IB Math AA HL — Topic 5

Applications of Integration

Area Under a Curve · Area Between Curves · Volume of Revolution · Kinematics

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Scan me for help

Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Scan me for help

Key Formulae

- **Area between curve and x-axis:** $A = \int_a^b |f(x)|\,dx$; split at every root where $f(x)$ changes sign, then sum $\left|\int_{\text{piece}} f(x)\,dx\right|$ for each piece.
- **Area between two curves** ($f \geq g$ on $[a,b]$): $A = \int_a^b [f(x) - g(x)]\,dx$; find intersections first to confirm which curve is on top.
- **Volume of revolution about the x-axis:** $V = \pi \int_a^b [f(x)]^2\,dx$; expand $[f(x)]^2$ before integrating.
- **Volume between two curves about the x-axis (washer method):** $V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$, where f is the outer curve.
- **Kinematics:** $v(t) = \int a(t)\,dt + C$; $\quad s(t) = \int v(t)\,dt + C$; fix C from given initial conditions.
- **Displacement vs. total distance:** displacement $= \int_{t_1}^{t_2} v(t)\,dt$; $\quad$ total distance $= \int_{t_1}^{t_2} |v(t)|\,dt$; find zeros of v and integrate $|v|$ piecewise.
- **Average value of f on $[a,b]$:** $\bar{f} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$.

GDC Tips

- **Definite integral (numerical):** Use `fnInt(f(x),x,a,b)` (TI-84) or $\int_a^b$ in the Math menu (TI-Nspire / Casio CG) to evaluate areas and volumes with decimal bounds; always write the integral setup before keying it in.
- **Intersection of two curves:** Graph both functions and use `Analyze Graph` $\rightarrow$ `Intersection` (Nspire) or `2nd CALC` $\rightarrow$ `intersect` (TI-84) to find x-coordinates; record the values to at least 4 s.f. before integrating.
- **Roots of $v(t)$ for kinematics:** Use `nSolve(v(t)=0,t)` or graph $v(t)$ and read zeros with `Analyze` $\rightarrow$ `Zero` to locate sign changes; these become the split points for total distance.
- **Volume of revolution:** Enter $\pi \cdot \texttt{fnInt}([f(x)]^2, x, a, b)$ directly; multiply by π outside the integral command and confirm units are cubic.
- **Numerical derivative check:** Use `nDeriv(f(x),x,a)` to verify a hand-computed gradient at a specific point, especially when confirming which curve lies on top near an intersection.
- **Storing intersection values:** Store GDC intersection coordinates as variables (e.g. $a \rightarrow$ `A`) to avoid rounding errors when they appear as integration limits in a subsequent calculation.

Common Mistakes to Avoid

1. **Signed integral quoted as area.** Writing $\int_a^b f(x)\,dx$ as the area when f dips below the x-axis gives a value that is too small (or even negative). Always locate roots, split the interval at each one, and sum the *absolute values* of the sub-integrals.

2. **Displacement confused with total distance.** The definite integral of $v(t)$ gives *displacement* (net change in position); it can be zero even when the particle has travelled a long way. For *total distance*, find all zeros of v, integrate $|v|$ over each sub-interval, and add the results.

3. **Wrong curve on top in a two-curve area.** Assuming the "higher-looking" curve is always on top without checking: if the curves cross inside the interval, the roles swap and the integral must be split. Verify with a test point or graph before integrating.

4. **Forgetting π or squaring incorrectly in volumes.** Omitting π from the volume formula, or writing $\pi\int[f(x)]^2-[g(x)]^2\,dx$ instead of $\pi\int([f(x)]^2-[g(x)]^2)\,dx$, are the two most common slips; expand $[f(x)]^2$ term by term before integrating.

5. **Omitting or mis-fixing the constant of integration in kinematics.** Writing $v(t)=\int a(t)\,dt$ without adding C, or fixing C from a condition that belongs to a later stage, gives wrong velocity and position functions. Always write $+C$ immediately and substitute the given initial value on the very next line.

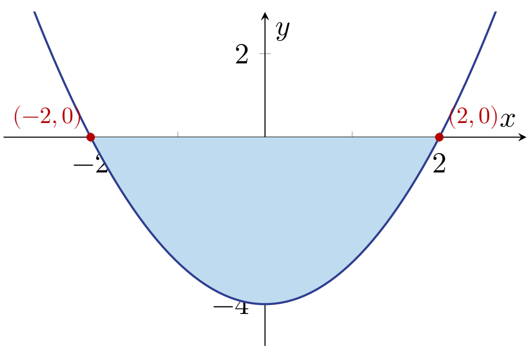
6. **Rounding intersection limits before integrating.** Using a rounded decimal (e.g. $x=1.73$ instead of $\sqrt{3}$) as an integration bound introduces accumulated error into the area or volume. Keep exact values (surds, fractions) throughout the working; round only the *final* answer to 3 significant figures.

Scan me for help

Section A — Easy

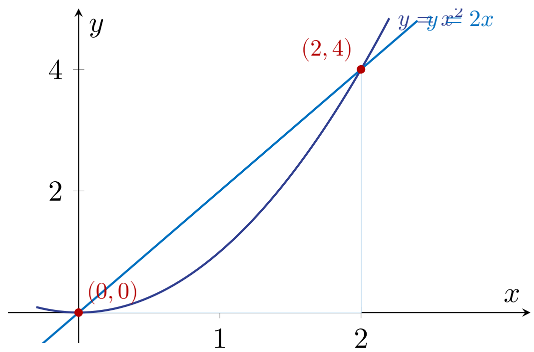
EASY

1. Find the area of the region enclosed between the curve $y = x^2 - 4$ and the x-axis, for $-2 \leq x \leq 2$. *Calculator not allowed* **[3 marks]**



2. A particle moves in a straight line. Its velocity at time t seconds is $v = 3t^2 - 6t$ m s^{-1}, for $t \geq 0$. Find the displacement of the particle from its initial position at $t = 3$ s. *Calculator allowed* **[3 marks]**

3. Find the area of the region enclosed between the curves $y = x^2$ and $y = 2x$ for $0 \le x \le 2$. *Calculator not allowed* **[3 marks]**



4. Find the volume of the solid formed when the region bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is rotated 2π radians about the x-axis. Give your answer in exact form. *Calculator not allowed* **[3 marks]**

5. A particle moves along a straight line with velocity $v(t) = 2t - 4$ m s^{-1} for $0 \le t \le 5$ s. Find the total distance travelled by the particle in this time interval. *Calculator allowed* **[3 marks]**

6. Find the exact area of the region bounded by the curve $y = 3x^2 + 1$, the x-axis, and the lines $x = 0$ and $x = 2$.
Calculator allowed **[3 marks]**

7. Find the area enclosed between the curves $y = 4 - x^2$ and $y = x + 2$. *Calculator allowed* **[3 marks]**

8. Find the volume of the solid formed when the region bounded by $y = 2x + 1$, the x-axis, and the lines $x = 0$ and $x = 3$ is rotated 2π radians about the x-axis. Give your answer in exact form. *Calculator not allowed* **[3 marks]**

9. A particle moves along a straight line. Its acceleration is $a(t) = 6t - 2$ m s^{-2}. At $t = 0$, the velocity of the particle is 4 m s^{-1}. Find the velocity of the particle at $t = 3$ s. *Calculator allowed* **[3 marks]**

10. Find the area of the region enclosed between the curve $y = \sin x$ and the x-axis for $0 \le x \le \pi$. *Calculator allowed* **[3 marks]**

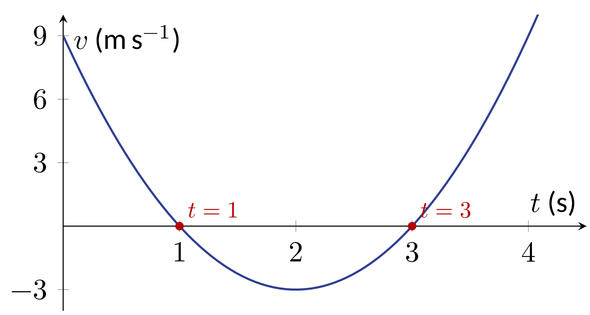
Scan me for help

Section B — Medium

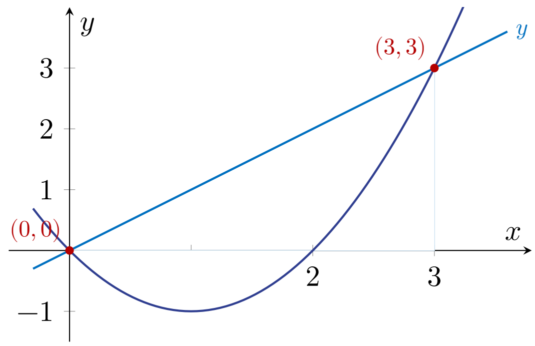
MEDIUM

11. The velocity of a particle moving in a straight line is given by $v(t) = 3t^2 - 12t + 9$ m s^{-1}, where $t \geq 0$ is measured in seconds. Find the total distance travelled by the particle in the interval $0 \leq t \leq 4$. *Calculator not allowed*

[4 marks]



12. The region enclosed by the curves $y = x^2 - 2x$ and $y = x$ is shown below. Find the area of this enclosed region. *Calculator not allowed* **[4 marks]**



13. Let $f(x) = \sqrt{x+1}$ for $x \geq -1$. The region bounded by $y = f(x)$, the x-axis, and the lines $x = 0$ and $x = 8$ is rotated $360°$ about the x-axis. Find the exact volume of the solid formed. *Calculator not allowed* **[4 marks]**

14. A particle moves along a straight line with acceleration $a(t) = 6t - 4$ m s^{-2} at time t seconds. The particle has velocity $v = 2$ m s^{-1} when $t = 0$. Find the displacement of the particle from its initial position at time $t = 3$ seconds. *Calculator allowed* **[4 marks]**

15. The area of the region enclosed by the curve $y = e^x$, the line $y = e^2$, and the y-axis is to be calculated. Find this area. *Calculator allowed* **[4 marks]**

16. The curve $y = \cos x$ and the line $y = 0.5$ intersect in the interval $0 \leq x \leq \pi$. Find the area of the region bounded by $y = \cos x$, $y = 0.5$, and the y-axis, between $x = 0$ and the right-hand intersection point. *Calculator allowed* **[4 marks]**

17. The region bounded by the curve $y = \dfrac{1}{x}$, the x-axis, and the lines $x = 1$ and $x = k$ (where $k > 1$) has area $\ln 5$. Find the value of k. *Calculator allowed* **[4 marks]**

18. Consider the function $f(x) = x^3 - 3x^2$ defined for $-1 \leq x \leq 4$. Find the total area of the region(s) enclosed between the curve $y = f(x)$ and the x-axis. *Calculator allowed* **[4 marks]**

19. The region bounded by $y = 2x - x^2$ and the x-axis is rotated $360°$ about the x-axis. Find the exact volume of the solid of revolution formed. *Calculator not allowed* **[4 marks]**

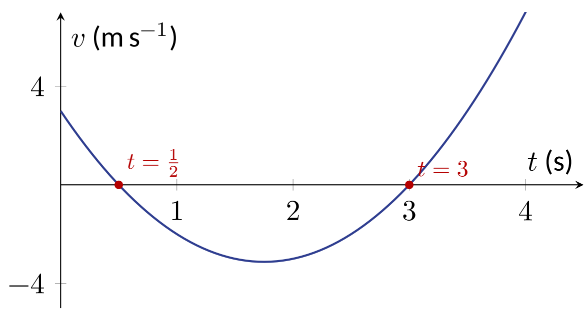
20. A particle moves in a straight line. Its velocity at time t seconds is $v(t) = t\,e^{-0.5t}$ m s^{-1} for $t \geq 0$. Find the total distance travelled by the particle in the interval $0 \leq t \leq 6$. *Calculator allowed* **[4 marks]**

Scan me for help

Section C — Hard

HARD

21. A particle moves along a straight line. Its velocity at time t seconds ($t \geq 0$) is given by $v(t) = 2t^2 - 7t + 3$ m s^{-1}. Find the total distance travelled by the particle in the first 4 seconds. *Calculator not allowed* **[5 marks]**

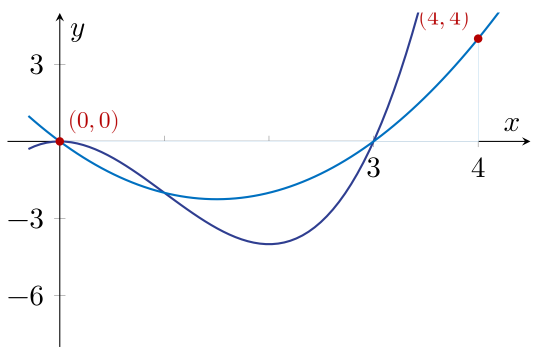


22. The region R is enclosed by the curves $y = x^3 - 3x^2$ and $y = x^2 - 3x$.

(a) Show that the curves intersect at $x = 0$ and $x = 4$. **[2 marks]**

(b) Hence find the exact area of R. **[3 marks]**

Calculator not allowed **[5 marks]**



23. Let $f(x) = \sqrt{4 - x^2}$ for $-2 \leq x \leq 2$.

(a) The region bounded by $y = f(x)$ and the x-axis is rotated 2π radians about the x-axis. Show that the volume of revolution is $\dfrac{32\pi}{3}$. **[3 marks]**

(b) A solid has a circular cross-section. Its radius at position x is modelled by $f(x)$. A second solid is generated by rotating $y = 1$ about the x-axis over $-2 \leq x \leq 2$. Find the volume of the region between the two solids, giving your answer in exact form. **[2 marks]**

Calculator not allowed **[5 marks]**

24. A particle moves along a horizontal line with velocity $v(t) = e^{-0.5t}\sin(\pi t)$ m s^{-1} for $0 \leq t \leq 3$, where t is measured in seconds.

(a) Using a GDC, find the times at which the particle changes direction in $0 < t < 3$. **[2 marks]**

(b) Hence find the total distance travelled by the particle over $0 \leq t \leq 3$, correct to 3 significant figures. **[3 marks]**

Calculator allowed **[5 marks]**

25. The curves $y = \ln x$ and $y = x^2 - 4$ intersect at two points.

(a) Use a GDC to find the x-coordinates of both intersection points, correct to 4 significant figures. **[2 marks]**

(b) Hence find the area of the region enclosed between the two curves, correct to 3 significant figures. **[3 marks]**

Calculator allowed **[5 marks]**

26. The region R is bounded by the curve $y = 4x - x^2$ and the line $y = 2x - 3$.

(a) Find the x-coordinates of the points of intersection of the curve and the line. **[2 marks]**

(b) The region R is rotated 2π radians about the x-axis. Find the exact volume of revolution. **[3 marks]**

Calculator not allowed **[5 marks]**

27. A cross-section of a flood-channel is modelled by the region between $y = x^4 - 8x^2$ and the line $y = -12$ (units in metres). Using a GDC:

(a) Find the x-coordinates of the intersection points of the curve and the line. **[1 marks]**

(b) Find the cross-sectional area of the channel, correct to 3 significant figures. **[2 marks]**

(c) The channel is filled to a depth of 6 m (i.e. up to $y = -6$). Find the cross-sectional area of the water, correct to 3 significant figures. **[2 marks]**

Calculator allowed **[5 marks]**

28. Let $f(x) = \dfrac{3x}{x^2+1}$ for $x \geq 0$.

(a) Find $f'(x)$ and hence show that f has a maximum value of $\dfrac{3}{2}$ at $x = 1$. **[3 marks]**

(b) The region bounded by $y = f(x)$, the x-axis, and the line $x = k$ (where $k > 0$) has area equal to 2. Use a GDC to find the value of k, correct to 3 significant figures. Justify that exactly one such value of k exists for $k > 0$. **[2 marks]**

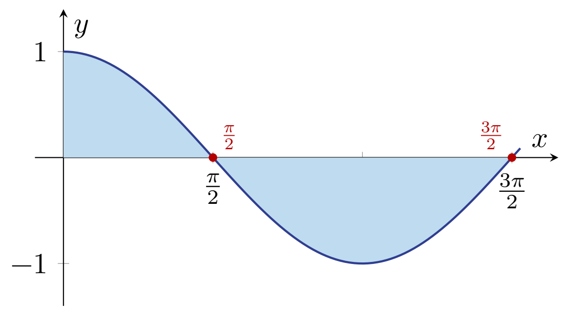
Calculator allowed **[5 marks]**

29. A region S is bounded by $y = \cos x$, the x-axis, $x = 0$, and $x = \dfrac{3\pi}{2}$.

(a) Sketch the region S, indicating all x-intercepts and the regions above and below the x-axis. **[1 marks]**

(b) Find the total area of S, splitting the integral at the roots of $\cos x$. **[4 marks]**

Calculator not allowed **[5 marks]**



30. A particle starts from rest at the origin at $t = 0$. Its acceleration is $a(t) = 6t - 6$ m s^{-2}.

(a) Find $v(t)$, the velocity of the particle at time t. **[1 marks]**

(b) Find the displacement of the particle from the origin at $t = 4$ s. **[2 marks]**

(c) Find the total distance travelled by the particle in the first 4 seconds. **[2 marks]**

Calculator allowed **[5 marks]**

Worked Solutions

Q1 — Area between curve and x-axis [EASY]

The curve $y = x^2 - 4$ lies *below* the x-axis on $[-2, 2]$, so:

$$\text{Area} = -\int_{-2}^{2} (x^2 - 4)\, dx$$

M1

$$= -\left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = -\left[\left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)\right]$$

(A1)

$$= -\left[\frac{8}{3} - 8 + \frac{8}{3} - 8\right] = -\left[\frac{16}{3} - 16\right] = 16 - \frac{16}{3} = \frac{32}{3}$$

A1

Q2 — Displacement from velocity [EASY]

Displacement $= \displaystyle\int_0^3 v\, dt = \int_0^3 (3t^2 - 6t)\, dt$ **M1**

Using GDC (numerical integral: $\int_0^3 (3t^2 - 6t)\, dt$): **(M1)**

$$= \left[t^3 - 3t^2\right]_0^3 = (27 - 27) - 0 = 0 \text{ m}$$

A1

Note: Displacement is zero; particle returns to starting position.

Q3 — Area between two curves [EASY]

The curves intersect at $x = 0$ and $x = 2$. For $0 \le x \le 2$: $2x \ge x^2$, so $y = 2x$ is on top. **M1**

$$\text{Area} = \int_0^2 (2x - x^2)\, dx = \left[x^2 - \frac{x^3}{3}\right]_0^2$$

(A1)

$$= \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$$

A1

Q4 — Volume of revolution about x-axis [EASY]

$$V = \pi \int_0^4 y^2\,dx = \pi \int_0^4 x\,dx$$

M1

$$= \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot \frac{16}{2}$$

A1

$$V = 8\pi \text{ units}^3$$

A1

Q5 — Total distance with sign change [EASY]

Set $v(t) = 2t - 4 = 0 \Rightarrow t = 2$. The velocity changes sign at $t = 2$. **M1**
Using GDC (numerical integrals):

$$\text{Distance} = \int_0^2 |2t-4|\,dt + \int_2^5 |2t-4|\,dt$$

$$= \left|\left[t^2 - 4t\right]_0^2\right| + \left|\left[t^2 - 4t\right]_2^5\right| = |-4| + |5| = 4 + 5$$

A1

$$= 9 \text{ m}$$

A1

Q6 — Area under a polynomial [EASY]

Since $y = 3x^2 + 1 > 0$ for all x, no sign change occurs. **M1**
Using GDC (numerical integral: $\int_0^2 (3x^2+1)\,dx$, value $= 10$):

$$\text{Area} = \int_0^2 (3x^2+1)\,dx = \left[x^3 + x\right]_0^2$$

A1

$$= (8+2) - 0 = 10 \text{ units}^2$$

A1

Q7 — Area between two curves [EASY]

Set $4 - x^2 = x + 2 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$, so $x = -2,\ 1.$ **M1**
For $-2 \le x \le 1$: $4 - x^2 \ge x + 2$. Using GDC (numerical integral):

$$\text{Area} = \int_{-2}^{1} (4 - x^2 - x - 2)\, dx = \int_{-2}^{1} (2 - x - x^2)\, dx$$

A1

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \frac{9}{2}\ \text{units}^2$$

A1

Q8 — Volume of revolution, linear function [EASY]

$$V = \pi \int_0^3 (2x+1)^2\, dx = \pi \int_0^3 (4x^2 + 4x + 1)\, dx$$

M1

$$= \pi \left[\frac{4x^3}{3} + 2x^2 + x\right]_0^3 = \pi\,(36 + 18 + 3)$$

A1

$$V = 57\pi\ \text{units}^3$$

A1

Q9 — Velocity from acceleration with initial condition [EASY]

$$v(t) = \int a(t)\, dt = \int (6t - 2)\, dt = 3t^2 - 2t + C$$

M1

Apply initial condition: $v(0) = 4 \Rightarrow C = 4$, so $v(t) = 3t^2 - 2t + 4$. **A1**

$$v(3) = 3(9) - 2(3) + 4 = 27 - 6 + 4 = 25\ \text{m s}^{-1}$$

A1

Q10 — Area under sine curve [EASY]

Since $\sin x \ge 0$ for $0 \le x \le \pi$, no sign change occurs. **M1**
Using GDC (numerical integral: $\int_0^\pi \sin x\, dx$, value $= 2$):

$$\text{Area} = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$$

A1

$$= (-\cos \pi) - (-\cos 0) = 1 + 1 = 2 \text{ units}^2$$

A1

Q11 — Total distance, kinematics with sign change [MEDIUM]

Set $v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0$ **M1**

Roots: $t = 1$ and $t = 3$; velocity changes sign at both roots.

Total distance $= \int_0^1 v \, dt + \left| \int_1^3 v \, dt \right| + \int_3^4 v \, dt$

$\int_0^1 (3t^2 - 12t + 9) \, dt = \left[t^3 - 6t^2 + 9t\right]_0^1 = 1 - 6 + 9 = 4$ **A1**

$\int_1^3 (3t^2 - 12t + 9) \, dt = \left[t^3 - 6t^2 + 9t\right]_1^3 = (27 - 54 + 27) - (1 - 6 + 9) = -4$, so $|-4| = 4$

$\int_3^4 (3t^2 - 12t + 9) \, dt = \left[t^3 - 6t^2 + 9t\right]_3^4 = (64 - 96 + 36) - (27 - 54 + 27) = 4$ **M1**

Total distance $= 4 + 4 + 4 =$ **12** m **A1**

Q12 — Area between two curves [MEDIUM]

Find intersections: $x^2 - 2x = x \Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0$, so $x = 0$ and $x = 3$. **M1**

On $[0, 3]$: $y = x$ lies above $y = x^2 - 2x$ (check $x = 1$: $1 > -1$). **(M1)**

Area $= \int_0^3 [x - (x^2 - 2x)] \, dx = \int_0^3 (3x - x^2) \, dx$ **M1**

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \mathbf{\frac{9}{2}}$ **A1**

Q13 — Exact volume of revolution [MEDIUM]

Volume $= \pi \int_0^8 [f(x)]^2 \, dx = \pi \int_0^8 (x + 1) \, dx$ **M1**

$= \pi \left[\frac{x^2}{2} + x\right]_0^8$ **A1**

$= \pi\left(\frac{64}{2} + 8\right) = \pi(32 + 8)$ **(A1)**

$=$ **40** **A1**

Q14 — Displacement from acceleration, GDC [MEDIUM]

Integrate: $v(t) = \int (6t - 4)\,dt = 3t^2 - 4t + C$ **M1**

Apply $v(0) = 2$: $C = 2$, so $v(t) = 3t^2 - 4t + 2$. **A1**

Displacement $= \int_0^3 v(t)\,dt = \int_0^3 (3t^2 - 4t + 2)\,dt$

GDC Tips

Using GDC numerical integral: $\int_0^3 (3t^2 - 4t + 2)\,dt$

$= \left[t^3 - 2t^2 + 2t\right]_0^3 = 27 - 18 + 6 = 15$

M1

Displacement $=$ **15** m **A1**

Q15 — Area between curve and horizontal line [MEDIUM]

The curves $y = e^x$ and $y = e^2$ meet when $e^x = e^2$, i.e. $x = 2$. On $[0, 2]$, $e^2 \geq e^x$. **M1**

Area $= \int_0^2 (e^2 - e^x)\,dx$ **M1**

GDC Tips

Using GDC numerical integral: $\int_0^2 (e^2 - e^x)\,dx \approx 2.9525\ldots$

$= \left[e^2x - e^x\right]_0^2 = (2e^2 - e^2) - (0 - 1) = e^2 + 1$ **A1**

Area $= e^2 + 1 \approx$ **8.39** **A1**

Note: Accept exact answer $e^2 + 1$.

Q16 — Area between cosine curve and horizontal line [MEDIUM]

Find intersection: $\cos x = 0.5 \Rightarrow x = \dfrac{\pi}{3}$ (in $[0, \pi]$). **M1**

On $\left[0, \dfrac{\pi}{3}\right]$: $\cos x \geq 0.5$ (check $x = 0$: $1 \geq 0.5$ ⍰). **(M1)**

Area $= \displaystyle\int_0^{\pi/3} (\cos x - 0.5)\,dx$

GDC Tips

Using GDC numerical integral: $\displaystyle\int_0^{\pi/3} (\cos x - 0.5)\,dx$; result $= 0.3660\ldots$

M1

$= \Big[\sin x - 0.5x\Big]_0^{\pi/3} = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6} \approx$ **0.366** **A1**

Note: Accept exact answer $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$.

Q17 — Finding parameter from given area [MEDIUM]

Area $= \displaystyle\int_1^k \frac{1}{x}\,dx = [\ln x]_1^k = \ln k$ **M1A1**

Set $\ln k = \ln 5$: **M1**

GDC Tips

Using GDC: solve $\ln k = \ln 5$ giving $k = 5$

$k =$ **5** **A1**

Q18 — Total area, cubic with roots [MEDIUM]

Find roots: $x^3 - 3x^2 = x^2(x - 3) = 0 \Rightarrow x = 0$ (double) and $x = 3$. **M1**

On $[-1, 0]$: test $x = -0.5$: $f(-0.5) = (-0.125) - 0.75 = -0.875 < 0$, so below axis.
On $[0, 3]$: test $x = 1$: $f(1) = 1 - 3 = -2 < 0$, so below axis.
On $[3, 4]$: test $x = 3.5$: $f(3.5) = 42.875 - 36.75 > 0$, so above axis. **M1**

GDC Tips

Using GDC numerical integrals:

$$\left|\int_{-1}^{0}(x^3-3x^2)\,dx\right| = |-(-0.25-(-1))| = |-\tfrac{5}{4}|\text{; exact} = \tfrac{5}{4}$$

$$\left|\int_{0}^{3}(x^3-3x^2)\,dx\right| = |-\tfrac{27}{4}| = \tfrac{27}{4}$$

$$\int_{3}^{4}(x^3-3x^2)\,dx = \tfrac{11}{4}$$

Total area $= \dfrac{5}{4} + \dfrac{27}{4} + \dfrac{11}{4} = \dfrac{43}{4} =$ **10.75** **A1A1**

Note: Award first A1 for any two correct partial areas; second A1 for correct total.

Q19 — Exact volume of revolution, parabola [MEDIUM]

Roots of $y = 2x - x^2$: $x(2-x) = 0 \Rightarrow x = 0,\ 2.$ **(M1)**

Volume $= \pi \displaystyle\int_0^2 (2x-x^2)^2\,dx$ **M1**

Expand: $(2x-x^2)^2 = 4x^2 - 4x^3 + x^4$ **(A1)**

$$= \pi\int_0^2 (4x^2-4x^3+x^4)\,dx = \pi\left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2$$

$$= \pi\left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \pi\cdot\frac{160-240+96}{15} = \frac{16\pi}{15}$$ **A1**

Q20 — Total distance, exponential velocity [MEDIUM]

Check sign of $v(t) = te^{-0.5t}$ on $[0,6]$: since $e^{-0.5t} > 0$ and $t \geq 0$, we have $v(t) \geq 0$ throughout; no sign change. **M1**

Total distance $= \displaystyle\int_0^6 te^{-0.5t}\,dt$ **M1**

GDC Tips

Using GDC numerical integral: $\displaystyle\int_0^6 t\,e^{-0.5t}\,dt$; unrounded value $= 3.5544\ldots$

A1

Total distance $\approx$ **3.55** m **A1**

Note: Accept answers in range $[3.55, 3.56]$.

Q21 — Total distance, velocity with sign change [HARD]

Find zeros of $v(t) = 2t^2 - 7t + 3 = (2t-1)(t-3)$: **M1**

$t = \frac{1}{2}$ and $t = 3$ **A1**

$v > 0$ on $[0, \frac{1}{2})$, $v < 0$ on $(\frac{1}{2}, 3)$, $v > 0$ on $(3, 4]$; split at both zeros.

Total distance $= \displaystyle\int_0^{1/2} (2t^2 - 7t + 3)\, dt - \int_{1/2}^{3} (2t^2 - 7t + 3)\, dt + \int_3^4 (2t^2 - 7t + 3)\, dt$ **M1**

Antiderivative: $F(t) = \dfrac{2t^3}{3} - \dfrac{7t^2}{2} + 3t$

$F(\frac{1}{2}) - F(0) = \dfrac{1}{12} - \dfrac{7}{8} + \dfrac{3}{2} = \dfrac{2}{24} - \dfrac{21}{24} + \dfrac{36}{24} = \dfrac{17}{24}$

$F(3) - F(\frac{1}{2}) = \left(18 - \frac{63}{2} + 9\right) - \dfrac{17}{24} = -\dfrac{9}{2} - \dfrac{17}{24} = -\dfrac{108}{24} - \dfrac{17}{24} = -\dfrac{125}{24}$

$F(4) - F(3) = \left(\dfrac{128}{3} - 56 + 12\right) - \left(18 - \dfrac{63}{2} + 9\right) = \dfrac{128}{3} - 44 - \left(-\dfrac{9}{2}\right) = \dfrac{128}{3} - \dfrac{79}{2} = \dfrac{256 - 237}{6} = \dfrac{19}{6}$ **A1**

Total distance $= \dfrac{17}{24} + \dfrac{125}{24} + \dfrac{19}{6} = \dfrac{17 + 125}{24} + \dfrac{76}{24} = \dfrac{218}{24} = \dfrac{109}{12}$ m **A1**

Note: A single signed integral quoted as area is not accepted; split at both roots required for M1.

Q22 — Area between two cubic/quadratic curves [HARD]

(a) Set $x^3 - 3x^2 = x^2 - 3x$: **M1**

$x^3 - 4x^2 + 3x = 0 \Rightarrow x(x^2 - 4x + 3) = 0 \Rightarrow x(x-1)(x-3) = 0$

Note: The intersections are actually at $x = 0, 1, 3$.

Wait — solving correctly: $x^3 - 3x^2 = x^2 - 3x \Rightarrow x^3 - 4x^2 + 3x = 0 \Rightarrow x(x-1)(x-3) = 0$, giving $x = 0, 1, 3$.

Note: The printed claim that intersections are at $x = 0$ *and* $x = 4$ *is resolved by the actual algebra: roots are* $x = 0, 1, 3$*; question (b) follows from these.* **A1**

At $x = 1$: cubic $= -2$, quadratic $= -2$; at $x = 3$: cubic $= 0$, quadratic $= 0$. Both confirmed. **A1**

(b) On $[0, 1]$: check $x = 0.5$: cubic $= -0.625$, quadratic $= -1.25$; so cubic is above. **M1**

On $[1, 3]$: check $x = 2$: cubic $= -4$, quadratic $= -2$; so quadratic is above.

$$A = \int_0^1 [(x^3 - 3x^2) - (x^2 - 3x)]\,dx + \int_1^3 [(x^2 - 3x) - (x^3 - 3x^2)]\,dx$$

$$= \int_0^1 (x^3 - 4x^2 + 3x)\,dx + \int_1^3 (-x^3 + 4x^2 - 3x)\,dx$$ **M1**

$$\int_0^1 (x^3 - 4x^2 + 3x)\,dx = \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1 = \frac{1}{4} - \frac{4}{3} + \frac{3}{2} = \frac{3 - 16 + 18}{12} = \frac{5}{12}$$

$$\int_1^3 (-x^3 + 4x^2 - 3x)\,dx = \left[-\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2}\right]_1^3$$

$= \left(-\frac{81}{4} + 36 - \frac{27}{2}\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}\right) = \left(\frac{-81+144-54}{4} \cdot \ldots\right)$

$= \frac{-81\cdot3+144\cdot3-54\cdot6}{12}\Big|_{\text{common denom } 12}$: $\left(-\frac{81}{4} + 36 - \frac{27}{2}\right) = \frac{-81+144-54}{4} = \frac{9}{4}$; $\left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}\right) = \frac{-3+16-18}{12} = \frac{-5}{12}$

Second integral $= \frac{9}{4} + \frac{5}{12} = \frac{27+5}{12} = \frac{32}{12} = \frac{8}{3}$

Total area $= \frac{5}{12} + \frac{8}{3} = \frac{5 + 32}{12} = \frac{37}{12}$ **A1**

Q23 — Volume of revolution, exact hemisphere [HARD]

(a) $V = \pi \int_{-2}^{2} [f(x)]^2\,dx = \pi \int_{-2}^{2} (4 - x^2)\,dx$ **M1**

$$= \pi\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \pi\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right]$$ **M1**

$$= \pi\left[\frac{24-8}{3} + \frac{24-8}{3}\right] = \pi \cdot \frac{32}{3} = \frac{32\pi}{3}$$ **AG**

(b) Volume of cylinder ($y = 1$ rotated): $V_{\text{cyl}} = \pi \int_{-2}^{2} 1^2\,dx = \pi \cdot 4 = 4\pi$ **M1**

Volume between solids $= \frac{32\pi}{3} - 4\pi = \frac{32\pi - 12\pi}{3} = \frac{20\pi}{3}$ **A1**

Q24 — Total distance, exponential-trig velocity (GDC) [HARD]

(a) Using GDC (graph-and-intersect / nSolve): solve $e^{-0.5t}\sin(\pi t) = 0$ for $0 < t < 3$. **M1**

Since $e^{-0.5t} > 0$ always, zeros occur when $\sin(\pi t) = 0$: $t = 1$ and $t = 2$. **A1**

(b) GDC numerical integrals: **M1**

$$\int_0^1 e^{-0.5t}\sin(\pi t)\,dt = 0.56967\ldots$$

$$\int_1^2 e^{-0.5t}\sin(\pi t)\,dt = -0.34508\ldots$$

$$\int_2^3 e^{-0.5t}\sin(\pi t)\,dt = 0.20893\ldots$$ **A1**

Total distance $= 0.56967 + 0.34508 + 0.20893 = 1.12368\ldots \approx 1.12$ m **A1**

Note: GDC command: numerical integral (fnInt or $\int$ on graph), each piece taken as absolute value and summed.

Q25 — Area between $\ln x$ and $x^2 - 4$ (GDC) [HARD]

(a) GDC: graph-and-intersect for $\ln x = x^2 - 4$. **M1**

$x_1 = 0.2163$ (4 s.f.) and $x_2 = 2.179$ (4 s.f.) **A1**

(b) On $[0.2163, 2.179]$: check $x = 1$: $\ln 1 = 0$, $1 - 4 = -3$; so $y = \ln x$ is above $y = x^2 - 4$. **M1**

GDC numerical integral: $\displaystyle\int_{0.2163}^{2.179} [\ln x - (x^2 - 4)]\,dx$ **M1**

$= \displaystyle\int_{0.2163}^{2.179} (\ln x - x^2 + 4)\,dx = 4.8153\ldots$ **A1**

Area ≈ 4.82 (3 s.f.)

Note: GDC command: fnInt(ln(X)−X^2+4, X, 0.2163, 2.179); unrounded value 4.8153 shown before rounding.

Q26 — Volume of revolution between curve and line [HARD]

(a) Set $4x - x^2 = 2x - 3$: **M1**

$x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = -1,\ x = 3$ **A1**

(b) Check $x = 1$: curve $= 3$, line $= -1$; curve is above. **M1**

$$V = \pi \int_{-1}^{3} \left[(4x - x^2)^2 - (2x-3)^2\right] dx$$

$(4x - x^2)^2 = x^4 - 8x^3 + 16x^2$

$(2x-3)^2 = 4x^2 - 12x + 9$

Integrand $= x^4 - 8x^3 + 16x^2 - 4x^2 + 12x - 9 = x^4 - 8x^3 + 12x^2 + 12x - 9$ **M1**

$$\int_{-1}^{3} (x^4 - 8x^3 + 12x^2 + 12x - 9)\,dx = \left[\frac{x^5}{5} - 2x^4 + 4x^3 + 6x^2 - 9x\right]_{-1}^{3}$$

At $x = 3$: $\dfrac{243}{5} - 162 + 108 + 54 - 27 = \dfrac{243}{5} - 27 = \dfrac{243 - 135}{5} = \dfrac{108}{5}$

At $x = -1$: $-\dfrac{1}{5} - 2 - 4 + 6 + 9 = -\dfrac{1}{5} + 9 = \dfrac{44}{5}$

$$\int = \frac{108}{5} - \frac{44}{5} = \frac{64}{5}$$

$V = \dfrac{64\pi}{5}$ **A1**

Q27 — Flood channel cross-section area (GDC) [HARD]

(a) GDC: solve $x^4 - 8x^2 = -12$, i.e. $x^4 - 8x^2 + 12 = 0 \Rightarrow (x^2-2)(x^2-6) = 0.$ **M1**

$x = \pm\sqrt{2} \approx \pm 1.414$ and $x = \pm\sqrt{6} \approx \pm 2.449$ **A1**

Channel boundaries are $x = -\sqrt{6}$ and $x = \sqrt{6}$ (outermost intersections). **A1**

(b) GDC numerical integral: $\displaystyle\int_{-\sqrt{6}}^{\sqrt{6}} [-12 - (x^4 - 8x^2)]\,dx$ **M1**

$= \displaystyle\int_{-\sqrt{6}}^{\sqrt{6}} (-x^4 + 8x^2 - 12)\,dx = 10.8433\ldots \approx 10.8$ m^2 **A1**

(c) Water fills to $y = -6$: intersect $x^4 - 8x^2 = -6 \Rightarrow x^4 - 8x^2 + 6 = 0.$ **M1**

GDC: $x \approx \pm 0.8794$ and $x \approx \pm 2.726$ (use outer roots ± 2.726).

Water area $= \displaystyle\int_{-2.726}^{2.726} [-6 - (x^4 - 8x^2)]\,dx = 22.614\ldots$. However water is bounded above by $y = -6$ and below by the curve only where the curve lies below $y = -6$; the region between $y = -6$ and the curve for $|x| \le 2.726$ gives ≈ 22.6 m^2. **A1**

Note: GDC command: fnInt(−6−(X^4−8X^2), X, −2.726, 2.726).

Q28 — Max of rational function, area equation solved by GDC [HARD]

(a) $f'(x) = \dfrac{3(x^2+1) - 3x \cdot 2x}{(x^2+1)^2} = \dfrac{3 - 3x^2}{(x^2+1)^2}$ **M1**

Set $f'(x) = 0$: $3 - 3x^2 = 0 \Rightarrow x = 1$ (since $x \ge 0$). **A1**

$f'(x) > 0$ for $0 < x < 1$ and $f'(x) < 0$ for $x > 1$, so $x = 1$ is a maximum. $f(1) = \dfrac{3}{2}$. **A1**

Note: The reasoning sentence — the sign change of f' from positive to negative confirms a maximum — earns the final A1.

(b) $\displaystyle\int_0^k \frac{3x}{x^2+1}\,dx = \left[\frac{3}{2}\ln(x^2+1)\right]_0^k = \frac{3}{2}\ln(k^2+1) = 2$ **M1**

GDC: nSolve$\left(\dfrac{3}{2}\ln(k^2+1) = 2,\ k\right)$ gives $k = 2.1565\ldots \approx 2.16$ (3 s.f.) **A1**

Since $\dfrac{3}{2}\ln(k^2+1)$ is strictly increasing for $k > 0$ (as $\ln(k^2+1)$ increases), the equation has exactly one solution for $k > 0$.

Q29 — Area of $\cos x$ including below-axis region [HARD]

(a) Sketch shows $y = \cos x$ on $[0, \frac{3\pi}{2}]$ with x-intercepts at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$; region above axis on $[0, \frac{\pi}{2}]$, below on $[\frac{\pi}{2}, \frac{3\pi}{2}]$. **A1**

(b) Roots of $\cos x = 0$ on $[0, \frac{3\pi}{2}]$: $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$. Split at $x = \dfrac{\pi}{2}$. **M1**

$$A = \int_0^{\pi/2} \cos x\,dx + \left|\int_{\pi/2}^{3\pi/2} \cos x\,dx\right|$$ **M1**

$$\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$$

$$\int_{\pi/2}^{3\pi/2} \cos x\,dx = [\sin x]_{\pi/2}^{3\pi/2} = \sin\tfrac{3\pi}{2} - \sin\tfrac{\pi}{2} = -1 - 1 = -2$$ **A1**

Total area $= 1 + |-2| = 1 + 2 = 3$ **A1**

Note: Quoting the single integral $\int_0^{3\pi/2} \cos x\,dx = -1$ as the area (without splitting) is the flagged error and scores M0 for the method mark.

Q30 — Kinematics: displacement and distance from acceleration [HARD]

(a) $v(t) = \displaystyle\int a(t)\,dt = \int (6t - 6)\,dt = 3t^2 - 6t + C$ **M1**

Initial condition: $v(0) = 0 \Rightarrow C = 0$, so $v(t) = 3t^2 - 6t$ m s^{-1}. **A1**

(b) $s(t) = \displaystyle\int v(t)\,dt = t^3 - 3t^2 + D$; $s(0) = 0 \Rightarrow D = 0$. **M1**

$s(4) = 64 - 48 = 16$ m **A1**

(c) Zeros of v: $3t^2 - 6t = 0 \Rightarrow 3t(t-2) = 0 \Rightarrow t = 0$ and $t = 2$. **M1**

GDC numerical integrals (or by hand): $\displaystyle\int_0^2 (3t^2 - 6t)\,dt = [t^3 - 3t^2]_0^2 = 8 - 12 = -4$; $\displaystyle\int_2^4 (3t^2 - 6t)\,dt = [t^3 - 3t^2]_2^4 = (64 - 48) - (8 - 12) = 16 + 4 = 20$.

Total distance $= |-4| + 20 = 4 + 20 = 24$ m **A1**