

IB Math AA HL — Topic 5

Integration Toolkit

Antiderivatives & Definite Integrals · Substitution · By Parts · Partial Fractions

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name: _____ Date: _____
Score: _____/120



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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.



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Key Formulae

- **Power rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1; \quad \int \frac{1}{x} dx = \ln|x| + C$
- **Standard forms:** $\int e^{kx} dx = \frac{e^{kx}}{k} + C; \quad \int \sin(kx) dx = -\frac{\cos(kx)}{k} + C; \quad \int \cos(kx) dx = \frac{\sin(kx)}{k} + C; \quad \int \sec^2(x) dx = \tan x + C$
- **Inverse trig:** $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C; \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$
- **Substitution:** $\int f(g(x)) g'(x) dx = \int f(u) du$ with $u = g(x)$; for definite integrals the limits *must* be changed: $x = a \Rightarrow u = g(a), x = b \Rightarrow u = g(b)$
- **Integration by parts:** $\int u dv = uv - \int v du$; choose u so that $\frac{du}{dx}$ simplifies (LIATE priority: Logarithm, Inverse trig, Algebraic, Trig, Exponential)
- **Partial fractions (two distinct linear factors):** $\frac{px + q}{(ax + b)(cx + d)} = \frac{A}{ax + b} + \frac{B}{cx + d}$; then $\int \frac{A}{ax + b} dx = \frac{A}{a} \ln|ax + b| + C$
- **Fundamental Theorem of Calculus:** $\frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x)) \cdot g'(x); \quad \int_a^b f(x) dx = F(b) - F(a)$
where $F' = f$

GDC Tips

- **Numerical definite integral:** Use `fnInt(f(x), x, a, b)` (TI) or $\int_a^b$ template (Casio/TI-Nspire) to verify an exact answer or evaluate an awkward integral; always write the setup, e.g. `fnInt(x^2*e^x, x, 0, 2)`.
- **Graph-and-intersect for limits:** Plot both curves, use `Intersect` (or `G-Solv` $\rightarrow$ `Intersection`) to find the x -values of intersection before integrating to find an enclosed area.
- **Numerical solve (nSolve/solve):** Use `nSolve(F(x)=k, x)` to find a variable upper limit when a definite integral equals a given value.
- **Verify by-parts or substitution:** After finding an antiderivative $F(x)$ analytically, confirm with $d/dx(F(x))$ (numerical derivative at a test point) that it returns $f(x)$ to at least 6 significant figures.
- **Checking partial-fraction integrals:** Evaluate the original integrand and the decomposed form at several x -values using `STO` $\rightarrow$ to confirm they are identical before integrating.

Common Mistakes to Avoid

- Omitting the constant of integration.** Every indefinite integral requires $+C$; omitting it loses the accuracy mark and is penalised every time it occurs, even in multi-step working.
- Forgetting to change limits in a definite substitution.** When $u = g(x)$, the limits $x = a$ and $x = b$ must become $u = g(a)$ and $u = g(b)$ before evaluating; substituting x -limits into a u -expression gives a wrong numerical answer.
- Incorrect sign in integration by parts.** The formula is $uv - \int v du$; a missing minus sign, especially in repeated applications, changes every subsequent term and produces a wrong final answer.
- Wrong cover-up / sign error in partial fractions.** When finding constants A and B , substituting the root of the *wrong* factor, or mis-reading a sign in $ax + b = 0$, gives incorrect numerators and hence wrong logarithm coefficients.
- Dropping the chain-rule factor in the FTC with a function upper limit.** $\frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x)) \cdot g'(x)$; omitting $g'(x)$ earns no method mark for that step.
- Applying $\int x^n dx$ with $n = -1$.** The formula $\frac{x^{n+1}}{n+1}$ is undefined for $n = -1$; the correct antiderivative of x^{-1} is $\ln|x| + C$, and the absolute value signs must appear whenever the domain can include negative values.



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Section A — Easy

EASY

1. Find $\int \left(3x^2 - \frac{2}{x} + e^{4x} \right) dx$. *Calculator not allowed*

[3 marks]

2. Find $\int_0^2 (5x^4 - 3x + 1) dx$, giving your answer as an exact value. *Calculator allowed*

[3 marks]

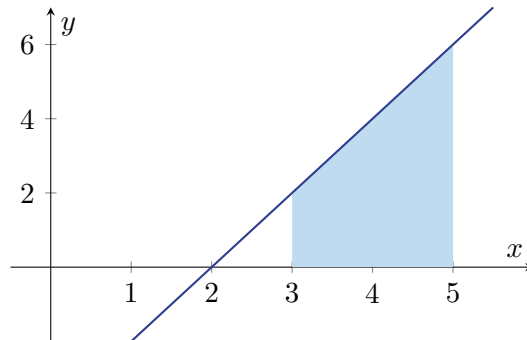
3. Find $\int \frac{1}{\sqrt{9-x^2}} dx$. *Calculator not allowed*

[3 marks]

4. The graph of $y = f'(x)$ is shown below, where $f'(x) = 2x - 4$. The shaded region is bounded by $y = f'(x)$, the x -axis, $x = 3$, and $x = 5$.

Find the exact value of $\int_3^5 (2x - 4) dx$. *Calculator allowed*

[3 marks]



5. Use the substitution $u = 3x + 1$ to find $\int (3x + 1)^5 dx$. Express your answer in terms of x . *Calculator not allowed*

[3 marks]

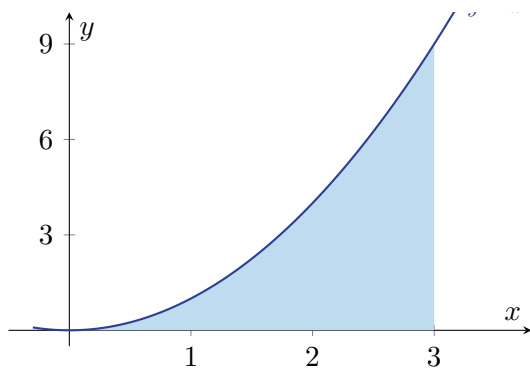
6. Find $\int x e^{2x} dx$. *Calculator allowed*

[3 marks]

10. The region R is enclosed between the curve $y = x^2$, the x -axis, $x = 0$ and $x = 3$, as shown below.

Find the exact area of region R . *Calculator allowed*

[3 marks]





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Section B — Medium

MEDIUM

11. Find $\int \frac{3x + 1}{(x + 2)(x - 1)} dx$. *Calculator not allowed*

[4 marks]

12. Find $\int x \sin(2x) dx$. *Calculator not allowed*

[4 marks]

13. Let $I = \int_0^4 \sqrt{2x+1} \, dx$.

Use the substitution $u = 2x + 1$ to find the exact value of I . *Calculator not allowed*

[4 marks]

14. The derivative of a function f is given by $f'(x) = \frac{4}{x^2 + 4}$.

Given that $f(0) = 3$, find $f(x)$. *Calculator allowed*

[4 marks]

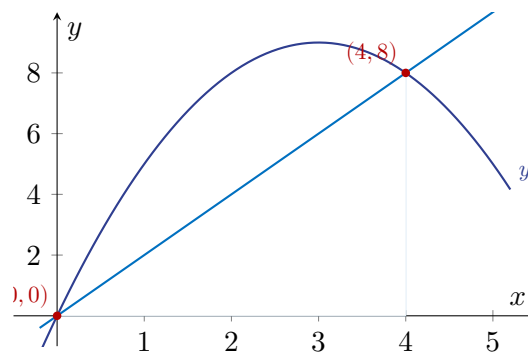
15. Let $g(x) = \int_1^{x^2} \frac{1}{1+t^3} \, dt$.

Find $g'(x)$. *Calculator allowed*

[4 marks]

- 16.** A particle moves along a straight line. Its velocity at time t seconds ($t \geq 0$) is given by $v(t) = te^{-t} \text{ m s}^{-1}$. Find the displacement of the particle from $t = 0$ to $t = 3$, giving your answer correct to 3 significant figures. *Calculator allowed* [4 marks]

- 17.** The graph of $y = 6x - x^2$ and the line $y = 2x$ are shown below. The shaded region is enclosed between the curve and the line.



- Find the exact area of the shaded region. *Calculator allowed* [4 marks]

18. It is given that $\int_0^3 f(x) \, dx = 10$ and $\int_0^3 g(x) \, dx = 4$.

(a) Write down the value of $\int_0^3 [2f(x) - 3g(x)] \, dx$. [2 marks]

(b) Given also that $\int_0^5 f(x) \, dx = 15$, find $\int_3^5 f(x) \, dx$. [2 marks]

Calculator allowed

19. The region bounded by the curve $y = \frac{1}{x}$, the x -axis, and the lines $x = 1$ and $x = k$ (where $k > 1$) has area $\ln 5$.

Find the value of k . *Calculator allowed* [4 marks]

20. Find $\int x^2 e^x \, dx$. *Calculator not allowed*

[4 marks]



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Section C — Hard

HARD

21. *Calculator not allowed*

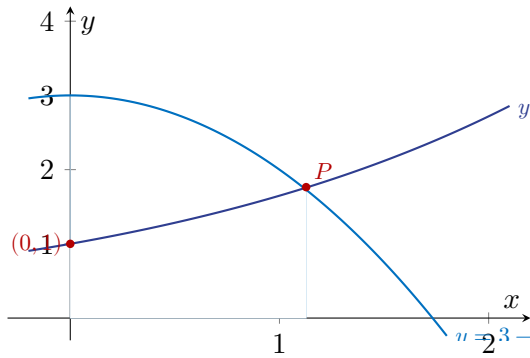
21. Calculator not allowed
Let $f(x) = \frac{3x + 1}{(x - 1)(x + 3)}$.

- (a) Express $f(x)$ in the form $\frac{A}{x-1} + \frac{B}{x+3}$, where A and B are constants to be found. [2 marks]

- (b) Hence find $\int f(x) dx$, giving your answer in the form $\ln \left| \frac{(x-1)^a}{(x+3)^b} \right| + C$, where a and b are rational numbers. [3 marks]

22. Calculator allowed

The diagram shows the region R enclosed between the curves $y = e^{x/2}$ and $y = 3 - x^2$ for $x \geq 0$.



- (a) Use your GDC to find the x -coordinate of P , the intersection point for $x > 0$, correct to 3 significant figures.
[2 marks]

- (b) Hence find the area of R , correct to 3 significant figures.

[3 marks]

23. *Calculator not allowed*

(a) Show that $\int x^2 e^{3x} dx = e^{3x} \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) + C$.

[4 marks]

(b) Hence write down $\int_0^1 x^2 e^{3x} dx$ in exact form.

[1 marks]

24. *Calculator allowed*

Let $g(x) = \frac{\ln x}{x^2}$ for $x > 0$.

(a) Find $\int g(x) dx$. [3 marks]

(b) The region bounded by $y = g(x)$, the x -axis, and the lines $x = 1$ and $x = e^2$ has area A . Find the exact value of A . [1 marks]

(c) Verify your answer to part (b) using your GDC, stating the command used. [1 marks]

25. *Calculator allowed*

Let $F(x) = \int_1^{x^3} \frac{\sqrt{t}}{1+t} dt$ for $x > 0$.

(a) Find $F'(x)$, simplifying your answer. [3 marks]

(b) Use your GDC to find the value of x for which $F'(x) = 1$, correct to 3 significant figures. [2 marks]

26. *Calculator not allowed*

Use the substitution $u = 1 + \cos \theta$ to find $\int \frac{\sin \theta}{(1 + \cos \theta)^3} d\theta$.

Show clearly how the substitution transforms the integral, and give your final answer in terms of θ . *Calculator not allowed* [5 marks]

27. *Calculator allowed*

The rate of change of the volume, $V \text{ cm}^3$, of water in a tank at time t seconds is modelled by

$$\frac{dV}{dt} = \frac{12t}{(t^2 + 4)^2}.$$

At $t = 0$, $V = 5$.

(a) Use the substitution $u = t^2 + 4$ to show that $\int \frac{12t}{(t^2 + 4)^2} dt = \frac{-6}{t^2 + 4} + C$. [3 marks]

(b) Hence write down an expression for V in terms of t . [1 marks]

(c) Find the volume of water in the tank as $t \rightarrow \infty$, and justify whether this represents a maximum or minimum value of V . [1 marks]

28. *Calculator allowed*

Let $I = \int_0^{\pi/2} x \cos(2x) dx$.

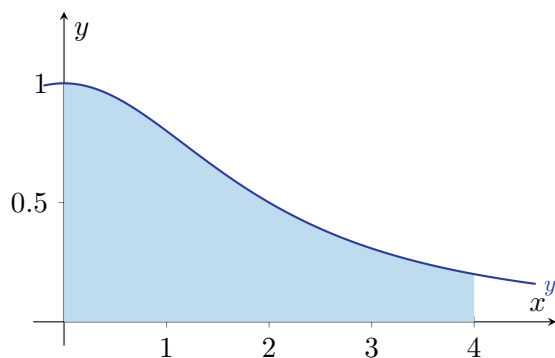
(a) Find I by integration by parts, using $u = x$ and $\frac{dv}{dx} = \cos(2x)$. [3 marks]

(b) Verify your exact answer using the numerical integration feature of your GDC, stating the command and values entered. [1 marks]

(c) Explain why the choice $u = \cos(2x)$ and $\frac{dv}{dx} = x$ would be a less efficient starting point for integration by parts. [1 marks]

29. *Calculator allowed*

The diagram below shows the shaded region S between $y = \frac{4}{x^2 + 4}$ and the x -axis, from $x = 0$ to $x = 4$.



(a) Show that $\int \frac{4}{x^2 + 4} dx = 2 \arctan\left(\frac{x}{2}\right) + C$. [2 marks]

(b) Hence find the exact area of S . [2 marks]

(c) Use your GDC to verify the area of S , correct to 3 significant figures. [1 marks]

30. Calculator allowed

A particle moves along a straight line. Its velocity at time t seconds ($t \geq 0$) is given by $v(t) = t^2 e^{-t} \text{ m s}^{-1}$.

- (a) Find the exact displacement of the particle from $t = 0$ to $t = 3$. [3 marks]
- (b) Using your GDC, find the total distance travelled by the particle from $t = 0$ to $t = 3$, correct to 3 significant figures. [1 marks]
- (c) Explain why the displacement and the total distance travelled are equal in this case. [1 marks]

Worked Solutions

Q1 — Antiderivative of mixed terms [EASY]

Integrate each term separately:

$$\int \left(3x^2 - \frac{2}{x} + e^{4x} \right) dx$$

M1

$$= x^3 - 2 \ln |x| + \frac{1}{4} e^{4x} + C$$

A1 A1

Note: Award **A1** for the first two terms correct and **A1** for $\frac{1}{4}e^{4x} + C$. Deduct one mark if $+C$ is absent (non-negotiable omission).

Q2 — Definite integral, polynomial [EASY]

Antiderivative: $x^5 - \frac{3}{2}x^2 + x$

M1

Evaluate at limits:

$$\left[x^5 - \frac{3}{2}x^2 + x \right]_0^2 = (32 - 6 + 2) - 0$$

A1

$$= 28$$

A1

Note: GDC confirmation: $\text{fnInt}(5x^4 - 3x + 1, x, 0, 2) = 28$.

Q3 — Antiderivative, arcsin form [EASY]

Recognise the form $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$, here $a = 3$.

M1

$$\int \frac{1}{\sqrt{9 - x^2}} dx = \arcsin\left(\frac{x}{3}\right) + C$$

A1 A1

Note: Award **A1** for correct arcsin form and **A1** for $+C$. Penalise missing $+C$.

Q4 — Definite integral from graph [EASY]

Antiderivative: $x^2 - 4x$

M1

$$[x^2 - 4x]_3^5 = (25 - 20) - (9 - 12)$$

A1

$$= 5 - (-3) = 8$$

A1

Note: GDC confirmation: $\text{fnInt}(2x-4, x, 3, 5) = 8$.

Q5 — Substitution, linear inner function [EASY]

Let $u = 3x + 1$, so $\frac{du}{dx} = 3$, giving $dx = \frac{du}{3}$.

M1

$$\int u^5 \cdot \frac{du}{3} = \frac{1}{3} \cdot \frac{u^6}{6} + C = \frac{u^6}{18} + C$$

A1

Returning to x :

$$= \frac{(3x + 1)^6}{18} + C$$

A1

Note: Deduct one mark if $+C$ is absent.

Q6 — Integration by parts, xe^{2x} [EASY]

Choose $u = x$ (simplifies on differentiation) and $dv = e^{2x} dx$, so $du = dx$ and $v = \frac{1}{2}e^{2x}$.

M1

$$\int x e^{2x} dx = \frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C$$

A1

$$= \frac{e^{2x}}{4} (2x - 1) + C$$

A1

Note: GDC check: numerical derivative of answer at $x = 1$ should equal $e^2 \approx 7.389$.

Q7 — FTC linearity property [EASY]

Using linearity of the definite integral:

M1

$$\int_1^4 [2g(x) - h(x)] \, dx = 2 \int_1^4 g(x) \, dx - \int_1^4 h(x) \, dx$$

A1

$$= 2(10) - 3 = 17$$

A1

Q8 — FTC with function upper limit [EASY]

By the Fundamental Theorem of Calculus, $\frac{d}{dx} \int_0^{x^2} \cos t \, dt = \cos(x^2) \cdot \frac{d}{dx}(x^2)$ (chain rule factor shown explicitly).

M1 A1

$$F'(x) = 2x \cos(x^2)$$

A1

Q9 — Antiderivative from kinematics context [EASY]

Integrate $v(t)$:

$$s(t) = \int (6t^2 - 2t) \, dt = 2t^3 - t^2 + C$$

M1 A1

Apply initial condition $s(1) = 4$:

$$2(1)^3 - (1)^2 + C = 4 \implies 1 + C = 4 \implies C = 3$$

$$s(t) = 2t^3 - t^2 + 3$$

A1

Note: GDC check: $\text{fnInt}(6t^2 - 2t, t, 0, 1) + s(0)$ consistent with $s(1) = 4$ for $C = 3$.

Q10 — Definite integral as area [EASY]

$$\text{Area} = \int_0^3 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^3$$

M1 A1

$$= \frac{27}{3} - 0 = 9$$

A1

Note: GDC confirmation: $\text{fnInt}(x^2, x, 0, 3) = 9$.

Q11 — Partial fractions then integrate [MEDIUM]

Write $\frac{3x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$.

M1

Cover-up: $x = 1: B = \frac{4}{3}; \quad x = -2: A = \frac{-5}{-3} = \frac{5}{3}$.

A1

$$\int \frac{3x+1}{(x+2)(x-1)} dx = \frac{5}{3} \ln|x+2| + \frac{4}{3} \ln|x-1| + C$$

A1A1

Note: Award A1 for each correct logarithmic term. Deduct 1 mark if $+C$ is omitted.

Q12 — Integration by parts, $x \sin(2x)$ [MEDIUM]

Let $u = x, dv = \sin(2x) dx$ (u simplifies on differentiation).

M1

Then $du = dx, v = -\frac{1}{2} \cos(2x)$.

$$\int x \sin(2x) dx = -\frac{x}{2} \cos(2x) + \int \frac{1}{2} \cos(2x) dx$$

A1

$$= -\frac{x}{2} \cos(2x) + \frac{1}{4} \sin(2x) + C$$

A1A1

Note: Award A1 for the first term and A1 for the second term with $+C$. Deduct 1 mark if $+C$ is omitted.

Q13 — Substitution with limit change [MEDIUM]

Let $u = 2x + 1$, so $du = 2 dx$, i.e. $dx = \frac{du}{2}$.

M1

Limits change: $x = 0 \Rightarrow u = 1; \quad x = 4 \Rightarrow u = 9$.

A1

$$I = \int_1^9 \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2} \right]_1^9 = \frac{1}{3} (27 - 1)$$

M1

$$I = \frac{26}{3}$$

A1

Note: If limits are not changed and the student returns to x before substituting limits, award full marks. Award M1A0 if limits are left as x -values throughout.

Q14 — Antiderivative of $1/(a^2 + x^2)$ with initial condition [MEDIUM]

Recognise $\int \frac{4}{x^2 + 4} dx = \frac{4}{2} \arctan\left(\frac{x}{2}\right) + C = 2 \arctan\left(\frac{x}{2}\right) + C.$ **M1A1**

Apply $f(0) = 3$: $2 \arctan(0) + C = 3 \Rightarrow C = 3.$ **M1**

$f(x) = 2 \arctan\left(\frac{x}{2}\right) + 3$ **A1**

Note: GDC may be used to verify by checking $f'(x)$ numerically at a test point.

Q15 — FTC with function upper limit [MEDIUM]

By the Fundamental Theorem of Calculus, $\frac{d}{dx} \int_1^{h(x)} \frac{1}{1+t^3} dt = \frac{1}{1+[h(x)]^3} \cdot h'(x).$ **M1**

Here $h(x) = x^2$, so $h'(x) = 2x.$ **A1**

Apply chain rule factor explicitly: $g'(x) = \frac{1}{1+(x^2)^3} \cdot 2x$ **M1**

$g'(x) = \frac{2x}{1+x^6}$ **A1**

Note: GDC verification: compare numerical derivative of g at $x = 2$ with $\frac{4}{65} \approx 0.0615.$

Q16 — Displacement by parts, GDC check [MEDIUM]

Displacement $= \int_0^3 t e^{-t} dt.$ **M1**

By parts: $u = t, dv = e^{-t} dt \Rightarrow v = -e^{-t}:$

$= [-te^{-t}]_0^3 + \int_0^3 e^{-t} dt = -3e^{-3} + [-e^{-t}]_0^3 = -3e^{-3} - e^{-3} + 1$ **A1**

Exact value: $1 - 4e^{-3}.$ **A1**

GDC: $\text{fnInt}(t \cdot e^{-t}, t, 0, 3) = 0.800846 \dots \approx 0.801 \text{ m}.$ **A1**

Note: Accept exact form $1 - 4e^{-3}$ or decimal 0.801 m for the final A1. Both the exact working and numerical confirmation are required.

Q17 — Area between curve and line [MEDIUM]

Intersection: $6x - x^2 = 2x \Rightarrow x(4 - x) = 0 \Rightarrow x = 0, x = 4.$ **M1**

Area $= \int_0^4 [(6x - x^2) - 2x] dx = \int_0^4 (4x - x^2) dx$ **A1**

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3}$ **A1**

$$= \frac{32}{3}$$

A1

Note: GDC: $\text{fnInt}(4x - x^2, x, 0, 4) = 10.666... = \frac{32}{3}$. Award A1A1 for correct integrand and correct antiderivative; final A1 for exact answer.

Q18 — Properties of definite integrals [MEDIUM]

(a)

$$\int_0^3 [2f(x) - 3g(x)] dx = 2 \times 10 - 3 \times 4$$

$$= 20 - 12 = 8$$

M1

A1

(b)

$$\int_3^5 f(x) dx = \int_0^5 f(x) dx - \int_0^3 f(x) dx = 15 - 10$$

$$= 5$$

M1

A1

Q19 — Solve for upper limit using logarithm [MEDIUM]

$$\int_1^k \frac{1}{x} dx = \ln 5$$

M1

$$[\ln x]_1^k = \ln k - \ln 1 = \ln k$$

A1

So $\ln k = \ln 5$, giving $k = 5$.

M1A1

Note: GDC verification: $\text{fnInt}(1/x, x, 1, 5) = 1.6094... = \ln 5 \approx 1.6094$. Final answer must be exact integer.

Q20 — Integration by parts twice, x^2e^x [MEDIUM]

Let $u = x^2$, $dv = e^x dx$ (u simplifies on differentiation).

M1

$\Rightarrow du = 2x dx$, $v = e^x$.

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$

A1

Apply by parts again to $\int 2x e^x dx$: $u = 2x$, $dv = e^x dx$:

$$= 2x e^x - 2e^x$$

M1

$$\int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + C = e^x(x^2 - 2x + 2) + C$$

A1

Note: Deduct 1 mark if $+C$ is omitted. Accept unsimplified form $x^2 e^x - 2x e^x + 2e^x + C$.

Q21 — Partial fractions then integrate [HARD]

(a) Write $\frac{3x+1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$, so $3x+1 = A(x+3) + B(x-1)$. **M1**

Cover-up $x = 1$: $4 = 4A \Rightarrow A = 1$. Cover-up $x = -3$: $-8 = -4B \Rightarrow B = 2$. **A1**

(b) $\int \frac{1}{x-1} + \frac{2}{x+3} dx = \ln|x-1| + 2\ln|x+3| + C$ **M1**

$$= \ln|x-1| + \ln(x+3)^2 + C = \ln\left|\frac{(x-1)}{(x+3)^2}\right| + C$$

Written in demanded form: $a = 1$, $b = -2$, i.e. $\ln\left|\frac{(x-1)^1}{(x+3)^{-2}}\right| + C$. **A1**

Accept equivalent: $\ln|x-1| + 2\ln|x+3| + C$ fully simplified with modulus signs shown. **A1**

Note: award the final A1 only if $+C$ is present on the indefinite integral.

Q22 — Area between two curves, GDC [HARD]

(a) Using GDC graph-and-intersect with $y_1 = e^{x/2}$ and $y_2 = 3 - x^2$: unrounded value $x = 1.12749 \dots$ **M1**
 $x_P = 1.13$ (3 s.f.) **A1**

(b) Area = $\int_0^{1.1275} [(3 - x^2) - e^{x/2}] dx$ **M1**

Using GDC numerical integral (e.g. $\text{fnInt}(3 - x^2 - e^{x/2}, x, 0, 1.1275)$): unrounded = 0.97003 ... **A1**

Area = 0.970 cm² (3 s.f.) **A1**

Note: accept limits using the candidate's value from (a); follow-through on the intersection point applies for the final A1.

Q23 — Integration by parts twice, show that [HARD]

(a) Apply by parts twice with $u = x^2$, $dv = e^{3x} dx$ (choose $u = x^2$ since it simplifies on differentiation). **M1**

$$\int x^2 e^{3x} dx = \frac{x^2 e^{3x}}{3} - \int \frac{2xe^{3x}}{3} dx$$

Second by parts: $u = 2x/3$, $dv = e^{3x} dx$: $\int \frac{2xe^{3x}}{3} dx = \frac{2xe^{3x}}{9} - \int \frac{2e^{3x}}{9} dx = \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27}$ **M1**

Combining: $\frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + C = e^{3x} \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) + C$ **A1**

Since this equals the printed target, the result is established. **AG**

Note: AG; award no marks if only differentiation is used to verify.

$$\begin{aligned} \text{(b)} \int_0^1 x^2 e^{3x} dx &= \left[e^{3x} \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27} \right) \right]_0^1 = e^3 \left(\frac{1}{3} - \frac{2}{9} + \frac{2}{27} \right) - \frac{2}{27} & \text{M1} \\ &= \frac{11e^3}{27} - \frac{2}{27} & \text{A1} \end{aligned}$$

Q24 — Integration by parts with $\ln x$ [HARD]

(a) $\int \frac{\ln x}{x^2} dx$: choose $u = \ln x$, $dv = x^{-2} dx$ (since $\ln x$ simplifies on differentiation). **M1**

$du = \frac{1}{x} dx$, $v = -x^{-1}$, so $\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C$ **A1A1**

Note: award first A1 for $-\frac{\ln x}{x} + \int \frac{1}{x^2} dx$; second A1 for $-\frac{1}{x} + C$ with $+C$ present.

(b) $A = \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^{e^2} = \left(-\frac{2}{e^2} - \frac{1}{e^2} \right) - (0 - 1) = 1 - \frac{3}{e^2}$ **A1**

(c) GDC: $\text{fnInt}(\ln(x)/x^2, x, 1, e^2)$ gives 0.59399 ...; and $1 - 3e^{-2} = 0.59399 \dots$ **A1**

Q25 — FTC with function upper limit [HARD]

(a) By the Fundamental Theorem of Calculus with upper limit x^3 , apply the chain rule: **M1**

$$\begin{aligned} F'(x) &= \frac{\sqrt{x^3}}{1+x^3} \cdot \frac{d}{dx}(x^3) & \text{M1} \\ &= \frac{x^{3/2}}{1+x^3} \cdot 3x^2 = \frac{3x^{7/2}}{1+x^3} & \text{A1} \end{aligned}$$

(b) Solve $\frac{3x^{7/2}}{1+x^3} = 1$ using GDC: graph-and-intersect (or nSolve) with $y_1 = 3x^{3.5}/(1+x^3)$ and $y_2 = 1$; unrounded $x = 0.687156 \dots$ **M1**

$x = 0.687$ (3 s.f.) **A1**

Q26 — Substitution with trig integrand [HARD]

Let $u = 1 + \cos \theta$, so $\frac{du}{d\theta} = -\sin \theta$, i.e. $\sin \theta d\theta = -du$. **M1**

Substituting: $\int \frac{\sin \theta}{(1 + \cos \theta)^3} d\theta = \int \frac{-du}{u^3} = -\int u^{-3} du$ **A1**

$$= -\frac{u^{-2}}{-2} + C = \frac{1}{2u^2} + C$$
 M1

Returning to θ : $= \frac{1}{2(1 + \cos \theta)^2} + C$

A1A1

Note: first A1 for correct integration in u ; second A1 for correct back-substitution and $+C$.

Q27 — Substitution in context, limiting behaviour [HARD]

(a) Let $u = t^2 + 4$, so $\frac{du}{dt} = 2t$, i.e. $12t dt = 6 du$.

M1

$$\int \frac{12t}{(t^2 + 4)^2} dt = \int \frac{6}{u^2} du = 6 \int u^{-2} du = -\frac{6}{u} + C = \frac{-6}{t^2 + 4} + C$$

A1A1

Since this equals the printed target, the result is established.

AG

Note: award first A1 for correct substitution giving $\int 6u^{-2} du$; second A1 for completing to $-6/u + C$ and back-substituting.

(b) $V = \frac{-6}{t^2 + 4} + C$. At $t = 0$: $5 = -\frac{6}{4} + C \Rightarrow C = 5 + \frac{3}{2} = \frac{13}{2}$, so $V = \frac{13}{2} - \frac{6}{t^2 + 4}$.

A1

(c) As $t \rightarrow \infty$, $\frac{6}{t^2 + 4} \rightarrow 0$, so $V \rightarrow \frac{13}{2}$. Since $\frac{dV}{dt} = \frac{12t}{(t^2 + 4)^2} \geq 0$ for all $t \geq 0$, V is increasing, so $\frac{13}{2}$ is a supremum (approached from below) — this is the maximum volume the tank approaches but never exceeds.

R1

Q28 — By parts definite integral, GDC verify [HARD]

(a) $u = x \Rightarrow du = dx$; $\frac{dv}{dx} = \cos(2x) \Rightarrow v = \frac{\sin(2x)}{2}$ (choose $u = x$ since it simplifies on differentiation).

M1

$$\begin{aligned} I &= \left[\frac{x \sin(2x)}{2} \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin(2x)}{2} dx = 0 - \left[-\frac{\cos(2x)}{4} \right]_0^{\pi/2} \\ &= -\left(\frac{\cos \pi}{4} - \frac{\cos 0}{4} \right) = -\left(-\frac{1}{4} - \frac{1}{4} \right) = -\frac{1}{2} \end{aligned}$$

M1

A1

(b) GDC: $\text{fnInt}(x \cos(2x), x, 0, \pi/2)$ gives $-0.500000 \dots$, confirming $I = -\frac{1}{2}$.

A1

(c) Choosing $u = \cos(2x)$ gives $v = x^2/2$ after integration, which is more complex than the original integrand and does not simplify the integral — the resulting $\int x^2 \sin(2x) dx$ is harder than the original.

R1

Q29 — Arctan antiderivative, show that, GDC verify [HARD]

(a) Write $\frac{4}{x^2 + 4} = \frac{4}{4\left(\frac{x^2}{4} + 1\right)} = \frac{1}{\left(\frac{x}{2}\right)^2 + 1}$; use the standard form $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

with $a = 2$:

$$\int \frac{4}{x^2 + 4} dx = \frac{4}{2} \arctan\left(\frac{x}{2}\right) + C = 2 \arctan\left(\frac{x}{2}\right) + C$$

This equals the printed target.

M1

A1

AG

(b) Area = $\left[2 \arctan\left(\frac{x}{2}\right)\right]_0^4 = 2 \arctan(2) - 2 \arctan(0) = 2 \arctan(2)$

M1

Exact area = $2 \arctan(2)$

A1

(c) GDC: $\text{fnInt}(4/(x^2 + 4), x, 0, 4)$ gives 2.2143 ...; and $2 \arctan(2) = 2.214$ (3 s.f.)

A1

Q30 — By parts twice in kinematics context [HARD]

(a) Displacement = $\int_0^3 t^2 e^{-t} dt$. Apply by parts twice; choose $u = t^2$, $dv = e^{-t} dt$.

M1

$$\text{First by parts: } = [-t^2 e^{-t}]_0^3 + \int_0^3 2te^{-t} dt$$

$$\text{Second by parts (} u = 2t, dv = e^{-t} dt \text{): } \int_0^3 2te^{-t} dt = [-2te^{-t}]_0^3 + \int_0^3 2e^{-t} dt = -6e^{-3} + [-2e^{-t}]_0^3$$

$$= -6e^{-3} + (-2e^{-3} + 2) = -8e^{-3} + 2$$

$$\text{Displacement} = -9e^{-3} + (-8e^{-3} + 2) = 2 - 17e^{-3} \text{ m}$$

A1

(b) Since $v(t) = t^2 e^{-t} \geq 0$ for all $t \in [0, 3]$, the particle never changes direction, so total distance = displacement = $2 - 17e^{-3}$; GDC $\text{fnInt}(t^2 e^{-t}, t, 0, 3)$ gives 1.1537 ... ≈ 1.15 m (3 s.f.).

A1

(c) Since $v(t) = t^2 e^{-t} \geq 0$ for all $t \in [0, 3]$, the particle always moves in the positive direction and never reverses, so the total distance travelled equals the displacement.

R1