

IB Math AA HL — Topic 5

Differential Equations

Separable Equations · Integrating Factor · Particular Solutions · Modelling

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50



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Name: _____ Date: _____
Score: _____/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

- **Separable DE (general form):** If $\frac{dy}{dx} = f(x)g(y)$, separate as $\int \frac{1}{g(y)} dy = \int f(x) dx$ and introduce one constant C at the moment of integration.
- **First-order linear DE (standard form):** $\frac{dy}{dx} + P(x)y = Q(x)$; integrating factor $\mu(x) = e^{\int P(x) dx}$; solution $\mu y = \int \mu Q(x) dx + C$.
- **Product-rule recognition:** $\frac{d}{dx}[\mu(x)y] = \mu \frac{dy}{dx} + \mu P(x)y$, so the left side after multiplying by μ is always $\frac{d}{dx}[\mu y]$.
- **Exponential growth/decay:** $\frac{dN}{dt} = kN \implies N = N_0 e^{kt}$; $k > 0$ growth, $k < 0$ decay.
- **Newton's law of cooling:** $\frac{dT}{dt} = -k(T - T_\infty) \implies T = T_\infty + (T_0 - T_\infty)e^{-kt}$, $k > 0$.
- **Logistic (separable, two linear factors):** $\frac{dP}{dt} = kP(L - P)$; partial fractions $\frac{1}{P(L - P)} = \frac{1}{L} \left(\frac{1}{P} + \frac{1}{L - P} \right)$; limiting value $P \rightarrow L$ as $t \rightarrow \infty$.
- **Particular solution:** Substitute the initial condition into the general solution immediately after integrating to determine the constant C .

GDC Tips

- **Solve for a constant numerically:** After obtaining the general solution, use nSolve(equation, variable) (Casio: SOLVE) to find k or C from a data point when exact algebra is awkward.
- **Solve for time t :** Enter the explicit solution as Y_1 and the threshold value as Y_2 , then use Graph $\rightarrow$ Intersection (TI: 2nd CALC 5; Casio: G-SOLV ISCT) to read off t to 3 s.f.
- **Numerical integration check:** Use fnInt(f(x),x,a,b) (TI) or $\int_a^b$ on the Casio to verify a definite integral obtained analytically.
- **Numerical derivative verification:** Use nDeriv(f(x),x,x₀) (TI) or d/dx at a point (Casio) to confirm that a proposed particular solution satisfies the DE at a given x .
- **Fitting two constants from data:** Store two data equations in the equation solver (SIMULTANEOUS on Casio or nSolve with a list on TI-Nspire) to find both k and A simultaneously.
- **Table of values for long-term behaviour:** Generate a TABLE (TI: 2nd TABLE; Casio: TABLE) for large t to confirm the solution is approaching its limiting value numerically.

Common Mistakes to Avoid

- 1. Skipping the separation step.** Integrating both sides directly without first writing all y -terms with dy and all x -terms with dx on opposite sides is an invalid operation. Always show the separated line $\frac{1}{g(y)} dy = f(x) dx$ before integrating.
- 2. Forgetting to divide through to standard form.** Applying the integrating factor to $a(x)\frac{dy}{dx} + P(x)y = Q(x)$ without first dividing every term by $a(x)$ gives the wrong μ . The coefficient of $\frac{dy}{dx}$ must equal 1 before computing $\mu = e^{\int P dx}$.
- 3. Introducing two constants of integration.** Placing a constant on each side of the equation after integrating (e.g. $\ln|y| + C_1 = x^2 + C_2$) is redundant. One constant $C = C_2 - C_1$ is introduced on one side only, at the moment of integration.
- 4. Mishandling the modulus in $\ln|y|$.** Writing $e^{\ln|y|} = y$ without discussion, or dropping the modulus entirely, loses the sign. Exponentiate as $|y| = e^{\text{RHS}}$, then absorb the $\pm$ into an arbitrary constant A ($A \neq 0$), giving $y = Ae^{\dots}$.
- 5. Not recognising the left side as $\frac{d}{dx}[\mu y]$.** After multiplying a linear DE by μ , students sometimes try to integrate the two terms on the left separately. The entire point of the integrating factor is that the left side equals $\frac{d}{dx}[\mu y]$; this line must be written explicitly before integrating.
- 6. Applying the initial condition before integrating.** Substituting the initial condition into the unsolved (pre-integration) form to eliminate C is meaningless. The constant C only exists after integration; substitute the initial condition into the general solution to obtain the particular solution.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)



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1. **EASY** No Calculator

[3 marks]

Consider the differential equation $\frac{dy}{dx} = 3x^2y$, where $y > 0$.

By separating variables, find the general solution, expressing your answer in the form $y = f(x)$.

2. **EASY** Calculator

[3 marks]

A population P (in thousands) of bacteria changes over time t (hours) according to

$$\frac{dP}{dt} = 0.4P, \quad P(0) = 2.$$

Find the value of P when $t = 5$. Give your answer to 3 significant figures.

3. **EASY** No Calculator

[3 marks]

Consider the differential equation

$$\frac{dy}{dx} + 2y = 6, \quad y(0) = 1.$$

Show that $y = 3 - 2e^{-2x}$ is the particular solution satisfying this initial condition.

4. **EASY** **Calculator**

[3 marks]

The temperature T ($^{\circ}\text{C}$) of a cup of coffee at time t (minutes) satisfies Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - 20),$$

where k is a positive constant and the room temperature is 20°C . The following data are recorded.

t (min)	0	10
T ($^{\circ}\text{C}$)	90	60

Find the value of k . Give your answer to 3 significant figures.

5. **EASY** **No Calculator**

[3 marks]

Consider the differential equation $\frac{dy}{dx} = \frac{2x}{y}$, where $y > 0$.

By separating variables, find the general solution, expressing your answer in the form $y^2 = f(x)$.

6. **EASY** **Calculator**

[3 marks]

A particle moves along a straight line. Its velocity v (m s^{-1}) at time t (seconds) satisfies

$$\frac{dv}{dt} = -0.5v, \quad v(0) = 12.$$

Find the time t at which $v = 3$. Give your answer to 3 significant figures.

7. **EASY** **Calculator**

[3 marks]

Consider the first-order linear differential equation

$$\frac{dy}{dx} + y = e^x.$$

(a) Write down an integrating factor for this equation. [1]

(b) Hence find the general solution. [2]

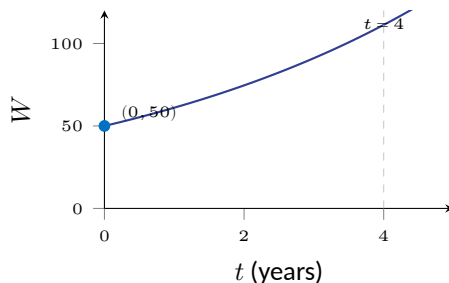
8. **EASY** **Calculator**

[3 marks]

The number of wolves W in a nature reserve at time t (years) satisfies

$$\frac{dW}{dt} = 0.2W, \quad W(0) = 50.$$

Find the number of wolves after 4 years. Give your answer to the nearest whole number.



9. **EASY** **Calculator**

[3 marks]

A tank contains salt water. The concentration C (g L^{-1}) of salt at time t (minutes) satisfies

$$\frac{dC}{dt} = -0.1C, \quad C(0) = 80.$$

Find the time at which the concentration first reaches 20 g L^{-1} . Give your answer to 3 significant figures.

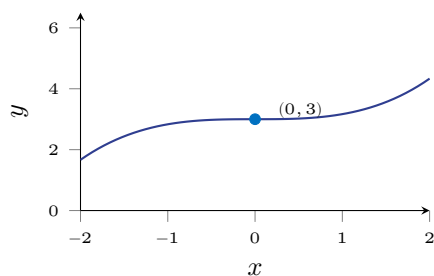
10. **EASY** **Calculator**

[3 marks]

Consider the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2}{2}, \quad y(0) = 3.$$

The solution curve is shown below.



Find the particular solution, expressing your answer in the form $y = f(x)$.

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)



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11. **MEDIUM** **No Calculator** [4 marks]

Consider the differential equation $\frac{dy}{dx} = \frac{x^2}{y}$, $y > 0$.

Find the general solution, expressing your answer in the form $y^2 = f(x)$.

12. **MEDIUM** **Calculator** [4 marks]

A population of bacteria, N (thousands), grows according to

$$\frac{dN}{dt} = 0.4N,$$

where t is measured in hours. Initially $N = 3$.

(a) Find the particular solution for N in terms of t . [2]

(b) Find the time at which $N = 20$, giving your answer to 3 significant figures. [2]

13. **MEDIUM** **No Calculator**

[4 marks]

Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = x^3, \quad x > 0,$$

expressing your answer in the form $y = f(x)$.

14. **MEDIUM** **Calculator**

[4 marks]

A cup of coffee has temperature T °C at time t minutes after being placed in a room of constant temperature 20 °C. The temperature satisfies Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - 20),$$

where $k > 0$ is a constant. The following data are recorded.

t (min)	0	8
T (°C)	90	55

(a) Show that $T = 20 + 70e^{-kt}$. [1]

(b) Find the value of k , giving your answer to 3 significant figures. [2]

(c) Find the time at which $T = 30$ °C, giving your answer to 3 significant figures. [1]

15. **MEDIUM** No Calculator

[4 marks]

Consider the differential equation

$$(1 + x^2) \frac{dy}{dx} = xy + x.$$

(a) Show that the equation is separable and write it in separated form. [1]

(b) Find the general solution, expressing your answer in the form $y = f(x)$. [3]

16. **MEDIUM** Calculator

[4 marks]

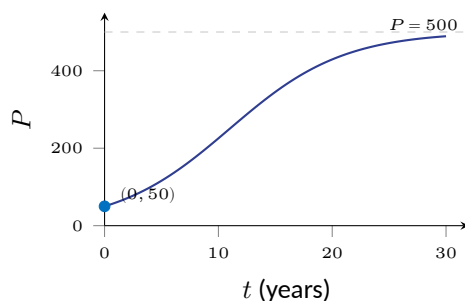
The number of individuals P in a wildlife reserve satisfies the logistic-type equation

$$\frac{dP}{dt} = \frac{P(500 - P)}{2500}, \quad 0 < P < 500,$$

where t is measured in years. When $t = 0$, $P = 50$.

(a) Using partial fractions, separate and integrate to find P as a function of t . [3]

(b) State the limiting value of P as $t \rightarrow \infty$, justifying your answer from the solution. [1]



17. **MEDIUM** **No Calculator**

[4 marks]

Find the general solution of the differential equation

$$\frac{dy}{dx} - y = e^{2x},$$

expressing your answer in the form $y = f(x)$.

18. **MEDIUM** **Calculator**

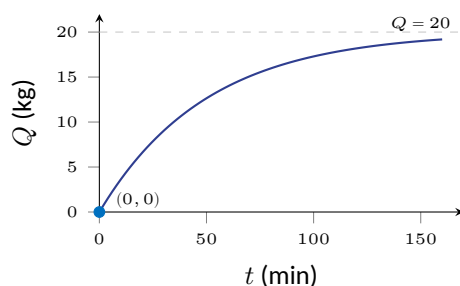
[4 marks]

A tank initially contains 100 L of pure water. Brine containing 0.2 kg of salt per litre flows in at 2 L min^{-1} , and the well-mixed solution drains out at 2 L min^{-1} . Let Q kg be the mass of salt in the tank at time t minutes.

(a) Show that $\frac{dQ}{dt} = 0.4 - \frac{Q}{50}$. [1]

(b) Find the particular solution for Q , given $Q(0) = 0$. [2]

(c) Find the value of Q when $t = 30$, giving your answer to 3 significant figures. [1]



19. MEDIUM Calculator

[4 marks]

Consider the differential equation

$$\frac{dy}{dx} + y \cos x = \cos x.$$

(a) Find the integrating factor. [1]

(b) Hence find the particular solution satisfying $y(0) = -3$, expressing your answer in the form $y = f(x)$. [3]

20. MEDIUM Calculator

[4 marks]

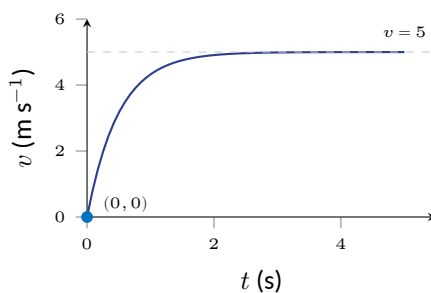
The velocity $v \text{ m s}^{-1}$ of a particle satisfies

$$\frac{dv}{dt} = 10 - 2v, \quad t \geq 0,$$

where t is measured in seconds. Initially $v = 0$.

(a) Find the particular solution for v in terms of t . [3]

(b) Find the time at which $v = 4 \text{ m s}^{-1}$, giving your answer to 3 significant figures. [1]



Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)



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21. **HARD** No Calculator

[5 marks]

Consider the differential equation $\frac{dy}{dx} = \frac{2xy}{x^2 + 1}$, $y > 0$.

(a) By separating variables, show that the general solution is $y = A(x^2 + 1)$, where A is a positive constant.
[3]

(b) Find the particular solution satisfying $y(0) = 5$, and state the value of y when $x = 3$. [2]

22. **HARD** Calculator

[5 marks]

A population P (measured in thousands) of bacteria grows according to the logistic differential equation

$$\frac{dP}{dt} = 0.4P \left(1 - \frac{P}{20} \right), \quad P(0) = 2,$$

where t is time in hours. Using partial fractions, find the particular solution for P in terms of t .

23. **HARD** No Calculator

[5 marks]

Consider the first-order linear differential equation

$$(x^2 + 1) \frac{dy}{dx} + 2xy = x^2 + 1, \quad x \in \mathbb{R}.$$

(a) Write the equation in standard form and hence find the integrating factor. [2]

(b) Find the general solution $y = f(x)$. [3]

24. **HARD** Calculator

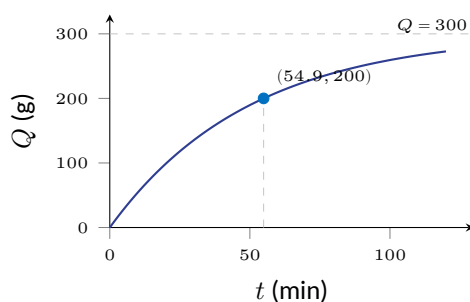
[5 marks]

A tank initially contains 100 litres of pure water. Brine containing 3 grams of salt per litre flows in at 2 litres per minute, and the well-mixed solution drains out at 2 litres per minute. Let Q (grams) be the amount of salt in the tank at time t (minutes).

(a) Write down the differential equation satisfied by Q . [1]

(b) Solve the differential equation with the initial condition $Q(0) = 0$ to find $Q(t)$. [3]

(c) Find, to 3 significant figures, the time at which the tank contains 200 grams of salt. [1]



25. **HARD** **Calculator**

[5 marks]

A cup of coffee is placed in a room held at a constant temperature of 20°C . According to Newton's law of cooling, the temperature T ($^\circ\text{C}$) of the coffee satisfies

$$\frac{dT}{dt} = -k(T - 20),$$

where t is time in minutes and $k > 0$ is a constant. The table below gives measured values.

t (min)	0	5	10	20
T ($^\circ\text{C}$)	90	71	56	35

(a) Solve the differential equation to show that $T = 20 + 70e^{-kt}$. [2]

(b) Use the data at $t = 5$ to find k , giving your answer to 3 significant figures. [2]

(c) Hence find the time at which $T = 30^\circ\text{C}$, to 3 significant figures. [1]

26. **HARD** **No Calculator**

[5 marks]

Show that $y = 2x^2e^{-2x}$ satisfies the differential equation

$$\frac{dy}{dx} + 2y = 4xe^{-2x},$$

and verify that $y(0) = 0$.

27. **HARD** No Calculator

[5 marks]

Consider the differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\ln x}{x^2}, \quad x > 0.$$

- (a) Find the integrating factor and hence show that the general solution is

$$y = \frac{x \ln x - x + C}{x^2},$$

where C is an arbitrary constant.

[4]

- (b) Find the value of C for which $y(1) = 2$, and state the limiting value of y as $x \rightarrow \infty$, justifying your answer from the solution.

[1]

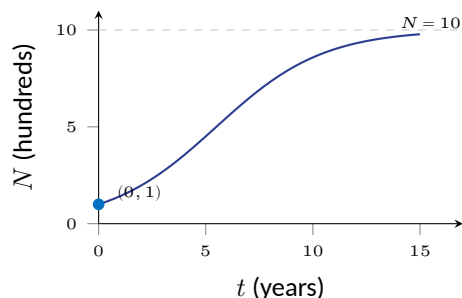
28. **HARD** Calculator

[5 marks]

A species of fish is introduced to a lake. Let N (hundreds of fish) be the population at time t (years). The population satisfies

$$\frac{dN}{dt} = \frac{N(10 - N)}{25}, \quad N(0) = 1.$$

- (a) Using partial fractions, find the particular solution for N in terms of t . [4]
(b) State the long-term population of fish (in hundreds), justifying your answer from the solution. [1]



29. **HARD** **Calculator**

[5 marks]

Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2 - 1}{2y}, \quad y > 1, \quad y(0) = 2.$$

- (a) Separate the variables, showing the separation step clearly, and integrate both sides to find an implicit relation between x and y . [3]
- (b) Hence find the value of y when $x = 1$, giving your answer to 3 significant figures. Use a GDC to solve numerically. [2]

30. **HARD** **Calculator**

[5 marks]

A chemical concentration C (mol L⁻¹) changes over time t (hours) according to

$$\frac{dC}{dt} + \frac{2}{t+1}C = \frac{6}{(t+1)^2}, \quad t \geq 0, \quad C(0) = 1.$$

- (a) Find the integrating factor $\mu(t)$ and hence find the general solution for C . [3]
- (b) Apply the initial condition to find the particular solution, and determine the limiting value of C as $t \rightarrow \infty$, justifying your answer. [2]

Mark Schemes

Q1 — Separable Equation - General Solution **EASY**

Separate variables: $\frac{1}{y} dy = 3x^2 dx$ **M1**
 Integrate both sides: $\ln |y| = x^3 + C$ **A1**
 Exponentiate, absorbing the constant: $y = Ae^{x^3}$, where $A > 0$ is an arbitrary constant. **A1**

Q2 — Exponential Growth - Evaluate at t

The general solution of $\frac{dP}{dt} = 0.4P$ is $P = Ae^{0.4t}$. Applying $P(0) = 2$ gives $A = 2$, so $P = 2e^{0.4t}$. **M1**
 Substitute $t = 5$: $P = 2e^2 = 14.778 \dots$ **(A1)**
 $P = 14.8$ (thousands) **A1**
 Note: Accept use of GDC nSolve or direct evaluation of $2e^2$.

Q3 — Verify Particular Solution by Substitution **EASY**

Differentiate: $\frac{dy}{dx} = 4e^{-2x}$ **M1**
 Substitute into LHS: $4e^{-2x} + 2(3 - 2e^{-2x}) = 4e^{-2x} + 6 - 4e^{-2x} = 6 = \text{RHS} \checkmark$ **A1**
 Check initial condition: $y(0) = 3 - 2e^0 = 3 - 2 = 1 \checkmark$ **A1**
 Note: Both the DE verification and the initial condition check are required for full marks. AG.

Q4 — Newton's Cooling - Find k **EASY**

The general solution is $T = 20 + Ae^{-kt}$. Applying $T(0) = 90$: $A = 70$, so $T = 20 + 70e^{-kt}$. **M1**
 Apply $T(10) = 60$: $60 = 20 + 70e^{-10k}$, so $e^{-10k} = \frac{40}{70} = \frac{4}{7}$ **(A1)**
 $k = -\frac{1}{10} \ln\left(\frac{4}{7}\right) = 0.05596 \dots \approx 0.0560$ **A1**
 Note: GDC: evaluate $-\frac{1}{10} \ln(4/7)$; unrounded value must be shown.

Q5 — Separable Equation dy/dx

Separate variables: $y \, dy = 2x \, dx$ M1

Integrate both sides: $\frac{y^2}{2} = x^2 + C$ A1

Rearrange: $y^2 = 2x^2 + A$, where A is an arbitrary constant. A1

Q6 — Exponential Decay - Solve for t EASY

The general solution is $v = Ae^{-0.5t}$. Applying $v(0) = 12$: $v = 12e^{-0.5t}$. M1

Set $v = 3$: $3 = 12e^{-0.5t}$, so $e^{-0.5t} = 0.25$, giving $t = \frac{-\ln(0.25)}{0.5} = 2 \ln 4 = 2.7725 \dots$ A1

$t = 2.77 \, \text{s}$ A1

Note: GDC: $\text{nSolve}(12e^{-0.5t} = 3, t)$ gives $t = 2.77 \, \text{s}$. Unrounded value required.

Q7 — Integrating Factor Method EASY

(a) The equation is already in standard form. Integrating factor: $e^{\int 1 \, dx} = e^x$ A1

(b) Multiply both sides by e^x : $\frac{d}{dx}(e^x y) = e^{2x}$ M1

Integrate: $e^x y = \frac{1}{2}e^{2x} + C$, so $y = \frac{1}{2}e^x + Ce^{-x}$, where C is an arbitrary constant. A1

Q8 — Wolf Population - Evaluate at t

General solution: $W = Ae^{0.2t}$. Apply $W(0) = 50$: $W = 50e^{0.2t}$. M1

Substitute $t = 4$: $W = 50e^{0.8} = 50 \times 2.2255 \dots = 111.27 \dots$ (A1)

$W = 111$ wolves A1

Note: GDC: evaluate $50e^{0.8}$; answer rounded to nearest whole number.

Q9 — Salt Concentration - Solve for t EASY

General solution: $C = Ae^{-0.1t}$. Apply $C(0) = 80$: $C = 80e^{-0.1t}$. M1

Set $C = 20$: $20 = 80e^{-0.1t}$, so $e^{-0.1t} = 0.25$, giving $t = \frac{-\ln(0.25)}{0.1} = 10 \ln 4 = 13.862 \dots$ A1

$t = 13.9 \, \text{min}$ A1

Note: GDC: $\text{nSolve}(80e^{-0.1t} = 20, t)$ gives $t = 13.9 \, \text{min}$.

Q10 — Particular Solution by Direct Integration **EASY**

Integrate both sides with respect to x : $y = \int \frac{x^2}{2} dx = \frac{x^3}{6} + C$ **M1**

Apply $y(0) = 3$: $3 = 0 + C$, so $C = 3$ **A1**

$y = \frac{x^3}{6} + 3$ **A1**

Q11 — Separable DE - General Solution **MEDIUM**

Separate variables: $y dy = x^2 dx$ **M1**

Integrate both sides: $\int y dy = \int x^2 dx \Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C$ **A1**

Multiply through by 2: $y^2 = \frac{2x^3}{3} + A$, $A \in \mathbb{R}$ **A1**

Note: Award M1 only if separation is shown as a genuine step with $y dy$ on one side and $x^2 dx$ on the other. A1A1 for correct integration and correct final form; accept $A = 2C$.

Q12 — Exponential Growth Model **MEDIUM**

(a) The equation $\frac{dN}{dt} = 0.4N$ is separable. Separate: $\frac{dN}{N} = 0.4 dt$, integrate: $\ln N = 0.4t + C$.
Exponentiate: $N = Ae^{0.4t}$. Apply $N(0) = 3$: $A = 3$. **M1**

$N = 3e^{0.4t}$ **A1**

(b) Set $3e^{0.4t} = 20 \Rightarrow e^{0.4t} = \frac{20}{3}$. Using GDC $\ln(3e^{0.4t} = 20)$: unrounded $t = 4.7428 \dots$ **M1**

$t = 4.74$ hours (3 s.f.) **A1**

Q13 — First-Order Linear DE - Integrating Factor **MEDIUM**

Equation is already in standard form $\frac{dy}{dx} + \frac{2}{x}y = x^3$. **M1**

Integrating factor: $e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$. **A1**

Multiply through: $\frac{d}{dx}(x^2y) = x^5$. Integrate both sides: $x^2y = \frac{x^6}{6} + C$. **M1**

$y = \frac{x^4}{6} + \frac{C}{x^2}$ **A1**

Note: Award M1 for identifying and computing the integrating factor, second M1 for recognising the left side as $\frac{d}{dx}(x^2y)$ and integrating. A1A1 for accuracy.

Q14 — Newton's Law of Cooling - Data Fitting MEDIUM

(a) The equation is separable: $\frac{dT}{T-20} = -k dt$, so $T - 20 = Ae^{-kt}$. Apply $T(0) = 90$: $A = 70$, giving $T = 20 + 70e^{-kt}$. **AG**

(b) Apply $T(8) = 55$: $55 = 20 + 70e^{-8k} \Rightarrow e^{-8k} = \frac{35}{70} = 0.5 \Rightarrow k = \frac{\ln 2}{8}$. GDC confirmation: $k = 0.0866433 \dots$ **M1**

$k = 0.0866$ (3 s.f.) **A1**

(c) Set $T = 30$: $30 = 20 + 70e^{-kt} \Rightarrow e^{-kt} = \frac{1}{7}$. Using GDC nSolve: $t = \frac{\ln 7}{k} = 22.4589 \dots \approx 22.5$ min (3 s.f.) **A1**

Q15 — Separable DE with ln - General Solution MEDIUM

(a) Rewrite: $\frac{dy}{dx} = \frac{x(y+1)}{1+x^2}$. Separated form: $\frac{dy}{y+1} = \frac{x}{1+x^2} dx$ **AG (M1)**

(b) Integrate both sides: $\ln|y+1| = \frac{1}{2} \ln(1+x^2) + C$. **M1**

Exponentiate: $y+1 = A\sqrt{1+x^2}$, where A absorbs e^C and the modulus. **A1**

$y = A\sqrt{1+x^2} - 1$, $A \in \mathbb{R}$ **A1**

Note: Award M1 for correct integration of both sides. Deduct a mark if the constant is introduced at the wrong stage or if modulus is ignored without absorption.

Q16 — Logistic Growth - Partial Fractions MEDIUM

(a) Separate: $\frac{dP}{P(500-P)} = \frac{dt}{2500}$. Partial fractions: $\frac{1}{P(500-P)} = \frac{1}{500} \left(\frac{1}{P} + \frac{1}{500-P} \right)$. Integrate: $\frac{1}{500} (\ln P - \ln(500-P)) = \frac{t}{2500} + C$. **M1**

So $\ln \frac{P}{500-P} = \frac{t}{5} + C_1$. Exponentiate: $\frac{P}{500-P} = Be^{t/5}$. Apply $P(0) = 50$: $B = \frac{50}{450} = \frac{1}{9}$. **M1**

Solve for P : $P(500-P)^{-1} = \frac{1}{9}e^{t/5}$, so $P = \frac{500}{1+9e^{-t/5}}$ **A1**

(b) As $t \rightarrow \infty$, $e^{-t/5} \rightarrow 0$, so $P \rightarrow \frac{500}{1+0} = 500$. The limiting value is 500 individuals. **R1**

Q17 — First-Order Linear DE - Integrating Factor (II) MEDIUM

Equation in standard form: $\frac{dy}{dx} - y = e^{2x}$. Integrating factor: $e^{\int (-1) dx} = e^{-x}$. **M1**

Multiply through: $e^{-x} \frac{dy}{dx} - e^{-x}y = e^x$. Recognise left side as $\frac{d}{dx}(e^{-x}y)$. **M1**

Integrate: $e^{-x}y = e^x + C$. **A1**

$y = e^{2x} + Ce^x, C \in \mathbb{R}$ **A1**

Note: Award M1M1 for correct IF and recognition of product-rule structure. Both accuracy marks require correct simplification.

Q18 — Mixing/Draining Tank Model **MEDIUM**

(a) Rate in = $2 \times 0.2 = 0.4 \text{ kg min}^{-1}$. Concentration in tank = $\frac{Q}{100}$; rate out = $2 \cdot \frac{Q}{100} = \frac{Q}{50}$. Hence $\frac{dQ}{dt} = 0.4 - \frac{Q}{50}$. **AG**

(b) This is a linear DE (or separable). Rewrite: $\frac{dQ}{dt} + \frac{Q}{50} = 0.4$. IF = $e^{t/50}$. $\frac{d}{dt}(Qe^{t/50}) = 0.4e^{t/50}$. Integrate: $Qe^{t/50} = 20e^{t/50} + C$. Apply $Q(0) = 0$: $C = -20$. **M1**

$Q = 20(1 - e^{-t/50})$ **A1**

(c) Substitute $t = 30$: $Q = 20(1 - e^{-0.6})$. GDC evaluates: $Q = 20(1 - 0.548812 \dots) = 9.0236 \dots$ $Q = 9.02$ kg (3 s.f.) **A1**

Q19 — First-Order Linear DE - Particular Solution **MEDIUM**

(a) Integrating factor: $e^{\int \cos x \, dx} = e^{\sin x}$. **A1**

(b) Multiply through: $\frac{d}{dx}(ye^{\sin x}) = \cos x e^{\sin x}$. Integrate: $ye^{\sin x} = \int \cos x e^{\sin x} \, dx = e^{\sin x} + C$ (substitution $u = \sin x$). **M1**

General solution: $y = 1 + Ce^{-\sin x}$. Apply $y(0) = -3$: $-3 = 1 + Ce^0 \Rightarrow C = -4$. **M1**

$y = 1 - 4e^{-\sin x}$ **A1**

Note: GDC may be used to verify by graphing $\frac{dy}{dx} + y \cos x$ against $\cos x$ for the particular solution; they should coincide.

Q20 — First-Order Separable DE - Velocity Model **MEDIUM**

(a) Separate: $\frac{dv}{10 - 2v} = dt$. Integrate: $-\frac{1}{2} \ln |10 - 2v| = t + C$. **M1**

Exponentiate: $10 - 2v = Ae^{-2t}$. Apply $v(0) = 0$: $10 = A$. So $v = 5(1 - e^{-2t})$. **M1**

$v = 5(1 - e^{-2t})$ **A1**

(b) Set $v = 4$: $4 = 5(1 - e^{-2t}) \Rightarrow e^{-2t} = 0.2$. GDC nSolve($5(1 - e^{-2t}) = 4, t$): $t = 0.804718 \dots$ $t = 0.805$ s (3 s.f.) **A1**

Q21 — Separable DE - Show That HARD

(a) Separate variables: $\frac{1}{y} dy = \frac{2x}{x^2 + 1} dx$ M1

Integrate both sides: $\ln |y| = \ln(x^2 + 1) + k$ A1

Exponentiate: $y = e^k(x^2 + 1) = A(x^2 + 1)$, where $A = e^k > 0$. AG

(b) $y(0) = 5$: $5 = A(0 + 1) \Rightarrow A = 5$, so $y = 5(x^2 + 1)$. A1

$y(3) = 5(9 + 1) = 50$. A1

Q22 — Logistic Growth - Partial Fractions (II) HARD

Write as $\frac{dP}{P(1 - P/20)} = 0.4 dt$, i.e. $\frac{20 dP}{P(20 - P)} = 0.4 dt$. M1

Partial fractions: $\frac{20}{P(20 - P)} = \frac{1}{P} + \frac{1}{20 - P}$. A1

Integrate: $\ln |P| - \ln |20 - P| = 0.4t + k$, so $\frac{P}{20 - P} = Ae^{0.4t}$. A1

$P(0) = 2$: $\frac{2}{18} = A \Rightarrow A = \frac{1}{9}$. A1

Solve for P : $9P = (20 - P)e^{0.4t} \Rightarrow P(9 + e^{0.4t}) = 20e^{0.4t}$, giving $P = \frac{20e^{0.4t}}{9 + e^{0.4t}} = \frac{20}{1 + 9e^{-0.4t}}$. A1

Q23 — Integrating-Factor Method HARD

(a) Divide by $(x^2 + 1)$: $\frac{dy}{dx} + \frac{2x}{x^2 + 1}y = 1$ (standard form). M1

Integrating factor: $\mu = e^{\int \frac{2x}{x^2+1} dx} = e^{\ln(x^2+1)} = x^2 + 1$. A1

(b) Recognise LHS as $\frac{d}{dx}[(x^2 + 1)y] = x^2 + 1$. M1

Integrate: $(x^2 + 1)y = \frac{x^3}{3} + x + C$. A1

General solution: $y = \frac{x^3/3 + x + C}{x^2 + 1} = \frac{x(x^2 + 3) + 3C}{3(x^2 + 1)}$. A1

Q24 — Salt-Tank Mixing Model HARD

(a) Rate in: $2 \times 3 = 6 \text{ g min}^{-1}$; rate out: $\frac{Q}{100} \times 2 = \frac{Q}{50} \text{ g min}^{-1}$. $\frac{dQ}{dt} = 6 - \frac{Q}{50}$. A1

(b) Standard form (already linear): integrating factor $e^{t/50}$. M1

$$\frac{d}{dt}(Qe^{t/50}) = 6e^{t/50}; \text{integrate: } Qe^{t/50} = 300e^{t/50} + C. \quad \text{A1}$$

$$Q(0) = 0: 0 = 300 + C \Rightarrow C = -300, \text{ so } Q(t) = 300(1 - e^{-t/50}). \quad \text{A1}$$

(c) Solve $300(1 - e^{-t/50}) = 200$: on GDC use $\text{nSolve}(300(1 - e^{-t/50}) = 200, t)$. Unrounded: $t = 50 \ln 3 \approx 54.931 \dots$; $t \approx 54.9$ min (3 s.f.). A1

Note: exact value $50 \ln 3$ accepted.

Q25 — Newton's Law of Cooling **HARD**

(a) Separate: $\frac{dT}{T - 20} = -k dt$; integrate: $\ln |T - 20| = -kt + c$. M1

Exponentiate with $T(0) = 90$: $T - 20 = 70e^{-kt}$, so $T = 20 + 70e^{-kt}$. AG

(b) $t = 5$, $T = 71$: $71 = 20 + 70e^{-5k} \Rightarrow e^{-5k} = \frac{51}{70}$. M1

$k = -\frac{1}{5} \ln \frac{51}{70}$; GDC evaluates: $k = 0.0633339 \dots \approx 0.0633$ (3 s.f.). A1

(c) $30 = 20 + 70e^{-kt} \Rightarrow e^{-kt} = \frac{1}{7} \Rightarrow t = \frac{\ln 7}{k}$. GDC: $\text{nSolve}(20 + 70e^{-0.0633339t} = 30, t)$ gives $t \approx 30.7$ min (3 s.f.). A1

Q26 — Verification by Substitution **HARD**

Let $y = 2x^2 e^{-2x}$. M1

Differentiate: $\frac{dy}{dx} = 4xe^{-2x} - 4x^2 e^{-2x} = 4x(1 - x)e^{-2x}$. A1

Compute $\frac{dy}{dx} + 2y$: $4xe^{-2x} - 4x^2 e^{-2x} + 2(2x^2 e^{-2x}) = 4xe^{-2x} - 4x^2 e^{-2x} + 4x^2 e^{-2x} = 4xe^{-2x}$. M1

LHS = $4xe^{-2x}$ = RHS ✓ A1

Check initial condition: $y(0) = 2(0)^2 e^0 = 0$ ✓ A1

Note: Both the DE verification and the initial condition check are required for full marks. AG.

Q27 — Integrating-Factor with ln - Limiting Value **HARD**

(a) Integrating factor: $\mu = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$. M1

Multiply through: $\frac{d}{dx}(x^2 y) = x^2 \cdot \frac{\ln x}{x^2} = \ln x$. A1

Integrate RHS by parts: $\int \ln x dx = x \ln x - x + C$. M1

So $x^2 y = x \ln x - x + C$, giving $y = \frac{x \ln x - x + C}{x^2}$. A1(AG)

(b) $y(1) = 2$: $\frac{1 \cdot 0 - 1 + C}{1} = 2 \Rightarrow C = 3$. As $x \rightarrow \infty$: $\frac{x \ln x}{x^2} = \frac{\ln x}{x} \rightarrow 0$, $\frac{x}{x^2} = \frac{1}{x} \rightarrow 0$, and $\frac{C}{x^2} \rightarrow 0$ (by L'Hôpital for the $\ln x/x$ term), so $y \rightarrow 0$. The limiting value is 0. A1

Q28 — Logistic Fish Population - Long-Term HARD

(a) Separate: $\frac{25 \, dN}{N(10 - N)} = dt$; partial fractions: $\frac{25}{N(10 - N)} = \frac{5}{2} \left(\frac{1}{N} + \frac{1}{10 - N} \right)$. M1

Integrate: $\frac{5}{2} (\ln N - \ln |10 - N|) = t + k$, so $\ln \frac{N}{10 - N} = \frac{2t}{5} + k'$. A1

Exponentiate: $\frac{N}{10 - N} = Ae^{2t/5}$. Apply $N(0) = 1$: $A = \frac{1}{9}$. A1

Solve for N : $9N = (10 - N)e^{2t/5}$; $N(9 + e^{2t/5}) = 10e^{2t/5}$, $N = \frac{10}{1 + 9e^{-2t/5}}$. A1

(b) As $t \rightarrow \infty$, $e^{-2t/5} \rightarrow 0$, so $N \rightarrow \frac{10}{1 + 0} = 10$ (hundreds of fish). The exponential term vanishes, confirming the carrying capacity. R1

Q29 — Separable DE - GDC Numerical Solve HARD

(a) Separate variables: $\frac{2y}{y^2 - 1} dy = dx$. M1

Integrate both sides: $\int \frac{2y}{y^2 - 1} dy = \int dx$. M1

LHS by substitution $u = y^2 - 1$: $\ln |y^2 - 1| = x + C$. Apply $y(0) = 2$: $\ln 3 = C$. Implicit relation: $\ln(y^2 - 1) = x + \ln 3$, i.e. $y^2 - 1 = 3e^x$. A1

(b) At $x = 1$: $y^2 = 1 + 3e^1 = 1 + 3e$. GDC: evaluate $\sqrt{1 + 3e}$ (since $y > 1$, take positive root). Unrounded: $y = \sqrt{1 + 3e} \approx 3.0257 \dots$; $y \approx 3.03$ (3 s.f.). A1

Alternatively, $\text{nSolve}(\ln(y^2 - 1) = 1 + \ln 3, y, y > 1)$ gives $y \approx 3.03$. A1

Q30 — Integrating Factor - Limiting Concentration HARD

(a) Standard form already: $P(t) = \frac{2}{t+1}$, $Q(t) = \frac{6}{(t+1)^2}$. Integrating factor: $\mu = e^{\int \frac{2}{t+1} dt} = e^{2 \ln(t+1)} = (t+1)^2$. M1

Multiply through: $\frac{d}{dt}[(t+1)^2 C] = (t+1)^2 \cdot \frac{6}{(t+1)^2} = 6$. A1

Integrate: $(t + 1)^2 C = 6t + K$, so $C = \frac{6t + K}{(t + 1)^2}$. **A1**

(b) $C(0) = 1$: $1 = \frac{0 + K}{1} \Rightarrow K = 1$. Particular solution: $C = \frac{6t + 1}{(t + 1)^2}$. **A1**

As $t \rightarrow \infty$: $C = \frac{6t + 1}{(t + 1)^2} \sim \frac{6t}{t^2} = \frac{6}{t} \rightarrow 0$. The concentration tends to 0 mol L^{-1} , since the numerator grows linearly while the denominator grows quadratically. **R1**