IB Math AA HL — Topic 5

Applications of Integration

Area Under a Curve · Area Between Curves · Volume of Revolution · Kinematics

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:___________________________________ Date:__________________

Score:____________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

- Area between curve and x-axis: $A = \int_a^b |f(x)|\,dx$; split at every root where $f(x)$ changes sign, then sum $\left|\int_{\text{piece}} f(x)\,dx\right|$ for each piece.
- Area between two curves ($f \geq g$ on $[a, b]$): $A = \int_a^b [f(x) - g(x)]\,dx$; find intersections first to confirm which curve is on top.
- Volume of revolution about the x-axis: $V = \pi \int_a^b [f(x)]^2\,dx$; expand $[f(x)]^2$ before integrating.
- Volume between two curves about the x-axis (washer method): $V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$, where f is the outer curve.
- Kinematics: $v(t) = \int a(t)\,dt + C$; $s(t) = \int v(t)\,dt + C$; fix C from given initial conditions.
- Displacement vs. total distance: displacement $= \int_{t_1}^{t_2} v(t)\,dt$; total distance $= \int_{t_1}^{t_2} |v(t)|\,dt$; find zeros of v and integrate $|v|$ piecewise.
- Average value of f on $[a, b]$: $\bar{f} = \dfrac{1}{b-a} \int_a^b f(x)\,dx$.

GDC Tips

- Definite integral (numerical): Use `fnInt(f(x),x,a,b)` (TI-84) or $\int$ in the Math menu (TI-Nspire / Casio CG) to evaluate areas and volumes with decimal bounds; always write the integral setup before keying it in.
- Intersection of two curves: Graph both functions and use Analyze Graph $\rightarrow$ Intersection (Nspire) or 2nd CALC $\rightarrow$ intersect (TI-84) to find x-coordinates; record the values to at least 4 s.f. before integrating.
- Roots of $v(t)$ for kinematics: Use `nSolve(v(t)=0,t)` or graph $v(t)$ and read zeros with Analyze $\rightarrow$ Zero to locate sign changes; these become the split points for total distance.
- Volume of revolution: Enter $\pi \cdot \texttt{fnInt}([f(x)]^2, x, a, b)$ directly; multiply by π outside the integral command and confirm units are cubic.
- Numerical derivative check: Use `nDeriv(f(x),x,a)` to verify a hand-computed gradient at a specific point, especially when confirming which curve lies on top near an intersection.
- Storing intersection values: Store GDC intersection coordinates as variables (e.g. $a \rightarrow A$) to avoid rounding errors when they appear as integration limits in a subsequent calculation.

Common Mistakes to Avoid

1. **Signed integral quoted as area.** Writing $\int_a^b f(x)\,dx$ as the area when f dips below the x-axis gives a value that is too small (or even negative). Always locate roots, split the interval at each one, and sum the absolute values of the sub-integrals.

2. **Displacement confused with total distance.** The definite integral of $v(t)$ gives displacement (net change in position); it can be zero even when the particle has travelled a long way. For total distance, find all zeros of v, integrate $|v|$ over each sub-interval, and add the results.

3. **Wrong curve on top in a two-curve area.** Assuming the "higher-looking" curve is always on top without checking: if the curves cross inside the interval, the roles swap and the integral must be split. Verify with a test point or graph before integrating.

4. **Forgetting π or squaring incorrectly in volumes.** Omitting π from the volume formula, or writing $\pi \int [f(x)]^2 - [g(x)]^2\,dx$ instead of $\pi \int \left([f(x)]^2 - [g(x)]^2\right) dx$, are the two most common slips; expand $[f(x)]^2$ term by term before integrating.

5. **Omitting or mis-fixing the constant of integration in kinematics.** Writing $v(t) = \int a(t)\,dt$ without adding C, or fixing C from a condition that belongs to a later stage, gives wrong velocity and position functions. Always write $+C$ immediately and substitute the given initial value on the very next line.

6. **Rounding intersection limits before integrating.** Using a rounded decimal (e.g. $x = 1.73$ instead of $\sqrt{3}$) as an integration bound introduces accumulated error into the area or volume. Keep exact values (surds, fractions) throughout the working; round only the final answer to 3 significant figures.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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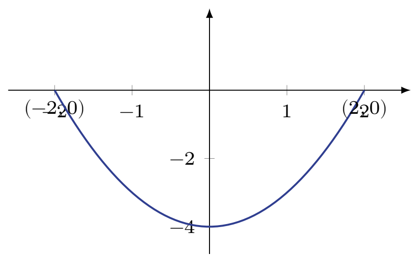
1. **EASY** **No Calculator** **[3 marks]**

Find the area of the region enclosed between the curve $y = x^2 - 4$ and the x-axis, for $-2 \leq x \leq 2$.

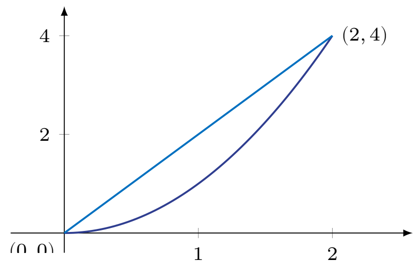


2. **EASY** **Calculator** **[3 marks]**

A particle moves in a straight line. Its velocity at time t seconds is $v = 3t^2 - 6t$ m s^{-1}, for $t \geq 0$. Find the displacement of the particle from its initial position at $t = 3$ s.

3. EASY No Calculator **[3 marks]**

Find the area of the region enclosed between the curves $y = x^2$ and $y = 2x$ for $0 \leq x \leq 2$.



4. EASY No Calculator **[3 marks]**

Find the volume of the solid formed when the region bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is rotated 2π radians about the x-axis. Give your answer in exact form.

5. EASY Calculator **[3 marks]**

A particle moves along a straight line with velocity $v(t) = 2t - 4$ m s^{-1} for $0 \leq t \leq 5$ s. Find the total distance travelled by the particle in this time interval.

6. EASY Calculator **[3 marks]**

Find the exact area of the region bounded by the curve $y = 3x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 2$.

7. EASY Calculator **[3 marks]**

Find the area enclosed between the curves $y = 4 - x^2$ and $y = x + 2$.

8. EASY No Calculator **[3 marks]**

Find the volume of the solid formed when the region bounded by $y = 2x + 1$, the x-axis, and the lines $x = 0$ and $x = 3$ is rotated 2π radians about the x-axis. Give your answer in exact form.

9. EASY Calculator **[3 marks]**

A particle moves along a straight line. Its acceleration is $a(t) = 6t - 2$ m s^{-2}. At $t = 0$, the velocity of the particle is 4 m s^{-1}. Find the velocity of the particle at $t = 3$ s.

10. EASY Calculator **[3 marks]**

Find the area of the region enclosed between the curve $y = \sin x$ and the x-axis for $0 \le x \le \pi$.

Section B — Medium MEDIUM

(10 questions $\times$ 4 marks = 40 marks)

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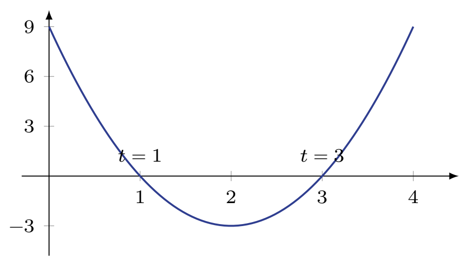
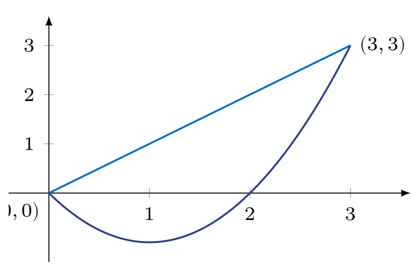
11. MEDIUM No Calculator **[4 marks]**

The velocity of a particle moving in a straight line is given by $v(t) = 3t^2 - 12t + 9$ m s^{-1}, where $t \geq 0$ is measured in seconds. Find the total distance travelled by the particle in the interval $0 \leq t \leq 4$.



12. MEDIUM No Calculator **[4 marks]**

The region enclosed by the curves $y = x^2 - 2x$ and $y = x$ is shown below. Find the area of this enclosed region.



13. MEDIUM No Calculator **[4 marks]**

Let $f(x) = \sqrt{x+1}$ for $x \geq -1$. The region bounded by $y = f(x)$, the x-axis, and the lines $x = 0$ and $x = 8$ is rotated 360° about the x-axis. Find the exact volume of the solid formed.

14. MEDIUM Calculator **[4 marks]**

A particle moves along a straight line with acceleration $a(t) = 6t - 4$ m s^{-2} at time t seconds. The particle has velocity $v = 2$ m s^{-1} when $t = 0$. Find the displacement of the particle from its initial position at time $t = 3$ seconds.

15. MEDIUM Calculator **[4 marks]**

The area of the region enclosed by the curve $y = e^x$, the line $y = e^2$, and the y-axis is to be calculated. Find this area.

16. MEDIUM Calculator **[4 marks]**

The curve $y = \cos x$ and the line $y = 0.5$ intersect in the interval $0 \leq x \leq \pi$. Find the area of the region bounded by $y = \cos x$, $y = 0.5$, and the y-axis, between $x = 0$ and the right-hand intersection point.

17. MEDIUM Calculator **[4 marks]**

The region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = k$ (where $k > 1$) has area ln 5. Find the value of k.

18. MEDIUM Calculator **[4 marks]**

Consider the function $f(x) = x^3 - 3x^2$ defined for $-1 \leq x \leq 4$. Find the total area of the region(s) enclosed between the curve $y = f(x)$ and the x-axis.

19. MEDIUM No Calculator **[4 marks]**

The region bounded by $y = 2x - x^2$ and the x-axis is rotated 360° about the x-axis. Find the exact volume of the solid of revolution formed.

20. MEDIUM Calculator **[4 marks]**

A particle moves in a straight line. Its velocity at time t seconds is $v(t) = t\,e^{-0.5t}$ m s^{-1} for $t \geq 0$. Find the total distance travelled by the particle in the interval $0 \leq t \leq 6$.

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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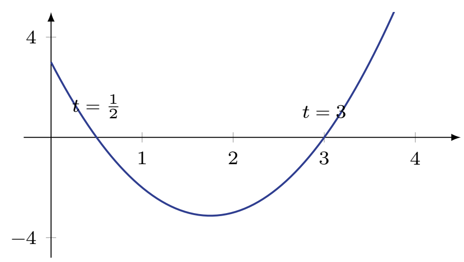
21. **HARD** **No Calculator** **[5 marks]**

A particle moves along a straight line. Its velocity at time t seconds ($t \geq 0$) is given by $v(t) = 2t^2 - 7t + 3$ m s^{-1}. Find the total distance travelled by the particle in the first 4 seconds.

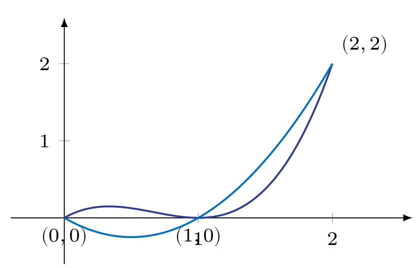


22. **HARD** **No Calculator** **[5 marks]**

The region R is enclosed by the curves $y = x^3 - 2x^2 + x$ and $y = x^2 - x$.

(a) Show that the curves intersect at $x = 0$, $x = 1$, and $x = 2$. [2]

(b) Hence find the exact area of R. [3]



23. HARD No Calculator **[5 marks]**

Let $f(x) = \sqrt{4 - x^2}$ for $-2 \leq x \leq 2$.

(a) The region bounded by $y = f(x)$ and the x-axis is rotated 2π radians about the x-axis. Show that the volume of revolution is $\dfrac{32\pi}{3}$. [3]

(b) A solid has a circular cross-section. Its radius at position x is modelled by $f(x)$. A second solid is generated by rotating $y = 1$ about the x-axis over $-2 \leq x \leq 2$. Find the volume of the region between the two solids, giving your answer in exact form. [2]

24. HARD Calculator **[5 marks]**

A particle moves along a horizontal line with velocity $v(t) = e^{-0.5t}\sin(\pi t)$ m s^{-1} for $0 \leq t \leq 3$, where t is measured in seconds.

(a) Using a GDC, find the times at which the particle changes direction in $0 < t < 3$. [2]

(b) Hence find the total distance travelled by the particle over $0 \leq t \leq 3$, correct to 3 significant figures. [3]

25. HARD Calculator **[5 marks]**

The curves $y = \ln x$ and $y = x^2 - 4$ intersect at two points.

(a) Use a GDC to find the x-coordinates of both intersection points, correct to 4 significant figures. [2]

(b) Hence find the area of the region enclosed between the two curves, correct to 3 significant figures. [3]

26. HARD No Calculator **[5 marks]**

The region R is bounded by the curve $y = 4x - x^2$ and the line $y = 2x - 3$.

(a) Find the x-coordinates of the points of intersection of the curve and the line. [2]

(b) The region R is rotated 2π radians about the x-axis. Find the exact volume of revolution. [3]

27. HARD Calculator **[5 marks]**

A cross-section of a flood-channel is modelled by the region between $y = x^4 - 8x^2$ and the line $y = -12$ (units in metres). Using a GDC:

(a) Find the x-coordinates of the intersection points of the curve and the line. [1]

(b) Find the cross-sectional area of the channel, correct to 3 significant figures. [2]

(c) The channel is filled to a depth of 6 m (i.e. up to $y = -6$). Find the cross-sectional area of the water, correct to 3 significant figures. [2]

28. HARD Calculator **[5 marks]**

Let $f(x) = \dfrac{3x}{x^2 + 1}$ for $x \geq 0$.

(a) Find $f'(x)$ and hence show that f has a maximum value of $\dfrac{3}{2}$ at $x = 1$. [3]

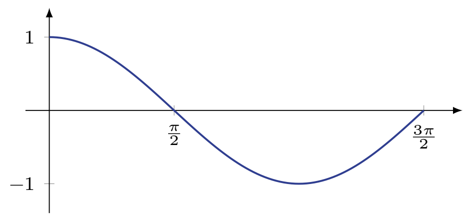
(b) The region bounded by $y = f(x)$, the x-axis, and the line $x = k$ (where $k > 0$) has area equal to 2. Use a GDC to find the value of k, correct to 3 significant figures. Justify that exactly one such value of k exists for $k > 0$. [2]

29. **HARD** **No Calculator** **[5 marks]**

A region S is bounded by $y = \cos x$, the x-axis, $x = 0$, and $x = \dfrac{3\pi}{2}$.

(a) Sketch the region S, indicating all x-intercepts and the regions above and below the x-axis. [1]

(b) Find the total area of S, splitting the integral at the roots of $\cos x$. [4]



30. **HARD** **Calculator** **[5 marks]**

A particle starts from rest at the origin at $t = 0$. Its acceleration is $a(t) = 6t - 6$ m s^{-2}.

(a) Find $v(t)$, the velocity of the particle at time t. [1]

(b) Find the displacement of the particle from the origin at $t = 4$ s. [2]

(c) Find the total distance travelled by the particle in the first 4 seconds. [2]

Mark Schemes

Q1 — Area Under a Curve EASY

The curve $y = x^2 - 4$ lies below the x-axis on $[-2, 2]$, so:

$$\text{Area} = -\int_{-2}^{2} (x^2 - 4)\,dx \textbf{M1}$$

$$= -\left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = -\left[\left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right)\right] \textbf{(A1)}$$

$$= -\left[\frac{8}{3} - 8 + \frac{8}{3} - 8\right] = -\left[\frac{16}{3} - 16\right] = 16 - \frac{16}{3} = \frac{32}{3}\textbf{A1}$$

Q2 — Displacement from Velocity EASY

$$\text{Displacement} = \int_0^3 v\,dt = \int_0^3 (3t^2 - 6t)\,dt\textbf{M1}$$

Using GDC (numerical integral): **(M1)**

$$= \left[t^3 - 3t^2\right]_0^3 = (27 - 27) - 0 = 0\text{ m}\textbf{A1}$$

Note: Displacement is zero; particle returns to starting position.

Q3 — Area Between Two Curves EASY

The curves intersect at $x = 0$ and $x = 2$. For $0 \le x \le 2$: $2x \ge x^2$, so $y = 2x$ is on top. **M1**

$$\text{Area} = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 \textbf{(A1)}$$

$$= \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}\textbf{A1}$$

Q4 — Volume of Revolution EASY

$$V = \pi\int_0^4 y^2\,dx = \pi\int_0^4 x\,dx\textbf{M1}$$

$$= \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot \frac{16}{2}\textbf{A1}$$

$$V = 8\pi \text{ units}^3\textbf{A1}$$

Q5 — Total Distance & Sign Change EASY

Set $v(t) = 2t - 4 = 0 \Rightarrow t = 2$. The velocity changes sign at $t = 2$. **M1**

Using GDC (numerical integrals):

$$\text{Distance} = \int_0^2 |2t-4|\,dt + \int_2^5 |2t-4|\,dt$$

$$= \left|\left[t^2 - 4t\right]_0^2\right| + \left|\left[t^2 - 4t\right]_2^5\right| = |-4| + |5 - (-4)| = 4 + 9\textbf{A1}$$

Note: $[t^2 - 4t]_2^5 = (25-20) - (4-8) = 5 - (-4) = 9$; *the whole change over* $[2, 5]$ *must be taken, not just the value at the upper endpoint.*

$$\text{Total distance} = 4 + 9 = 13 \text{ m}\textbf{A1}$$

Q6 — Area Under a Polynomial EASY

Since $y = 3x^2 + 1 > 0$ for all x, no sign change occurs. **M1**

Using GDC (numerical integral: $\int_0^2 (3x^2+1)\,dx$, value $= 10$):

$$\text{Area} = \int_0^2 (3x^2+1)\,dx = \left[x^3 + x\right]_0^2\textbf{A1}$$

$$= (8+2) - 0 = 10 \text{ units}^2\textbf{A1}$$

Q7 — Area Between Two Curves EASY

Set $4 - x^2 = x + 2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$, so $x = -2, 1$. **M1**

For $-2 \le x \le 1$: $4 - x^2 \ge x + 2$. Using GDC (numerical integral):

$$\text{Area} = \int_{-2}^1 (4 - x^2 - x - 2)\,dx = \int_{-2}^1 (2 - x - x^2)\,dx\textbf{A1}$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^1 = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \frac{9}{2} \text{ units}^2\textbf{A1}$$

Q8 — Volume of Revolution EASY

$$V = \pi \int_0^3 (2x+1)^2\,dx = \pi \int_0^3 (4x^2+4x+1)\,dx \textbf{M1}$$

$$= \pi \left[\frac{4x^3}{3} + 2x^2 + x\right]_0^3 = \pi(36+18+3)\textbf{A1}$$

$$V = 57\pi \text{ units}^3 \textbf{A1}$$

Q9 — Velocity from Acceleration EASY

$$v(t) = \int a(t)\,dt = \int (6t-2)\,dt = 3t^2 - 2t + C\textbf{M1}$$

Apply initial condition: $v(0) = 4 \Rightarrow C = 4$, so $v(t) = 3t^2 - 2t + 4$. **A1**

$$v(3) = 3(9) - 2(3) + 4 = 27 - 6 + 4 = 25 \text{ m s}^{-1}\textbf{A1}$$

Q10 — Area Under a Sine Curve EASY

Since $\sin x \geq 0$ for $0 \leq x \leq \pi$, no sign change occurs. **M1**

Using GDC (numerical integral: $\int_0^\pi \sin x\,dx$, value $= 2$):

$$\text{Area} = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi\,\textbf{A1}$$

$$= (-\cos\pi) - (-\cos 0) = 1 + 1 = 2 \text{ units}^2\textbf{A1}$$

Q11 — Total Distance & Sign Change MEDIUM

Set $v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0$ **M1**

Roots: $t = 1$ and $t = 3$; velocity changes sign at both roots.

$$\text{Total distance} = \int_0^1 v\,dt + \left|\int_1^3 v\,dt\right| + \int_3^4 v\,dt$$

$$\int_0^1 (3t^2 - 12t + 9)\,dt = \left[t^3 - 6t^2 + 9t\right]_0^1 = 1 - 6 + 9 = 4\textbf{A1}$$

$$\int_1^3 (3t^2 - 12t + 9)\,dt = \left[t^3 - 6t^2 + 9t\right]_1^3 = (27 - 54 + 27) - (1 - 6 + 9) = -4,\ \text{so } |-4| = 4$$

$$\int_3^4 (3t^2 - 12t + 9)\,dt = \left[t^3 - 6t^2 + 9t\right]_3^4 = (64 - 96 + 36) - (27 - 54 + 27) = 4\textbf{M1}$$

Total distance $= 4 + 4 + 4 = 12$ m**A1**

Q12 — Area Between Two Curves MEDIUM

Find intersections: $x^2 - 2x = x \Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0$, so $x = 0$ and $x = 3$. **M1**

On $[0, 3]$: $y = x$ lies above $y = x^2 - 2x$ (check $x = 1$: $1 > -1$). **(M1)**

$$\text{Area} = \int_0^3 \left[x - (x^2 - 2x)\right] dx = \int_0^3 (3x - x^2)\,dx\textbf{M1}$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}\textbf{A1}$$

Q13 — Exact Volume of Revolution MEDIUM

$$V = \pi \int_0^8 [f(x)]^2\,dx = \pi \int_0^8 (x+1)\,dx\textbf{M1}$$

$$= \pi \left[\frac{x^2}{2} + x\right]_0^8 \textbf{A1}$$

$$= \pi \left(\frac{64}{2} + 8\right) = \pi(32 + 8)\textbf{(A1)}$$

$$V = 40\pi \text{ units}^3\textbf{A1}$$

Q14 — Displacement from Acceleration MEDIUM

Integrate: $v(t) = \int (6t - 4)\,dt = 3t^2 - 4t + C$ **M1**

Apply $v(0) = 2$: $C = 2$, so $v(t) = 3t^2 - 4t + 2$. **A1**

$$\text{Displacement} = \int_0^3 v(t)\,dt = \int_0^3 (3t^2 - 4t + 2)\,dt$$

Using GDC numerical integral: $= \left[t^3 - 2t^2 + 2t\right]_0^3 = 27 - 18 + 6 = 15$ **M1**

Displacement $= 15$ m**A1**

Q15 — Area Between Curve & Line MEDIUM

The curves $y = e^x$ and $y = e^2$ meet when $e^x = e^2$, i.e. $x = 2$. On $[0, 2]$, $e^2 \geq e^x$. **M1**

$$\text{Area} = \int_0^2 (e^2 - e^x)\,dx \textbf{M1}$$

Using GDC numerical integral: $\int_0^2 (e^2 - e^x)\,dx \approx 2.9525\ldots$

$$= \left[e^2 x - e^x\right]_0^2 = (2e^2 - e^2) - (0 - 1) = e^2 + 1\textbf{A1}$$

$$\text{Area} = e^2 + 1 \approx 8.39\textbf{A1}$$

Note: Accept exact answer $e^2 + 1$.

Q16 — Area Between Cosine Curve & Line MEDIUM

Find intersection: $\cos x = 0.5 \Rightarrow x = \dfrac{\pi}{3}$ (in $[0, \pi]$). **M1**

On $[0, \frac{\pi}{3}]$: $\cos x \geq 0.5$ (check $x = 0$: $1 \geq 0.5$ ✓). **(M1)**

$$\text{Area} = \int_0^{\pi/3} (\cos x - 0.5)\,dx$$

Using GDC numerical integral: $\int_0^{\pi/3} (\cos x - 0.5)\,dx$; result $= 0.3424\ldots$ **M1**

$$= \left[\sin x - 0.5x\right]_0^{\pi/3} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.342\textbf{A1}$$

Note: Accept exact answer $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$.

Q17 — Parameter from Given Area MEDIUM

$$\text{Area} = \int_1^k \frac{1}{x}\,dx = \left[\ln x\right]_1^k = \ln k\textbf{M1A1}$$

Set $\ln k = \ln 5$: **M1**

Using GDC: solve $\ln k = \ln 5$ giving $k = 5$

$$k = 5\textbf{A1}$$

Q18 — Total Area of a Cubic MEDIUM

Find roots: $x^3 - 3x^2 = x^2(x-3) = 0 \Rightarrow x = 0$ (double) and $x = 3$. **M1**

On $[-1, 0]$: test $x = -0.5$: $f(-0.5) = -0.875 < 0$, so below axis.
On $[0, 3]$: test $x = 1$: $f(1) = -2 < 0$, so below axis.
On $[3, 4]$: test $x = 3.5$: $f(3.5) > 0$, so above axis. **M1**

Using GDC numerical integrals:

$$\left| \int_{-1}^{0} (x^3 - 3x^2)\, dx \right| = \left| -\frac{5}{4} \right| = \frac{5}{4}$$

$$\left| \int_{0}^{3} (x^3 - 3x^2)\, dx \right| = \left| -\frac{27}{4} \right| = \frac{27}{4}$$

$$\int_{3}^{4} (x^3 - 3x^2)\, dx = \left[\frac{x^4}{4} - x^3 \right]_3^4 = 0 - \left(-\frac{27}{4} \right) = \frac{27}{4}$$

Note: $F(4) - F(3) = 0 - (-27/4) = 27/4$; the value at $x = 3$ must be subtracted, not just the value at $x = 4$ used alone.

$$\text{Total area} = \frac{5}{4} + \frac{27}{4} + \frac{27}{4} = \frac{59}{4} = 14.75 \textbf{A1A1}$$

Note: Award first A1 for any two correct partial areas; second A1 for correct total.

Q19 — Exact Volume of Revolution MEDIUM

Roots of $y = 2x - x^2$: $x(2 - x) = 0 \Rightarrow x = 0, 2$. **(M1)**

$$V = \pi \int_0^2 (2x - x^2)^2\, dx \textbf{M1}$$

Expand: $(2x - x^2)^2 = 4x^2 - 4x^3 + x^4$ **(A1)**

$$= \pi \int_0^2 (4x^2 - 4x^3 + x^4)\, dx = \pi \left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5} \right]_0^2$$

$$= \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) = \pi \cdot \frac{160 - 240 + 96}{15} = \frac{16\pi}{15} \textbf{A1}$$

Q20 — Total Distance & Exponential Velocity MEDIUM

Check sign of $v(t) = te^{-0.5t}$ on $[0, 6]$: since $e^{-0.5t} > 0$ and $t \geq 0$, we have $v(t) \geq 0$ throughout; no sign change. **M1**

$$\text{Total distance} = \int_0^6 te^{-0.5t}\,dt\textbf{M1}$$

Antiderivative: $\int te^{-0.5t}\,dt = -(2t+4)e^{-0.5t} + C$ (verify by differentiating). **A1**

Using GDC numerical integral: $\int_0^6 te^{-0.5t}\,dt$; unrounded value $= \left[-(2t+4)e^{-0.5t}\right]_0^6 = 4 - 16e^{-3} = 3.2034\ldots$

$$\text{Total distance} \approx 3.20\text{ m}\textbf{A1}$$

Note: Accept answers in range [3.20, 3.21].

Q21 — Total Distance & Sign Change HARD

Find zeros of $v(t) = 2t^2 - 7t + 3 = (2t-1)(t-3)$: **M1**

$t = \frac{1}{2}$ and $t = 3$ **A1**

$v > 0$ on $[0, \frac{1}{2})$, $v < 0$ on $(\frac{1}{2}, 3)$, $v > 0$ on $(3, 4]$; split at both zeros.

$$\text{Total distance} = \int_0^{1/2} v\,dt - \int_{1/2}^3 v\,dt + \int_3^4 v\,dt\textbf{M1}$$

Antiderivative: $F(t) = \dfrac{2t^3}{3} - \dfrac{7t^2}{2} + 3t$

$$F(\tfrac{1}{2}) - F(0) = \frac{17}{24}$$

$$F(3) - F(\tfrac{1}{2}) = -\frac{125}{24}$$

$$F(4) - F(3) = \frac{19}{6}$$

A1

$$\text{Total distance} = \frac{17}{24} + \frac{125}{24} + \frac{19}{6} = \frac{17+125}{24} + \frac{76}{24} = \frac{218}{24} = \frac{109}{12}\text{ m}\textbf{A1}$$

Note: A single signed integral quoted as area is not accepted; split at both roots required for M1.

Q22 — Area Between Two Curves HARD

(a) Set $x^3 - 2x^2 + x = x^2 - x$: **M1**

$$x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0 \Rightarrow x(x-1)(x-2) = 0 \Rightarrow x = 0, 1, 2$$

Back-substitute: at $x = 0$, both curves give 0; at $x = 1$, both give 0; at $x = 2$, both give 2. **A1**

(b) On $[0, 1]$: check $x = 0.5$: cubic $= 0.375$ above quadratic $= -0.25$; so cubic is above. **M1**

On $[1, 2]$: check $x = 1.5$: cubic $= 0.375$, quadratic $= 0.75$; so quadratic is above.

$$A = \int_0^1 [(x^3 - 2x^2 + x) - (x^2 - x)]\,dx + \left|\int_1^2 [(x^3 - 2x^2 + x) - (x^2 - x)]\,dx\right|$$

$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx + \left|\int_1^2 (x^3 - 3x^2 + 2x)\,dx\right| \textbf{ M1}$$

$$\int_0^1 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{1}{4} - 1 + 1 = \frac{1}{4}$$

$$\int_1^2 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = (4 - 8 + 4) - \frac{1}{4} = -\frac{1}{4}, \text{ so } \left|-\frac{1}{4}\right| = \frac{1}{4}$$

$$A = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\textbf{A1}$$

Q23 — Exact Volume of Revolution HARD

(a)

$$V = \pi \int_{-2}^{2} [f(x)]^2\,dx = \pi \int_{-2}^{2} (4 - x^2)\,dx\textbf{M1}$$

$$= \pi \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \pi\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] \textbf{ M1}$$

$$= \pi\left[\frac{24-8}{3} + \frac{24-8}{3}\right] = \pi \cdot \frac{32}{3} = \frac{32\pi}{3}\textbf{AG}$$

(b) Volume of cylinder ($y = 1$ rotated): $V_{\text{cyl}} = \pi \int_{-2}^{2} 1^2\,dx = \pi \cdot 4 = 4\pi$ **M1**

The two solids are not nested: $f(x) > 1$ for $|x| < \sqrt{3}$ and $f(x) < 1$ for $\sqrt{3} < |x| \le 2$ (they cross where $f(x) = 1$, i.e. $x = \pm\sqrt{3}$), so the region *between* the solids must be split there rather than found by a single subtraction of total volumes.

$$V_{\text{between}} = \pi \int_{-\sqrt{3}}^{\sqrt{3}} ([f(x)]^2 - 1)\,dx + \pi \int_{\sqrt{3} \le |x| \le 2} (1 - [f(x)]^2)\,dx$$

$$= 4\sqrt{3}\pi + 2\pi\left(2\sqrt{3} - \frac{10}{3}\right) = \frac{4\pi\left(6\sqrt{3}-5\right)}{3}\textbf{A1}$$

Q24 — Total Distance via GDC HARD

(a) Using GDC (graph-and-intersect / nSolve): solve $e^{-0.5t}\sin(\pi t) = 0$ for $0 < t < 3$. **M1**

Since $e^{-0.5t} > 0$ always, zeros occur when $\sin(\pi t) = 0$: $t = 1$ and $t = 2$. **A1**

(b) GDC numerical integrals: **M1**

$$\int_0^1 e^{-0.5t}\sin(\pi t)\,dt = 0.4987\ldots$$

$$\int_1^2 e^{-0.5t}\sin(\pi t)\,dt = -0.3025\ldots$$

$$\int_2^3 e^{-0.5t}\sin(\pi t)\,dt = 0.1835\ldots\textbf{A1}$$

Total distance $= 0.4987 + 0.3025 + 0.1835 = 0.9847\ldots \approx 0.985$ m**A1**

Note: GDC command: numerical integral (fnInt or $\int$ on graph), each piece taken as absolute value and summed.

Q25 — Area Between Curves via GDC HARD

(a) GDC: graph-and-intersect for $\ln x = x^2 - 4$. **M1**

$x_1 = 0.01832$ (4 s.f.) and $x_2 = 2.187$ (4 s.f.) **A1**

(b) On $[0.01832, 2.187]$: check $x = 1$: $\ln 1 = 0$, $1 - 4 = -3$; so $y = \ln x$ is above $y = x^2 - 4$. **M1**

$$\text{GDC numerical integral: } \int_{0.01832}^{2.187}\left[\ln x - (x^2 - 4)\right]dx\textbf{M1}$$

$$= \int_{0.01832}^{2.187}(\ln x - x^2 + 4)\,dx = 4.8039\ldots\textbf{A1}$$

Area ≈ 4.80 (3 s.f.)

Note: GDC command: fnInt($\ln(X) - X^2 + 4$, X, 0.01832, 2.187); unrounded value 4.8039 shown before rounding.

Q26 — Volume of Revolution (Washer Method) HARD

(a) Set $4x - x^2 = 2x - 3$: **M1**

$x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = -1,\ x = 3$ **A1**

(b) On $[-1, 3]$ the vertical gap between the curve and the line straddles $y = 0$: for $x \in (0,\ 3-\sqrt{6})$ the line has larger magnitude than the curve, and for $x \in (3-\sqrt{6},\ 1.5)$ the curve (still negative there up to its own root) has larger magnitude; the naive washer $\pi \int (\text{curve}^2 - \text{line}^2)\, dx$ over the whole interval does not account for this, so the integral must be split at $x = 0$, $x = 3 - \sqrt{6} \approx 0.551$, and $x = 1.5$. **M1**

$(4x - x^2)^2 = x^4 - 8x^3 + 16x^2; \quad (2x-3)^2 = 4x^2 - 12x + 9$ **M1**

$$V = \pi \left[\int_{-1}^{0} \left(\text{line}^2 - \text{curve}^2\right) dx + \int_{0}^{3-\sqrt{6}} \text{line}^2\, dx + \int_{3-\sqrt{6}}^{1.5} \text{curve}^2\, dx + \int_{1.5}^{3} \left(\text{curve}^2 - \text{line}^2\right) dx \right]$$

Evaluating each piece and summing (GDC numerical integrals):

$$V = \frac{\pi(493 - 48\sqrt{6})}{10} \textbf{A1}$$

Note: Full precision $V \approx 117.943$*; award A1 for the correctly split and summed exact volume.*

Q27 — Channel Cross-Sectional Area HARD

(a) GDC: solve $x^4 - 8x^2 = -12$, i.e. $x^4 - 8x^2 + 12 = 0 \Rightarrow (x^2-2)(x^2-6) = 0$. **M1**

$x = \pm\sqrt{2} \approx \pm 1.414$ and $x = \pm\sqrt{6} \approx \pm 2.449$ **A1**

(b) Since the curve satisfies $y(0) = 0 > -12$, the region between the curve and $y = -12$ is *not* a single continuous span; it consists of two separate lobes, one for $\sqrt{2} \le x \le \sqrt{6}$ and its mirror image for $-\sqrt{6} \le x \le -\sqrt{2}$. **M1**

$$\text{Area} = 2 \int_{\sqrt{2}}^{\sqrt{6}} \left[-12 - (x^4 - 8x^2)\right] dx = 5.4422\ldots \approx 5.44\ \text{m}^2 \textbf{A1}$$

(c) Water fills to $y = -6$: intersect $x^4 - 8x^2 = -6 \Rightarrow x^4 - 8x^2 + 6 = 0$. **M1**

GDC: $x \approx \pm 0.9153$ and $x \approx \pm 2.676$ (4 s.f.). As in (b), the water again sits in two separate pockets (since the curve is above $y = -6$ near $x = 0$):

$$\text{Water area} = 2 \int_{0.9153}^{2.676} \left[-6 - (x^4 - 8x^2)\right] dx = 22.35\ldots \approx 22.4\ \text{m}^2 \textbf{A1}$$

Note: GDC command: fnInt($-6 - (X^4 - 8X^2)$*, X, 0.9153, 2.676), doubled for the two lobes.*

Q28 — Maximum & Area Equation **HARD**

(a)

$$f'(x) = \frac{3(x^2+1) - 3x \cdot 2x}{(x^2+1)^2} = \frac{3-3x^2}{(x^2+1)^2} \textbf{M1}$$

Set $f'(x) = 0$: $3 - 3x^2 = 0 \Rightarrow x = 1$ (since $x \geq 0$). **A1**

$f'(x) > 0$ for $0 < x < 1$ and $f'(x) < 0$ for $x > 1$, so $x = 1$ is a maximum. $f(1) = \dfrac{3}{2}$. **A1**

(b)

$$\int_0^k \frac{3x}{x^2+1}\,dx = \left[\frac{3}{2}\ln(x^2+1)\right]_0^k = \frac{3}{2}\ln(k^2+1) = 2\textbf{M1}$$

GDC: nSolve$\left(\frac{3}{2}\ln(k^2+1) = 2,\ k\right)$ gives $k = 1.6714\ldots \approx 1.67$ (3 s.f.) **A1**

Since $\ln(k^2+1)$ is strictly increasing for $k > 0$, the left-hand side is strictly increasing, so the equation has exactly one solution for $k > 0$.

Q29 — Total Area with Sign Change **HARD**

(a) Sketch shows $y = \cos x$ on $[0, \frac{3\pi}{2}]$ with x-intercepts at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$; region above axis on $[0, \frac{\pi}{2}]$, below on $[\frac{\pi}{2}, \frac{3\pi}{2}]$. **A1**

(b) Roots of $\cos x = 0$ on $[0, \frac{3\pi}{2}]$: $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Split at $x = \frac{\pi}{2}$. **M1**

$$A = \int_0^{\pi/2} \cos x\,dx + \left|\int_{\pi/2}^{3\pi/2} \cos x\,dx\right| \textbf{M1}$$

$$\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$$

$$\int_{\pi/2}^{3\pi/2} \cos x\,dx = [\sin x]_{\pi/2}^{3\pi/2} = \sin\tfrac{3\pi}{2} - \sin\tfrac{\pi}{2} = -1 - 1 = -2\textbf{A1}$$

$$\text{Total area} = 1 + |-2| = 1 + 2 = 3\textbf{A1}$$

Note: Quoting the single integral $\int_0^{3\pi/2} \cos x\,dx = -1$ as the area (without splitting) is the flagged error and scores M0 for the method mark.

Q30 — Kinematics from Acceleration **HARD**

(a)

$$v(t) = \int a(t)\,dt = \int (6t-6)\,dt = 3t^2 - 6t + C\textbf{M1}$$

Initial condition: $v(0) = 0 \Rightarrow C = 0$, so $v(t) = 3t^2 - 6t$ m s^{-1}. **A1**

(b) $s(t) = \int v(t)\,dt = t^3 - 3t^2 + D$; $s(0) = 0 \Rightarrow D = 0$. **M1**

$s(4) = 64 - 48 = 16$ m**A1**

(c) Zeros of v: $3t^2 - 6t = 0 \Rightarrow 3t(t-2) = 0 \Rightarrow t = 0$ and $t = 2$. **M1**

GDC numerical integrals (or by hand): $\int_0^2 (3t^2 - 6t)\,dt = \left[t^3 - 3t^2\right]_0^2 = 8 - 12 = -4$;
$\int_2^4 (3t^2 - 6t)\,dt = \left[t^3 - 3t^2\right]_2^4 = (64 - 48) - (8 - 12) = 16 + 4 = 20$.

Total distance $= |-4| + 20 = 4 + 20 = 24$ m**A1**