\int x\,\sin(2x)\,dx$.

13. MEDIUM No Calculator **[4 marks]**

Let $I = \displaystyle\int_0^4 \sqrt{2x+1}\,dx$.

Use the substitution $u = 2x+1$ to find the exact value of I.

14. MEDIUM Calculator **[4 marks]**

The derivative of a function f is given by $f'(x) = \dfrac{4}{x^2+4}$.

Given that $f(0) = 3$, find $f(x)$.

15. MEDIUM Calculator **[4 marks]**

Let $g(x) = \displaystyle\int_1^{x^2} \frac{1}{1+t^3}\,dt$.

Find $g'(x)$.

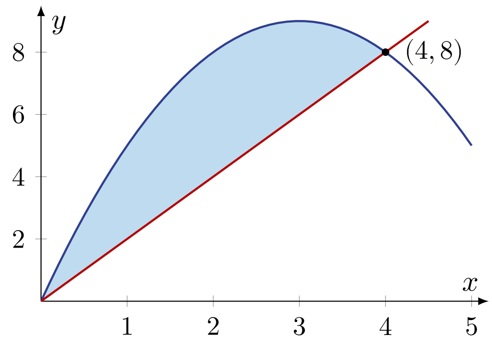
16. MEDIUM Calculator **[4 marks]**

A particle moves along a straight line. Its velocity at time t seconds ($t \geq 0$) is given by $v(t) = te^{-t}$ m s^{-1}. Find the displacement of the particle from $t = 0$ to $t = 3$, giving your answer correct to 3 significant figures.

17. **MEDIUM** **Calculator** **[4 marks]**

The graph of $y = 6x - x^2$ and the line $y = 2x$ are shown below. The shaded region is enclosed between the curve and the line.

Find the exact area of the shaded region.



18. **MEDIUM** **Calculator** **[4 marks]**

It is given that $\int_0^3 f(x)\,dx = 10$ and $\int_0^3 g(x)\,dx = 4$.

(a) Write down the value of $\int_0^3 [2f(x) - 3g(x)]\,dx$. [2]

(b) Given also that $\int_0^5 f(x)\,dx = 15$, find $\int_3^5 f(x)\,dx$. [2]

19. **MEDIUM** **Calculator** **[4 marks]**

The region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = k$ (where $k > 1$) has area ln 5.

Find the value of k.

20. **MEDIUM** **No Calculator** **[4 marks]**

Find $\displaystyle\int x^2 e^x \, dx$.

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD No Calculator **[5 marks]**

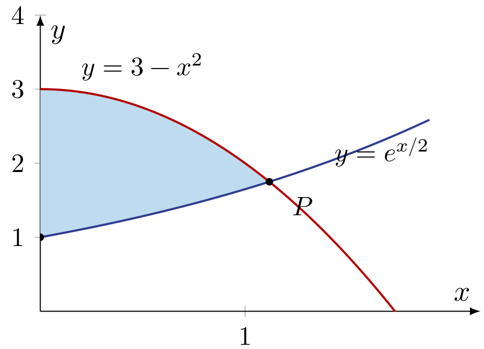
Let $f(x) = \dfrac{3x+1}{(x-1)(x+3)}$.

(a) Express $f(x)$ in the form $\dfrac{A}{x-1} + \dfrac{B}{x+3}$, where A and B are constants to be found. [2]

(b) Hence find $\displaystyle\int f(x)\,dx$, giving your answer in the form $\ln\left|\dfrac{(x-1)^a}{(x+3)^b}\right| + C$, where a and b are rational numbers. [3]

22. HARD Calculator **[5 marks]**

The diagram shows the region R enclosed between the curves $y = e^{x/2}$ and $y = 3 - x^2$ for $x \geq 0$.



(a) Use your GDC to find the x-coordinate of P, the intersection point for $x > 0$, correct to 3 significant figures. [2]

(b) Hence find the area of R, correct to 3 significant figures. [3]

23. HARD No Calculator **[5 marks]**

(a) Show that $\int x^2 e^{3x}\,dx = e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) + C.$ [4]

(b) Hence write down $\int_0^1 x^2 e^{3x}\,dx$ in exact form. [1]

24. HARD Calculator **[5 marks]**

Let $g(x) = \frac{\ln x}{x^2}$ for $x > 0$.

(a) Find $\int g(x)\,dx$. [3]

(b) The region bounded by $y = g(x)$, the x-axis, and the lines $x = 1$ and $x = e^2$ has area A. Find the exact value of A. [1]

(c) Verify your answer to part (b) using your GDC, stating the command used. [1]

25. HARD Calculator **[5 marks]**

Let $F(x) = \displaystyle\int_1^{x^3} \frac{\sqrt{t}}{1+t}\,dt$ for $x > 0$.

(a) Find $F'(x)$, simplifying your answer. [3]

(b) Use your GDC to find the value of x for which $F'(x) = 1$, correct to 3 significant figures. [2]

26. HARD No Calculator **[5 marks]**

Use the substitution $u = 1 + \cos\theta$ to find $\displaystyle\int \frac{\sin\theta}{(1+\cos\theta)^3}\,d\theta$.

Show clearly how the substitution transforms the integral, and give your final answer in terms of θ.

27. HARD Calculator **[5 marks]**

The rate of change of the volume, V cm^3, of water in a tank at time t seconds is modelled by

$$\frac{dV}{dt} = \frac{12t}{(t^2+4)^2}.$$

At $t = 0$, $V = 5$.

(a) Use the substitution $u = t^2 + 4$ to show that $\displaystyle\int \frac{12t}{(t^2+4)^2}\,dt = \frac{-6}{t^2+4} + C$. [3]

(b) Hence write down an expression for V in terms of t. [1]

(c) Find the volume of water in the tank as $t \to \infty$, and justify whether this represents a maximum or minimum value of V. [1]

28. HARD Calculator **[5 marks]**

Let $I = \int_0^{\pi/2} x\cos(2x)\,dx$.

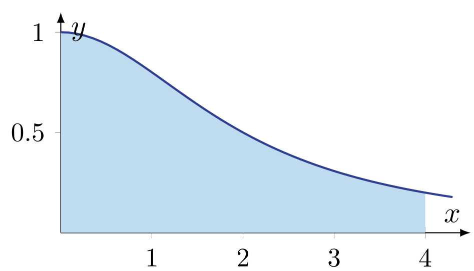
(a) Find I by integration by parts, using $u = x$ and $\dfrac{dv}{dx} = \cos(2x)$. [3]

(b) Verify your exact answer using the numerical integration feature of your GDC, stating the command and values entered. [1]

(c) Explain why the choice $u = \cos(2x)$ and $\dfrac{dv}{dx} = x$ would be a less efficient starting point for integration by parts. [1]

29. HARD Calculator **[5 marks]**

The diagram below shows the shaded region S between $y = \dfrac{4}{x^2+4}$ and the x-axis, from $x = 0$ to $x = 4$.



(a) Show that $\displaystyle\int \frac{4}{x^2+4}\,dx = 2\arctan\left(\frac{x}{2}\right) + C$. [2]

(b) Hence find the exact area of S. [2]

(c) Use your GDC to verify the area of S, correct to 3 significant figures. [1]

30. **HARD** **Calculator** **[5 marks]**

A particle moves along a straight line. Its velocity at time t seconds ($t \geq 0$) is given by $v(t) = t^2 e^{-t}$ m s^{-1}.

(a) Find the exact displacement of the particle from $t = 0$ to $t = 3$. [3]

(b) Using your GDC, find the total distance travelled by the particle from $t = 0$ to $t = 3$, correct to 3 significant figures. [1]

(c) Explain why the displacement and the total distance travelled are equal in this case. [1]

Worked Solutions

Q1 — Antiderivative of Mixed Terms EASY

Integrate each term separately: **M1**

$$\int \left(3x^2 - \frac{2}{x} + e^{4x}\right) dx = x^3 - 2\ln|x| + \frac{1}{4}e^{4x} + C$$ **A1 A1**

Note: Award A1 for the first two terms correct and A1 for $\frac{1}{4}e^{4x} + C$. Deduct one mark if $+C$ is absent (non-negotiable omission).

Flat-text check: `x^3 -2 ln|x| + e^{4x}/4 + C`

Q2 — Definite Integral of a Polynomial EASY

Antiderivative: $x^5 - \frac{3}{2}x^2 + x$ **M1**

Evaluate at limits: $\left[x^5 - \frac{3}{2}x^2 + x\right]_0^2 = (32 - 6 + 2) - 0$ **A1**

$= 28$ **A1**

Note: GDC confirmation: `fnInt(5x^4-3x+1, x, 0, 2)` $= 28$.

Q3 — Antiderivative in Arcsin Form EASY

Recognise the form $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + C$, here $a = 3$. **M1**

$$\int \frac{1}{\sqrt{9 - x^2}}\, dx = \arcsin\left(\frac{x}{3}\right) + C, \text{ i.e. } \arcsin(x/3) + C$$ **A1 A1**

Note: Award A1 for correct arcsin form and A1 for $+C$. Penalise missing $+C$.

Q4 — Definite Integral from a Graph EASY

Antiderivative: $x^2 - 4x$ **M1**

$\left[x^2 - 4x\right]_3^5 = (25 - 20) - (9 - 12)$ **A1**

$= 5 - (-3) = 8$ **A1**

Note: GDC confirmation: `fnInt(2x-4, x, 3, 5)` $= 8$.

Q5 — Substitution with a Linear Inner Function EASY

Let $u = 3x + 1$, so $\dfrac{du}{dx} = 3$, giving $dx = \dfrac{du}{3}$. **M1**

$\displaystyle\int u^5 \cdot \frac{du}{3} = \frac{1}{3} \cdot \frac{u^6}{6} + C = \frac{u^6}{18} + C$ **A1**

Returning to x: $= \dfrac{(3x+1)^6}{18} + C$, i.e. $(3x+1)^6/18 + C$ **A1**

Note: Deduct one mark if $+C$ is absent.

Q6 — Integration by Parts EASY

Choose $u = x$ (simplifies on differentiation) and $dv = e^{2x}dx$, so $du = dx$ and $v = \frac{1}{2}e^{2x}$. **M1**

$\displaystyle\int x\,e^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$ **A1**

$= \dfrac{e^{2x}}{4}(2x-1) + C$, i.e. $e^{2x}(2x-1)/4 + C$ **A1**

Note: GDC check: numerical derivative of answer at $x = 1$ should equal $e^2 \approx 7.389$.

Q7 — Linearity of the Definite Integral EASY

Using linearity of the definite integral: **M1**

$\displaystyle\int_1^4 [2g(x) - h(x)]\,dx = 2\int_1^4 g(x)\,dx - \int_1^4 h(x)\,dx$ **A1**

$= 2(10) - 3 = 17$ **A1**

Q8 — FTC with a Function Upper Limit EASY

By the Fundamental Theorem of Calculus, $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \cos t\,dt = \cos(x^2) \cdot \dfrac{d}{dx}(x^2)$ (chain rule factor shown explicitly). **M1 A1**

$F'(x) = 2x\cos(x^2)$ **A1**

Flat-text check: `2x cos(x^2)`

Q9 — Antiderivative from a Kinematics Context EASY

Integrate $v(t)$: $s(t) = \displaystyle\int (6t^2 - 2t)\,dt = 2t^3 - t^2 + C$ **M1 A1**

Apply initial condition $s(1) = 4$: $2(1)^3 - (1)^2 + C = 4 \Rightarrow 1 + C = 4 \Rightarrow C = 3$

$s(t) = 2t^3 - t^2 + 3$ **A1**

Note: GDC check: `fnInt(6t^2-2t, t, 0, 1)` $+ s(0)$ *consistent with* $s(1) = 4$ *for* $C = 3$.

Q10 — Definite Integral as an Area EASY

Area $= \int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3$ **M1 A1**

$= \frac{27}{3} - 0 = 9$ **A1**

Note: GDC confirmation: `fnInt(x^2, x, 0, 3)` $= 9$.

Q11 — Partial Fractions then Integrate MEDIUM

Write $\frac{3x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$. **M1**

Cover-up: $x = 1$: $B = \frac{4}{3}$; $x = -2$: $A = \frac{-5}{-3} = \frac{5}{3}$. **A1**

$\int \frac{3x+1}{(x+2)(x-1)}\,dx = \frac{5}{3}\ln|x+2| + \frac{4}{3}\ln|x-1| + C$ **A1 A1**

Note: Award A1 for each correct logarithmic term. Deduct 1 mark if $+C$ *is omitted. Coefficients:* $5/3$ *and* $4/3$.

Q12 — Integration by Parts MEDIUM

Let $u = x$, $dv = \sin(2x)\,dx$ (u simplifies on differentiation). **M1**

Then $du = dx$, $v = -\frac{1}{2}\cos(2x)$.

$\int x\sin(2x)\,dx = -\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx$ **A1**

$= -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$ **A1 A1**

Note: Award A1 for the first term and A1 for the second term with $+C$. *Deduct 1 mark if* $+C$ *is omitted.*

Flat-text check: `-(x/2) cos(2x) + (1/4) sin(2x) + C`

Q13 — Substitution with a Limit Change MEDIUM

Let $u = 2x + 1$, so $du = 2\,dx$, i.e. $dx = \frac{du}{2}$. **M1**

Limits change: $x = 0 \Rightarrow u = 1$; $x = 4 \Rightarrow u = 9$. **A1**

$I = \int_1^9 \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{1}{3}(27 - 1)$ **M1**

$I = 26/3$ **A1**

Note: If limits are not changed and the student returns to x before substituting limits, award full marks. Award M1A0 if limits are left as x-values throughout.

Q14 — Antiderivative with an Initial Condition MEDIUM

Recognise $\int \frac{4}{x^2+4}\,dx = \frac{4}{2}\arctan\left(\frac{x}{2}\right) + C = 2\arctan\left(\frac{x}{2}\right) + C$. **M1 A1**

Apply $f(0) = 3$: $2\arctan(0) + C = 3 \Rightarrow C = 3$. **M1**

$f(x) = 2\arctan\left(\frac{x}{2}\right) + 3$ **A1**

Note: GDC may be used to verify by checking $f'(x)$ numerically at a test point.

Flat-text check: `f(x) = 2 arctan(x/2) + 3`

Q15 — FTC with a Function Upper Limit MEDIUM

By the Fundamental Theorem of Calculus, $\frac{d}{dx}\int_1^{h(x)} \frac{1}{1+t^3}\,dt = \frac{1}{1+[h(x)]^3} \cdot h'(x)$. **M1**

Here $h(x) = x^2$, so $h'(x) = 2x$. **A1**

Apply chain rule factor explicitly: $g'(x) = \frac{1}{1+(x^2)^3} \cdot 2x$ **M1**

$g'(x) = \frac{2x}{1+x^6}$, i.e. $2x/(1+x^6)$ **A1**

Note: GDC verification: compare numerical derivative of g at $x = 2$ with $\frac{4}{65} \approx 0.0615$.

Q16 — Displacement by Parts with GDC Check MEDIUM

Displacement $= \int_0^3 te^{-t}\,dt$. **M1**

By parts: $u = t$, $dv = e^{-t}dt \Rightarrow v = -e^{-t}$:

$= \left[-te^{-t}\right]_0^3 + \int_0^3 e^{-t}\,dt = -3e^{-3} + \left[-e^{-t}\right]_0^3 = -3e^{-3} - e^{-3} + 1$ **A1**

Exact value: $1 - 4e^{-3}$. **A1**

GDC: `fnInt(t*e^(-t), t, 0, 3)` $= 0.800846\ldots \approx 0.801$ m. **A1**

Note: Accept exact form $1-4e^{-3}$ or decimal 0.801 m for the final A1. Both the exact working and numerical

confirmation are required.
Flat-text check: `1 - 4e^-3`

Q17 — Area Between a Curve and a Line MEDIUM

Intersection: $6x - x^2 = 2x \Rightarrow x(4-x) = 0 \Rightarrow x = 0,\ x = 4.$ **M1**

Area $= \displaystyle\int_0^4 [(6x - x^2) - 2x]\,dx = \int_0^4 (4x - x^2)\,dx$ **A1**

$= \left[2x^2 - \dfrac{x^3}{3}\right]_0^4 = 32 - \dfrac{64}{3}$ **A1**

$= 32/3$ **A1**

Note: GDC: `fnInt(4x-x^2, x, 0, 4)` $= 10.666\ldots = 32/3$. *Award A1A1 for correct integrand and correct antiderivative; final A1 for exact answer.*

Q18 — Properties of Definite Integrals MEDIUM

(a) $\displaystyle\int_0^3 [2f(x) - 3g(x)]\,dx = 2 \times 10 - 3 \times 4$ **M1**

$= 20 - 12 = 8$ **A1**

(b) $\displaystyle\int_3^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^3 f(x)\,dx = 15 - 10$ **M1**

$= 5$ **A1**

Q19 — Solving for an Upper Limit MEDIUM

$\displaystyle\int_1^k \frac{1}{x}\,dx = \ln 5$ **M1**

$[\ln x]_1^k = \ln k - \ln 1 = \ln k$ **A1**

So $\ln k = \ln 5$, giving $k = 5$. **M1 A1**

Note: GDC verification: `fnInt(1/x, x, 1, 5)` $= 1.6094\ldots = \ln 5 \approx 1.6094$. *Final answer must be exact integer.*

Q20 — Integration by Parts Twice MEDIUM

Let $u = x^2$, $dv = e^x\,dx$ (u simplifies on differentiation). **M1**

$\Rightarrow du = 2x\,dx$, $v = e^x$.

$$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx$$ **A1**

Apply by parts again to $\int 2xe^x\,dx$: $u = 2x$, $dv = e^x\,dx$:

$= 2xe^x - 2e^x$ **M1**

$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C = e^x(x^2-2x+2)+C$, i.e. $e^x(x^2-2x+2)+C$ **A1**

Note: Deduct 1 mark if $+C$ is omitted. Accept unsimplified form $x^2e^x - 2xe^x + 2e^x + C$.

Q21 — Partial Fractions then Integrate HARD

(a) Write $\dfrac{3x+1}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$, so $3x+1 = A(x+3)+B(x-1)$. **M1**

Cover-up $x=1$: $4=4A \Rightarrow A=1$. Cover-up $x=-3$: $-8=-4B \Rightarrow B=2$. **A1**

(b) $\displaystyle\int \frac{1}{x-1} + \frac{2}{x+3}\,dx = \ln|x-1| + 2\ln|x+3| + C$ **M1**

$$= \ln|x-1| + \ln(x+3)^2 + C = \ln\left|\frac{(x-1)}{(x+3)^{-2}}\right| + C$$

Written in demanded form: $a=1, b=-2$, i.e. $\ln\left|\dfrac{(x-1)^1}{(x+3)^{-2}}\right| + C$. **A1**

Accept equivalent: $\ln|x-1| + 2\ln|x+3| + C$ fully simplified with modulus signs shown. **A1**

Note: award the final A1 only if $+C$ is present on the indefinite integral.

Q22 — Area Between Two Curves via GDC HARD

(a) Using GDC graph-and-intersect with $y_1 = e^{x/2}$ and $y_2 = 3-x^2$: unrounded value $x = 1.11837\ldots$ **M1**

$x_P = 1.12$ (3 s.f.) **A1**

(b) Area $= \displaystyle\int_0^{1.1184} [(3-x^2) - e^{x/2}]\,dx$ **M1**

Using GDC numerical integral (e.g. `fnInt(3-x^2-e^(x/2), x, 0, 1.1184)`): unrounded $= 1.39035\ldots$ **A1**

Area $= 1.39$ (3 s.f.) **A1**

Note: accept limits using the candidate's value from (a); follow-through on the intersection point applies for the final A1.

Q23 — Integration by Parts Twice with a Show That HARD

(a) Apply by parts twice with $u = x^2$, $dv = e^{3x}dx$ (choose $u = x^2$ since it simplifies on differentiation). **M1**

$$\int x^2e^{3x}\,dx = \frac{x^2e^{3x}}{3} - \int \frac{2xe^{3x}}{3}\,dx$$

Second by parts: $u = \frac{2x}{3}$, $dv = e^{3x}dx$: $\displaystyle\int \frac{2xe^{3x}}{3}\,dx = \frac{2xe^{3x}}{9} - \int \frac{2e^{3x}}{9}\,dx = \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}$ **M1**

Combining: $\displaystyle\frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + C = e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right) + C$ **A1**

Since this equals the printed target, the result is established. **AG**

Note: AG; award no marks if only differentiation is used to verify.

(b) $\displaystyle\int_0^1 x^2e^{3x}\,dx = \left[e^{3x}\left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right)\right]_0^1 = e^3\left(\frac{1}{3} - \frac{2}{9} + \frac{2}{27}\right) - \frac{2}{27}$ **M1**

$\displaystyle= e^3\left(\frac{9}{27} - \frac{6}{27} + \frac{2}{27}\right) - \frac{2}{27} = e^3\left(\frac{5}{27}\right) - \frac{2}{27}$

$= 5e^3/27 - 2/27$ **A1**

Q24 — Integration by Parts with a Log Integrand HARD

(a) $\displaystyle\int \frac{\ln x}{x^2}\,dx$: choose $u = \ln x$, $dv = x^{-2}dx$ (since $\ln x$ simplifies on differentiation). **M1**

$du = \frac{1}{x}dx$, $v = -x^{-1}$, so $\displaystyle\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$ **A1 A1**

Note: award first A1 for $-\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$; second A1 for $-\frac{1}{x} + C$ with $+C$ present.

(b) $\displaystyle A = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^{e^2} = \left(-\frac{2}{e^2} - \frac{1}{e^2}\right) - (0 - 1) = 1 - \frac{3}{e^2}$ **A1**

(c) GDC: `fnInt(ln(x)/x^2, x, 1, e^2)` gives $0.59399\ldots$; and $1 - 3e^{-2} = 0.59399\ldots$ ✓ **A1**

Flat-text check: `1 - 3e^-2`

Q25 — FTC with a Function Upper Limit and GDC Solve HARD

(a) By the Fundamental Theorem of Calculus with upper limit x^3, apply the chain rule: **M1**

$$F'(x) = \frac{\sqrt{x^3}}{1+x^3}\cdot\frac{d}{dx}(x^3)$$ **M1**

$$= \frac{x^{3/2}}{1+x^3}\cdot 3x^2 = \frac{3x^{7/2}}{1+x^3}$$ **A1**

(b) Solve $\dfrac{3x^{7/2}}{1+x^3} = 1$ using GDC: graph-and-intersect (or nSolve) with $y_1 = 3x^{3.5}/(1+x^3)$ and $y_2 = 1$; the unique positive root (since F' is monotonically increasing for $x > 0$) gives unrounded $x = 0.831987\ldots$ **M1**

$x = 0.832$ (3 s.f.) **A1**

Flat-text check: `3x^(7/2)/(1+x^3)`

Q26 — Substitution with a Trig Integrand HARD

Let $u = 1 + \cos\theta$, so $\dfrac{du}{d\theta} = -\sin\theta$, i.e. $\sin\theta\, d\theta = -du$. **M1**

Substituting: $\displaystyle\int \frac{\sin\theta}{(1+\cos\theta)^3}\, d\theta = \int \frac{-du}{u^3} = -\int u^{-3}\, du$ **A1**

$= -\dfrac{u^{-2}}{-2} + C = \dfrac{1}{2u^2} + C$ **M1**

Returning to θ: $= \dfrac{1}{2(1+\cos\theta)^2} + C$ **A1 A1**

Note: first A1 for correct integration in u; second A1 for correct back-substitution and $+C$.

Flat-text check: `1/(2(1+cos(theta))^2)`

Q27 — Substitution in Context with Limiting Behaviour HARD

(a) Let $u = t^2 + 4$, so $\dfrac{du}{dt} = 2t$, i.e. $12t\, dt = 6\, du$. **M1**

$\displaystyle\int \frac{12t}{(t^2+4)^2}\, dt = \int \frac{6}{u^2}\, du = 6\int u^{-2}\, du = -\frac{6}{u} + C = \frac{-6}{t^2+4} + C$ **A1 A1**

Since this equals the printed target, the result is established. **AG**

Note: award first A1 for correct substitution giving $\int 6u^{-2}\, du$; second A1 for completing to $-6/u + C$ and back-substituting.

(b) $V = \dfrac{-6}{t^2+4} + C$. At $t = 0$: $5 = -\dfrac{6}{4} + C \Rightarrow C = 5 + \dfrac{3}{2} = \dfrac{13}{2}$, so $V = 13/2 - 6/(t^2+4)$. **A1**

(c) As $t \to \infty$, $\dfrac{6}{t^2+4} \to 0$, so $V \to \dfrac{13}{2}$. Since $\dfrac{dV}{dt} = \dfrac{12t}{(t^2+4)^2} \geq 0$ for all $t \geq 0$, V is increasing, so $13/2$ is a supremum (approached from below) — this is the maximum volume the tank approaches but never exceeds. **R1**

Q28 — By Parts Definite Integral with GDC Verification HARD

(a) $u = x \Rightarrow du = dx$; $\dfrac{dv}{dx} = \cos(2x) \Rightarrow v = \dfrac{\sin(2x)}{2}$ (choose $u = x$ since it simplifies on differentiation). **M1**

$$I = \left[\frac{x\sin(2x)}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin(2x)}{2}\,dx = 0 - \left[-\frac{\cos(2x)}{4}\right]_0^{\pi/2}$$ **M1**

$$= -\left(\frac{\cos\pi}{4} - \frac{\cos 0}{4}\right) = -\left(-\frac{1}{4} - \frac{1}{4}\right) = -1/2$$ **A1**

(b) GDC: `fnInt(x*cos(2x), x, 0, pi/2)` gives $-0.500000\ldots$, confirming $I = -\dfrac{1}{2}$. **A1**

(c) Choosing $u = \cos(2x)$ gives $v = x^2/2$ after integration, which is more complex than the original integrand and does not simplify the integral — the resulting $\int x^2 \sin(2x)\,dx$ is harder than the original. **R1**

Q29 — Arctan Antiderivative with a Show That HARD

(a) Write $\dfrac{4}{x^2+4} = \dfrac{4}{x^2\left(\frac{4}{x^2}+1\right)}$; use the standard form $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ with $a = 2$: **M1**

$$\int \frac{4}{x^2+4}\,dx = 4 \cdot \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C = 2\arctan\left(\frac{x}{2}\right) + C$$ **A1**

This equals the printed target. **AG**

(b) Area $= \left[2\arctan\left(\dfrac{x}{2}\right)\right]_0^4 = 2\arctan(2) - 2\arctan(0) = 2\arctan(2)$ **M1**

Exact area $= 2\arctan(2)$ **A1**

(c) GDC: `fnInt(4/(x^2+4), x, 0, 4)` gives $2.2143\ldots$; and $2\arctan(2) = 2.214$ (3 s.f.) ✓ **A1**

Flat-text check: `2 arctan(2)`

Q30 — By Parts Twice in a Kinematics Context HARD

(a) Displacement $= \displaystyle\int_0^3 t^2 e^{-t}\,dt$. Apply by parts twice; choose $u = t^2$, $dv = e^{-t}dt$. **M1**

First by parts: $= \left[-t^2 e^{-t}\right]_0^3 + \displaystyle\int_0^3 2te^{-t}\,dt$

Second by parts ($u = 2t$, $dv = e^{-t}dt$): $\displaystyle\int_0^3 2te^{-t}\,dt = \left[-2te^{-t}\right]_0^3 + \int_0^3 2e^{-t}\,dt = -6e^{-3} + \left[-2e^{-t}\right]_0^3$ **M1**

$= -6e^{-3} + (-2e^{-3} + 2) = -8e^{-3} + 2$

Displacement $= -9e^{-3} + (-8e^{-3} + 2) = 2 - 17e^{-3}$ m **A1**

(b) Since $v(t) = t^2e^{-t} \geq 0$ for all $t \in [0, 3]$, the particle never changes direction, so total distance $=$ displacement $= 2 - 17e^{-3}$; GDC `fnInt(t^2*e^(-t), t, 0, 3)` gives $1.1537\ldots \approx 1.15$ m (3 s.f.). **A1**

(c) Since $v(t) = t^2e^{-t} \geq 0$ for all $t \in [0, 3]$, the particle always moves in the positive direction and never reverses, so the total distance travelled equals the displacement. **R1**

Flat-text check: `2 - 17e^-3`