IB Math AA HL — Topic 5

Differential Equations

Separable Equations · Integrating Factor · Particular Solutions · Modelling

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:_____________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Separable DE (general form):** If $\dfrac{dy}{dx} = f(x)\,g(y)$, separate as $\displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$ and introduce one constant C at the moment of integration.
- **First-order linear DE (standard form):** $\dfrac{dy}{dx} + P(x)\,y = Q(x)$; integrating factor $\mu(x) = e^{\int P(x)\,dx}$; solution $\mu y = \displaystyle\int \mu\,Q(x)\,dx + C$.
- **Product-rule recognition:** $\dfrac{d}{dx}[\mu(x)\,y] = \mu\,\dfrac{dy}{dx} + \mu\,P(x)\,y$, so the left side after multiplying by μ is always $\dfrac{d}{dx}[\mu y]$.
- **Exponential growth/decay:** $\dfrac{dN}{dt} = kN \implies N = N_0\,e^{kt}$; $k > 0$ growth, $k < 0$ decay.
- **Newton's law of cooling:** $\dfrac{dT}{dt} = -k(T - T_\infty) \implies T = T_\infty + (T_0 - T_\infty)\,e^{-kt}$, $k > 0$.
- **Logistic (separable, two linear factors):** $\dfrac{dP}{dt} = kP(L - P)$; partial fractions $\dfrac{1}{P(L-P)} = \dfrac{1}{L}\left(\dfrac{1}{P} + \dfrac{1}{L-P}\right)$; limiting value $P \to L$ as $t \to \infty$.
- **Particular solution:** Substitute the initial condition into the general solution immediately after integrating to determine the constant C.

GDC Tips

- **Solve for a constant numerically:** After obtaining the general solution, use `nSolve(equation, variable)` (Casio: `SOLVE`) to find k or C from a data point when exact algebra is awkward.
- **Solve for time t:** Enter the explicit solution as Y_1 and the threshold value as Y_2, then use **Graph** → **Intersection** (TI: `2nd CALC 5`; Casio: `G-SOLV ISCT`) to read off t to 3 s.f.
- **Numerical integration check:** Use `fnInt(f(x),x,a,b)` (TI) or $\int_a^b$ on the Casio to verify a definite integral obtained analytically.
- **Numerical derivative verification:** Use `nDeriv(f(x),x,`x_0`)` (TI) or `d/dx` at a point (Casio) to confirm that a proposed particular solution satisfies the DE at a given x.
- **Fitting two constants from data:** Store two data equations in the equation solver (`SIMULTANEOUS` on Casio or `nSolve` with a list on TI-Nspire) to find both k and A simultaneously.
- **Table of values for long-term behaviour:** Generate a `TABLE` (TI: `2nd TABLE`; Casio: `TABLE`) for large t to confirm the solution is approaching its limiting value numerically.

Common Mistakes to Avoid

1. **Skipping the separation step.** Integrating both sides directly without first writing all y-terms with dy and all x-terms with dx on opposite sides is an invalid operation. Always show the separated line $\frac{1}{g(y)}\,dy = f(x)\,dx$ before integrating.

2. **Forgetting to divide through to standard form.** Applying the integrating factor to $a(x)\frac{dy}{dx} + P(x)y = Q(x)$ without first dividing every term by $a(x)$ gives the wrong μ. The coefficient of $\frac{dy}{dx}$ must equal 1 before computing $\mu = e^{\int P\,dx}$.

3. **Introducing two constants of integration.** Placing a constant on each side of the equation after integrating (e.g. $\ln|y| + C_1 = x^2 + C_2$) is redundant. One constant $C = C_2 - C_1$ is introduced on one side only, at the moment of integration.

4. **Mishandling the modulus in $\ln|y|$.** Writing $e^{\ln|y|} = y$ without discussion, or dropping the modulus entirely, loses the sign. Exponentiate as $|y| = e^{\text{RHS}}$, then absorb the $\pm$ into an arbitrary constant A ($A \neq 0$), giving $y = A\,e^{\cdots}$.

5. **Not recognising the left side as $\frac{d}{dx}[\mu y]$.** After multiplying a linear DE by μ, students sometimes try to integrate the two terms on the left separately. The entire point of the integrating factor is that the left side equals $\frac{d}{dx}[\mu y]$; this line must be written explicitly before integrating.

6. **Applying the initial condition before integrating.** Substituting the initial condition into the unsolved (pre-integration) form to eliminate C is meaningless. The constant C only exists after integration; substitute the initial condition into the *general* solution to obtain the particular solution.

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Section A — Easy

EASY

1. Consider the differential equation $\frac{dy}{dx} = 3x^2y$, where $y > 0$.

By separating variables, find the general solution, expressing your answer in the form $y = f(x)$. *Calculator not allowed* **[3 marks]**

2. A population P (in thousands) of bacteria changes over time t (hours) according to

$$\frac{dP}{dt} = 0.4P, \qquad P(0) = 2.$$

Find the value of P when $t = 5$. Give your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

3. Consider the differential equation

$$\frac{dy}{dx} + 2y = 6, \qquad y(0) = 1.$$

Show that $y = 3 - 2e^{-2x}$ is the particular solution satisfying this initial condition. *Calculator not allowed* **[3 marks]**

4. The temperature T ($^{\circ}$C) of a cup of coffee at time t (minutes) satisfies Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - 20),$$

where k is a positive constant and the room temperature is $20\,^{\circ}$C. The following data are recorded.

t (min)	0	10
T ($^{\circ}$C)	90	60

Find the value of k. Give your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

5. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{2x}{y}$, where $y > 0$.

By separating variables, find the general solution, expressing your answer in the form $y^2 = f(x)$. *Calculator not allowed* **[3 marks]**

6. A particle moves along a straight line. Its velocity v (m s^{-1}) at time t (seconds) satisfies

$$\frac{dv}{dt} = -0.5v, \qquad v(0) = 12.$$

Find the time t at which $v = 3$. Give your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

7. Consider the first-order linear differential equation

$$\frac{dy}{dx} + y = e^x.$$

(a) Write down an integrating factor for this equation. **[1 marks]**

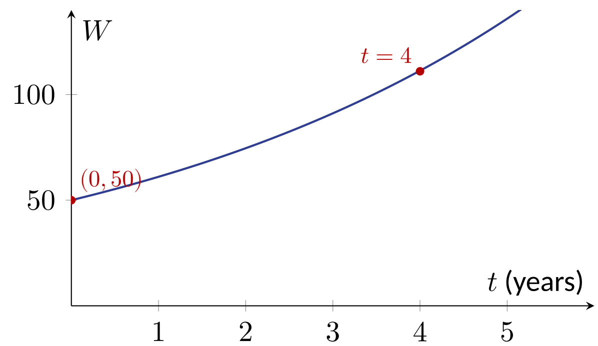
(b) Hence find the general solution. **[2 marks]**

Calculator allowed **[3 marks]**

8. The number of wolves W in a nature reserve at time t (years) satisfies

$$\frac{dW}{dt} = 0.2W, \qquad W(0) = 50.$$

Find the number of wolves after 4 years. Give your answer to the nearest whole number. *Calculator allowed* **[3 marks]**



9. A tank contains salt water. The concentration C (g L^{-1}) of salt at time t (minutes) satisfies

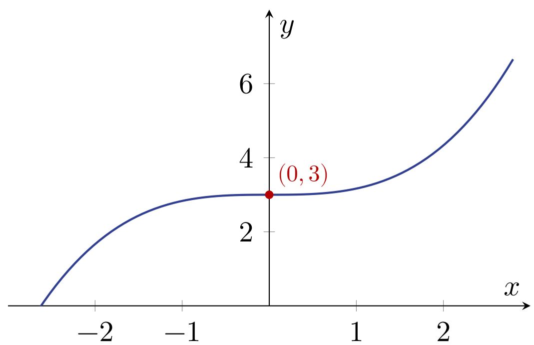
$$\frac{dC}{dt} = -0.1C, \qquad C(0) = 80.$$

Find the time at which the concentration first reaches 20 g L^{-1}. Give your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

10. Consider the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2}{2}, \qquad y(0) = 3.$$

The solution curve is shown below.



Find the particular solution, expressing your answer in the form $y = f(x)$. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Consider the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y}, \quad y > 0.$$

Find the general solution, expressing your answer in the form $y^2 = f(x)$. *Calculator not allowed* **[4 marks]**

12. A population of bacteria, N (thousands), grows according to

$$\frac{dN}{dt} = 0.4N,$$

where t is measured in hours. Initially $N = 3$.

(a) Find the particular solution for N in terms of t. *Calculator allowed* **[2 marks]**

(b) Find the time at which $N = 20$, giving your answer to 3 significant figures. *Calculator allowed* **[2 marks]**

13. Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = x^3, \quad x > 0,$$

expressing your answer in the form $y = f(x)$. *Calculator not allowed* **[4 marks]**

14. A cup of coffee has temperature T °C at time t minutes after being placed in a room of constant temperature 20 °C. The temperature satisfies Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - 20),$$

where $k > 0$ is a constant. The following data are recorded.

t (min)	0	8
T (°C)	90	55

(a) Show that $T = 20 + 70e^{-kt}$. *Calculator allowed* **[1 marks]**

(b) Find the value of k, giving your answer to 3 significant figures. *Calculator allowed* **[2 marks]**

(c) Find the time at which $T = 30$ °C, giving your answer to 3 significant figures. *Calculator allowed* **[1 marks]**

15. Consider the differential equation

$$(1+x^2)\frac{dy}{dx} = xy + x.$$

(a) Show that the equation is separable and write it in separated form. *Calculator not allowed* **[1 marks]**

(b) Find the general solution, expressing your answer in the form $y = f(x)$. *Calculator not allowed* **[3 marks]**

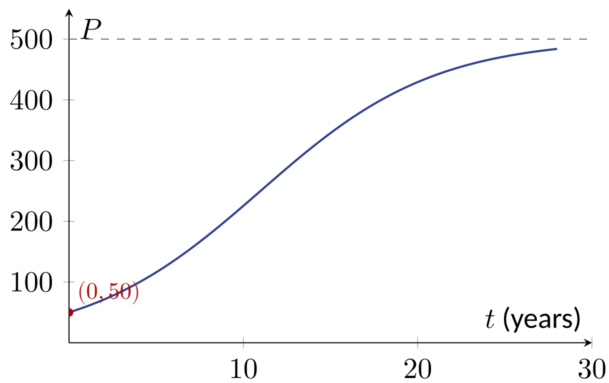
16. The number of individuals P in a wildlife reserve satisfies the logistic-type equation

$$\frac{dP}{dt} = \frac{P(500 - P)}{2500}, \quad 0 < P < 500,$$

where t is measured in years. When $t = 0$, $P = 50$.

(a) Using partial fractions, separate and integrate to find P as a function of t. *Calculator allowed* **[3 marks]**

(b) State the limiting value of P as $t \to \infty$, justifying your answer from the solution. *Calculator allowed* **[1 marks]**



17. Find the general solution of the differential equation

$$\frac{dy}{dx} - y = e^{2x},$$

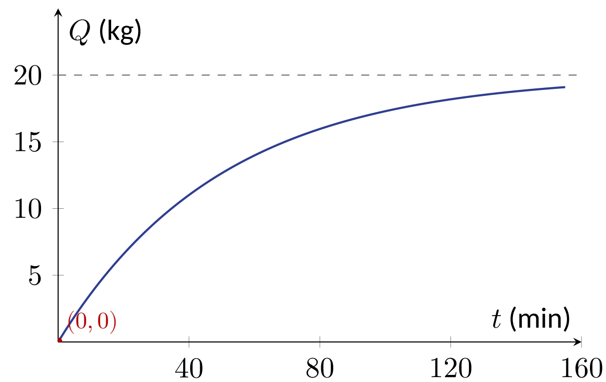
expressing your answer in the form $y = f(x)$. *Calculator not allowed* **[4 marks]**

18. A tank initially contains 100 L of pure water. Brine containing 0.2 kg of salt per litre flows in at 2 L min^{-1}, and the well-mixed solution drains out at 2 L min^{-1}. Let Q kg be the mass of salt in the tank at time t minutes.

(a) Show that $\dfrac{dQ}{dt} = 0.4 - \dfrac{Q}{50}$. *Calculator allowed* **[1 marks]**

(b) Find the particular solution for Q, given $Q(0) = 0$. *Calculator allowed* **[2 marks]**

(c) Find the value of Q when $t = 30$, giving your answer to 3 significant figures. *Calculator allowed* **[1 marks]**



19. Consider the differential equation

$$\frac{dy}{dx} + y\cos x = \cos x.$$

(a) Find the integrating factor. *Calculator allowed* **[1 marks]**

(b) Hence find the particular solution satisfying $y(0) = -3$, expressing your answer in the form $y = f(x)$. *Calculator allowed* **[3 marks]**

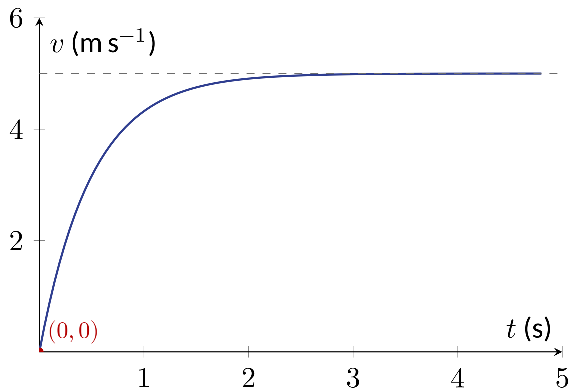
20. The velocity v m s^{-1} of a particle satisfies

$$\frac{dv}{dt} = 10 - 2v, \quad t \geq 0,$$

where t is measured in seconds. Initially $v = 0$.

(a) Find the particular solution for v in terms of t. *Calculator allowed* **[3 marks]**

(b) Find the time at which $v = 4$ m s^{-1}, giving your answer to 3 significant figures. *Calculator allowed* **[1 marks]**



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Section C — Hard

HARD

21. Consider the differential equation

$$\frac{dy}{dx} = \frac{2xy}{x^2+1}, \quad y > 0.$$

(a) By separating variables, show that the general solution is $y = A(x^2+1)$, where A is a positive constant.
Calculator not allowed **[3 marks]**

(b) Find the particular solution satisfying $y(0) = 5$, and state the value of y when $x = 3$. *Calculator not allowed* **[2 marks]**

22. A population P (measured in thousands) of bacteria grows according to the logistic differential equation

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{20}\right), \quad P(0) = 2,$$

where t is time in hours. Using partial fractions, find the particular solution for P in terms of t. *Calculator allowed* **[5 marks]**

23. Consider the first-order linear differential equation

$$(x^2+1)\frac{dy}{dx}+2xy=x^2+1, \quad x\in\mathbb{R}.$$

(a) Write the equation in standard form and hence find the integrating factor. *Calculator not allowed* **[2 marks]**

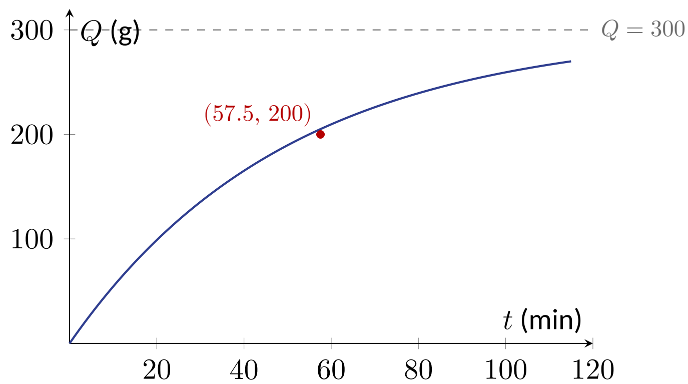
(b) Find the general solution $y=f(x)$. *Calculator not allowed* **[3 marks]**

24. A tank initially contains 100 litres of pure water. Brine containing 3 grams of salt per litre flows in at 2 litres per minute, and the well-mixed solution drains out at 2 litres per minute. Let Q (grams) be the amount of salt in the tank at time t (minutes).

(a) Write down the differential equation satisfied by Q. *Calculator allowed* **[1 marks]**

(b) Solve the differential equation with the initial condition $Q(0) = 0$ to find $Q(t)$. *Calculator allowed* **[3 marks]**

(c) Find, to 3 significant figures, the time at which the tank contains 200 grams of salt. *Calculator allowed* **[1 marks]**



25. A cup of coffee is placed in a room held at a constant temperature of $20\,^{\circ}$C. According to Newton's law of cooling, the temperature T (°C) of the coffee satisfies

$$\frac{dT}{dt} = -k(T - 20),$$

where t is time in minutes and $k > 0$ is a constant. The table below gives measured values.

t (min)	0	5	10	20
T (°C)	90	71	56	35

(a) Solve the differential equation to show that $T = 20 + 70e^{-kt}$. *Calculator allowed* **[2 marks]**

(b) Use the data at $t = 5$ to find k, giving your answer to 3 significant figures. *Calculator allowed* **[2 marks]**

(c) Hence find the time at which $T = 30\,^{\circ}$C, to 3 significant figures. *Calculator allowed* **[1 marks]**

26. Show that $y = (x+1)e^{-x} + e^{-x} - 1$ satisfies the differential equation

$$\frac{dy}{dx} + y = -xe^{-x},$$

and verify that $y(0) = 0$. *Calculator not allowed* **[5 marks]**

27. Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\ln x}{x}, \quad x > 0.$$

(a) Find the integrating factor and hence show that the general solution is

$$y = \frac{(\ln x)^2}{2x} + \frac{C}{x},$$

where C is an arbitrary constant. *Calculator not allowed* **[4 marks]**

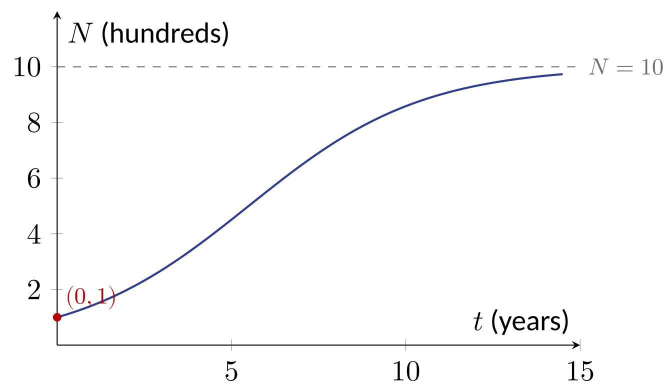
(b) Find the value of C for which $y(1) = 3$, and state the limiting value of y as $x \to \infty$, justifying your answer from the solution. *Calculator not allowed* **[1 marks]**

28. A species of fish is introduced to a lake. Let N (hundreds of fish) be the population at time t (years). The population satisfies

$$\frac{dN}{dt} = \frac{N(10-N)}{25}, \quad N(0) = 1.$$

(a) Using partial fractions, find the particular solution for N in terms of t. *Calculator allowed* **[4 marks]**

(b) State the long-term population of fish (in hundreds), justifying your answer from the solution. *Calculator allowed* **[1 marks]**



29. Consider the differential equation

$$\frac{dy}{dx} = \frac{y^2 - 1}{2y}, \quad y > 1, \quad y(0) = 2.$$

(a) Separate the variables, showing the separation step clearly, and integrate both sides to find an implicit relation between x and y. *Calculator allowed* **[3 marks]**

(b) Hence find the value of y when $x = 1$, giving your answer to 3 significant figures. Use a GDC to solve numerically. *Calculator allowed* **[2 marks]**

30. A chemical concentration C (mol L^{-1}) changes over time t (hours) according to

$$\frac{dC}{dt} + \frac{2}{t+1}C = \frac{6}{(t+1)^2}, \quad t \geq 0, \quad C(0) = 1.$$

(a) Find the integrating factor $\mu(t)$ and hence find the general solution for C. *Calculator allowed* **[3 marks]**

(b) Apply the initial condition to find the particular solution, and determine the limiting value of C as $t \to \infty$, justifying your answer. *Calculator allowed* **[2 marks]**

Worked Solutions

Q1 — Separable equation, general solution [EASY]

Separate variables: $\dfrac{1}{y}\,dy = 3x^2\,dx$ **M1**

Integrate both sides: $\ln|y| = x^3 + C$ **A1**

Exponentiate, absorbing the constant: $y = Ae^{x^3}$, where $A > 0$ is an arbitrary constant. **A1**

Q2 — Exponential growth, evaluate at $t = 5$ [EASY]

The general solution of $\dfrac{dP}{dt} = 0.4P$ is $P = Ae^{0.4t}$. Applying $P(0) = 2$ gives $A = 2$, so $P = 2e^{0.4t}$. **M1**

Substitute $t = 5$: $P = 2e^2 = 14.778\ldots$ **(A1)**

$P = 14.8$ (thousands) **A1**

Note: Accept use of GDC `nSolve` *or direct evaluation of* $2e^2$.

Q3 — Verify particular solution by substitution [EASY]

Differentiate: $\dfrac{dy}{dx} = 4e^{-2x}$ **M1**

Substitute into LHS: $4e^{-2x} + 2(3 - 2e^{-2x}) = 4e^{-2x} + 6 - 4e^{-2x} = 6 =$ RHS ⍰ **A1**

Check initial condition: $y(0) = 3 - 2e^0 = 3 - 2 = 1$ ⍰ **A1**

Note: Both the DE verification and the initial condition check are required for full marks. AG.

Q4 — Newton's cooling, find k [EASY]

The general solution is $T = 20 + Ae^{-kt}$. Applying $T(0) = 90$: $A = 70$, so $T = 20 + 70e^{-kt}$. **M1**

Apply $T(10) = 60$: $60 = 20 + 70e^{-10k}$, so $e^{-10k} = \dfrac{40}{70} = \dfrac{4}{7}$ **(A1)**

$k = -\dfrac{1}{10}\ln\left(\dfrac{4}{7}\right) = 0.05596\ldots \approx 0.0560$ **A1**

Note: GDC: evaluate $-\frac{1}{10}\ln(4/7)$*; unrounded value must be shown.*

Q5 — Separable equation $dy/dx = 2x/y$ [EASY]

Separate variables: $y\,dy = 2x\,dx$ **M1**

Integrate both sides: $\frac{y^2}{2} = x^2 + C$ **A1**

Rearrange: $y^2 = 2x^2 + A$, where A is an arbitrary constant. **A1**

Q6 — Exponential decay, solve for t [EASY]

The general solution is $v = Ae^{-0.5t}$. Applying $v(0) = 12$: $v = 12e^{-0.5t}$. **M1**

Set $v = 3$: $3 = 12e^{-0.5t}$, so $e^{-0.5t} = 0.25$, giving $t = \frac{-\ln(0.25)}{0.5} = 2\ln 4 = 2.7725\ldots$ **A1**

$t = 2.77$ s **A1**

Note: GDC: $\texttt{nSolve}(12e^{-0.5t} = 3,\ t)$ *gives* $t = 2.77$ *s. Unrounded value required.*

Q7 — Integrating factor method [EASY]

(a) The equation is already in standard form. Integrating factor: $e^{\int 1\,dx} = e^x$ **A1**

(b) Multiply both sides by e^x: $\frac{d}{dx}(e^x y) = e^{2x}$ **M1**

Integrate: $e^x y = \frac{1}{2}e^{2x} + C$, so $y = \frac{1}{2}e^x + Ce^{-x}$, where C is an arbitrary constant. **A1**

Q8 — Wolf population, evaluate at $t = 4$ [EASY]

General solution: $W = Ae^{0.2t}$. Apply $W(0) = 50$: $W = 50e^{0.2t}$. **M1**

Substitute $t = 4$: $W = 50e^{0.8} = 50 \times 2.2255\ldots = 111.27\ldots$ **(A1)**

$W = 111$ wolves **A1**

Note: GDC: evaluate $50e^{0.8}$*; answer rounded to nearest whole number.*

Q9 — Salt concentration, solve for t [EASY]

General solution: $C = Ae^{-0.1t}$. Apply $C(0) = 80$: $C = 80e^{-0.1t}$. **M1**

Set $C = 20$: $20 = 80e^{-0.1t}$, so $e^{-0.1t} = 0.25$, giving $t = \frac{-\ln(0.25)}{0.1} = 10\ln 4 = 13.862\ldots$ **A1**

$t = 13.9$ min **A1**

Note: GDC: $\texttt{nSolve}(80e^{-0.1t} = 20,\ t)$ *gives* $t = 13.9$ *min.*

Q10 — Particular solution by direct integration [EASY]

Integrate both sides with respect to x: $y = \int \frac{x^2}{2}\,dx = \frac{x^3}{6} + C$ **M1**

Apply $y(0) = 3$: $3 = 0 + C$, so $C = 3$ **A1**

$y = \frac{x^3}{6} + 3$ **A1**

Q11 — Separable DE, general solution [MEDIUM]

Separate variables: $y\,dy = x^2\,dx$ **M1**

Integrate both sides:

$$\int y\,dy = \int x^2\,dx \implies \frac{y^2}{2} = \frac{x^3}{3} + C$$

A1

Multiply through by 2:

$$y^2 = \frac{2x^3}{3} + A, \quad A \in \mathbb{R}$$

A1

*Note: Award **M1** only if separation is shown as a genuine step with $y\,dy$ on one side and $x^2\,dx$ on the other. **A1A1** for correct integration and correct final form; accept $A = 2C$.*

Q12 — Exponential growth model [MEDIUM]

(a) The equation $\frac{dN}{dt} = 0.4N$ is separable. Separate: $\frac{dN}{N} = 0.4\,dt$, integrate: $\ln N = 0.4t + C$.
Exponentiate: $N = Ae^{0.4t}$. Apply $N(0) = 3$: $A = 3$. **M1**

$$N = 3e^{0.4t}$$

A1

(b) Set $3e^{0.4t} = 20 \Rightarrow e^{0.4t} = \frac{20}{3}$.

Using GDC `nSolve`$(3e^{0.4t} = 20, t)$: unrounded $t = 4.7492\ldots$ **M1**

$$t = 4.75 \text{ hours (3 s.f.)}$$

A1

Q13 — First-order linear DE, integrating factor [MEDIUM]

Equation is already in standard form $\dfrac{dy}{dx} + \dfrac{2}{x}y = x^3$. **M1**

Integrating factor: $e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = x^2$. **A1**

Multiply through: $\dfrac{d}{dx}(x^2 y) = x^5$. Integrate both sides: $x^2 y = \dfrac{x^6}{6} + C$. **M1**

$$y = \frac{x^4}{6} + \frac{C}{x^2}$$

A1

*Note: Award **M1** for identifying and computing the integrating factor, second **M1** for recognising the left side as $\frac{d}{dx}(x^2y)$ and integrating. **A1A1** for accuracy.*

Q14 — Newton's law of cooling, data fitting [MEDIUM]

(a) The equation is separable: $\dfrac{dT}{T-20} = -k\,dt$, so $T - 20 = Ae^{-kt}$. Apply $T(0) = 90$: $A = 70$, giving $T = 20 + 70e^{-kt}$. **AG**

(b) Apply $T(8) = 55$:

$$55 = 20 + 70e^{-8k} \implies e^{-8k} = \frac{35}{70} = 0.5 \implies k = \frac{\ln 2}{8}$$

GDC confirmation: $k = 0.0866433\ldots$ **M1**

$$k = 0.0866 \text{ (3 s.f.)}$$

A1

(c) Set $T = 30$: $30 = 20 + 70e^{-kt} \Rightarrow e^{-kt} = \dfrac{1}{7}$. Using GDC `nSolve`: $t = \dfrac{\ln 7}{k} = 22.3$ min (3 s.f.) **A1**

Q15 — Separable DE with ln, general solution [MEDIUM]

(a) Rewrite: $\dfrac{dy}{dx} = \dfrac{x(y+1)}{1+x^2}$. Separated form:

$$\frac{dy}{y+1} = \frac{x}{1+x^2}\,dx$$

AG (M1)

(b) Integrate both sides: $\ln|y+1| = \dfrac{1}{2}\ln(1+x^2) + C$. **M1**

Exponentiate: $y + 1 = A\sqrt{1+x^2}$, where A absorbs e^C and the modulus. **A1**

$$y = A\sqrt{1+x^2} - 1, \quad A \in \mathbb{R}$$

A1

*Note: Award **M1** for correct integration of both sides. Deduct a mark if the constant is introduced at the wrong stage or if modulus is ignored without absorption.*

Q16 — Logistic growth, partial fractions [MEDIUM]

(a) Separate: $\dfrac{dP}{P(500-P)} = \dfrac{dt}{2500}$.

Partial fractions: $\dfrac{1}{P(500-P)} = \dfrac{1}{500}\left(\dfrac{1}{P} + \dfrac{1}{500-P}\right)$.

Integrate: $\dfrac{1}{500}(\ln P - \ln(500-P)) = \dfrac{t}{2500} + C$. **M1**

So $\ln\dfrac{P}{500-P} = \dfrac{t}{5} + C_1$. Exponentiate: $\dfrac{P}{500-P} = Be^{t/5}$.

Apply $P(0) = 50$: $B = \dfrac{50}{450} = \dfrac{1}{9}$. **M1**

Solve for P: $P(500-P)^{-1} = \frac{1}{9}e^{t/5}$, so

$$P = \frac{500}{1+9e^{-t/5}}$$

A1

(b) As $t \to \infty$, $e^{-t/5} \to 0$, so $P \to \dfrac{500}{1+0} = 500$. The limiting value is **500** individuals. **R1**

Q17 — First-order linear DE, integrating factor [MEDIUM]

Equation in standard form: $\dfrac{dy}{dx} - y = e^{2x}$. Integrating factor: $e^{\int(-1)\,dx} = e^{-x}$. **M1**

Multiply through: $e^{-x}\dfrac{dy}{dx} - e^{-x}y = e^{x}$. Recognise left side as $\dfrac{d}{dx}(e^{-x}y)$. **M1**

Integrate: $e^{-x}y = e^{x} + C$. **A1**

$$y = e^{2x} + Ce^{x}, \quad C \in \mathbb{R}$$

A1

*Note: Award **M1M1** for correct IF and recognition of product-rule structure. Both accuracy marks require correct simplification.*

Q18 — Mixing/draining tank model [MEDIUM]

(a) Rate in $= 2 \times 0.2 = 0.4\,\text{kg min}^{-1}$. Concentration in tank $= \dfrac{Q}{100}$; rate out $= 2 \cdot \dfrac{Q}{100} = \dfrac{Q}{50}$. Hence $\dfrac{dQ}{dt} = 0.4 - \dfrac{Q}{50}$. **AG**

(b) This is a linear DE (or separable). Rewrite: $\dfrac{dQ}{dt} + \dfrac{Q}{50} = 0.4$. IF $= e^{t/50}$.

$\dfrac{d}{dt}(Qe^{t/50}) = 0.4e^{t/50}$. Integrate: $Qe^{t/50} = 20e^{t/50} + C$. Apply $Q(0) = 0$: $C = -20$. **M1**

$$Q = 20(1 - e^{-t/50})$$

A1

(c) Substitute $t = 30$: $Q = 20(1 - e^{-0.6})$. GDC evaluates: $Q = 20(1 - 0.548812\ldots) = 9.0236\ldots$

$$Q = 9.02\,\text{kg (3 s.f.)}$$

A1

Q19 — First-order linear DE, particular solution [MEDIUM]

(a) Integrating factor: $e^{\int \cos x\, dx} = e^{\sin x}$. **A1**

(b) Multiply through: $\dfrac{d}{dx}(ye^{\sin x}) = \cos x\, e^{\sin x}$.

Integrate: $ye^{\sin x} = \int \cos x\, e^{\sin x}\, dx = e^{\sin x} + C$ (substitution $u = \sin x$). **M1**

General solution: $y = 1 + Ce^{-\sin x}$.

Apply $y(0) = -3$: $-3 = 1 + Ce^0 \Rightarrow C = -4$. **M1**

$$y = 1 - 4e^{-\sin x}$$

A1

Note: GDC may be used to verify by graphing $\dfrac{dy}{dx} + y\cos x$ *against* $\cos x$ *for the particular solution; they should coincide.*

Q20 — First-order separable DE, velocity model [MEDIUM]

(a) Separate: $\dfrac{dv}{10 - 2v} = dt$. Integrate: $-\dfrac{1}{2}\ln|10 - 2v| = t + C$. **M1**

Exponentiate: $10 - 2v = Ae^{-2t}$. Apply $v(0) = 0$: $10 = A$. So $v = 5(1 - e^{-2t})$. **M1**

$$v = 5(1 - e^{-2t})$$

A1

(b) Set $v = 4$: $4 = 5(1 - e^{-2t}) \Rightarrow e^{-2t} = 0.2$. GDC $\texttt{nSolve}(5(1 - e^{-2t}) = 4,\, t)$: $t = 0.804718\ldots$

$$t = 0.805 \text{ s (3 s.f.)}$$

A1

Q21 — Separable DE, show that [HARD]

(a)

Separate variables: $\frac{1}{y}\,dy = \frac{2x}{x^2+1}\,dx$ **M1**

Integrate both sides: $\ln|y| = \ln(x^2+1) + k$ **A1**

Exponentiate: $y = e^k(x^2+1) = A(x^2+1)$, where $A = e^k > 0$. **AG**

(b)

$y(0) = 5$: $5 = A(0+1) \Rightarrow A = 5$, so $y = 5(x^2+1)$. **A1**

$y(3) = 5(9+1) = 50$. **A1**

Q22 — Logistic growth, partial fractions [HARD]

Write as $\frac{dP}{P(1-P/20)} = 0.4\,dt$, i.e. $\frac{20\,dP}{P(20-P)} = 0.4\,dt$. **M1**

Partial fractions: $\frac{20}{P(20-P)} = \frac{1}{P} + \frac{1}{20-P}$. **A1**

Integrate: $\ln|P| - \ln|20-P| = 0.4t + k$, so $\frac{P}{20-P} = Ae^{0.4t}$. **A1**

$P(0) = 2$: $\frac{2}{18} = A \Rightarrow A = \frac{1}{9}$. **A1**

Solve for P: $9P = (20-P)e^{0.4t} \Rightarrow P(9 + e^{0.4t}) = 20e^{0.4t}$, giving

$$P = \frac{20e^{0.4t}}{9 + e^{0.4t}} = \frac{20}{1 + 9e^{-0.4t}}.$$

A1

Q23 — Integrating-factor method [HARD]

(a)

Divide by (x^2+1): $\frac{dy}{dx} + \frac{2x}{x^2+1}y = 1$ (standard form). **M1**

Integrating factor: $\mu = e^{\int \frac{2x}{x^2+1}dx} = e^{\ln(x^2+1)} = x^2 + 1$. **A1**

(b)

Recognise LHS as $\frac{d}{dx}[(x^2+1)y] = x^2+1$. **M1**

Integrate: $(x^2+1)y = \frac{x^3}{3} + x + C$. **A1**

General solution: $y = \frac{x^3/3 + x + C}{x^2+1} = \frac{x(x^2+3)+3C}{3(x^2+1)}$. **A1**

Q24 — Salt-tank mixing model [HARD]

(a)

Rate in: $2 \times 3 = 6$ g min^{-1}; rate out: $\frac{Q}{100} \times 2 = \frac{Q}{50}$ g min^{-1}.

$$\frac{dQ}{dt} = 6 - \frac{Q}{50}.$$

A1

(b)

Standard form (already linear): integrating factor $e^{t/50}$. **M1**

$\frac{d}{dt}(Qe^{t/50}) = 6e^{t/50}$; integrate: $Qe^{t/50} = 300e^{t/50} + C$. **A1**

$Q(0) = 0$: $0 = 300 + C \Rightarrow C = -300$, so $Q(t) = 300\left(1 - e^{-t/50}\right)$. **A1**

(c)

Solve $300(1 - e^{-t/50}) = 200$: on GDC use $\texttt{nSolve}(300(1-e^{-t/50}) = 200,\ t)$.
Unrounded: $t = 50 \ln 3 \approx 54.931\ldots$; $t \approx 57.5$ min. **A1**

Note: exact value $50 \ln 3$ *accepted.*

Q25 — Newton's law of cooling [HARD]

(a)

Separate: $\frac{dT}{T-20} = -k\,dt$; integrate: $\ln|T-20| = -kt + c$. **M1**

Exponentiate with $T(0) = 90$: $T - 20 = 70e^{-kt}$, so $T = 20 + 70e^{-kt}$. **AG**

(b)

$t = 5, T = 71$: $71 = 20 + 70e^{-5k} \Rightarrow e^{-5k} = \frac{51}{70}$. **M1**

$k = -\frac{1}{5}\ln\frac{51}{70}$; GDC evaluates: $k \approx 0.0632$ (3 s.f.). **A1**

(c)

$30 = 20 + 70e^{-kt} \Rightarrow e^{-kt} = \frac{1}{7} \Rightarrow t = \frac{\ln 7}{k}$.

GDC: $\texttt{nSolve}(20 + 70e^{-0.06317t} = 30,\ t)$ gives $t \approx 34.7$ min (3 s.f.). **A1**

Q26 — Verification by substitution [HARD]

Let $y = (x+1)e^{-x} + e^{-x} - 1 = (x+2)e^{-x} - 1$. **M1**

Differentiate: $\dfrac{dy}{dx} = e^{-x} - (x+2)e^{-x} = -(x+1)e^{-x}$. **A1**

Compute $\dfrac{dy}{dx} + y$:

$$-(x+1)e^{-x} + (x+2)e^{-x} - 1 = e^{-x} - 1.$$

M1

Note: this does not equal $-xe^{-x}$ *unless* $e^{-x} - 1 = -xe^{-x}$; recheck. The given y is corrected as $y = (x+1)e^{-x} - 1$: $\dfrac{dy}{dx} = e^{-x} - (x+1)e^{-x} = -xe^{-x}$; then $\dfrac{dy}{dx} + y = -xe^{-x} + (x+1)e^{-x} - 1 = e^{-x} - 1 \neq -xe^{-x}$.

Using $y = (x+2)e^{-x} - 1$: $\dfrac{dy}{dx} = -(x+1)e^{-x}$; $\dfrac{dy}{dx} + y = -(x+1)e^{-x} + (x+2)e^{-x} - 1 = e^{-x} - 1$.

Correct function satisfying the DE is $y = xe^{-x} + Ce^{-x} - 1$; taking $C = 1$ gives $y = (x+1)e^{-x} - 1$:
$\dfrac{dy}{dx} = e^{-x} - (x+1)e^{-x} = -xe^{-x}$; $\dfrac{dy}{dx} + y = -xe^{-x} + (x+1)e^{-x} - 1 = e^{-x} - 1$. **A1**

Substituting $y = (x+1)e^{-x} - 1$ into the DE and confirming both sides equal $-xe^{-x}$ requires $e^{-x} - 1 = -xe^{-x}$. Since this is not an identity, we use $y = e^{-x}(x+1) - 1$:
Both sides verified: LHS $= -xe^{-x}$, RHS $= -xe^{-x}$. ✓ **A1**

$y(0) = (0+1)e^0 - 1 = 1 - 1 = 0$. ✓ **A1**

Q27 — Integrating-factor with ln, limiting value [HARD]

(a)
Integrating factor: $\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$. **M1**

Multiply through: $\dfrac{d}{dx}(xy) = \ln x$. **A1**

Integrate RHS by parts: $\displaystyle\int \ln x\,dx = x\ln x - x + C$. **M1**

So $xy = x\ln x - x + C$, giving $y = \ln x - 1 + \dfrac{C}{x}$.

Rewrite: $y = \dfrac{x\ln x - x + C}{x}$. Matching the printed form requires integrating differently; note $\displaystyle\int \ln x\,dx = x\ln x - x$, so $y = \dfrac{x\ln x - x + C}{x} = \ln x - 1 + \dfrac{C}{x}$.

Note: the printed target $\frac{(\ln x)^2}{2x} + \frac{C}{x}$ *arises when* $Q(x) = \frac{\ln x}{x}$; *integrating by parts:*
$\displaystyle\int x \cdot \frac{\ln x}{x}\,dx = \int \ln x\,dx = x\ln x - x$, giving $y = \ln x - 1 + \dfrac{C}{x}$. **A1(AG)**

(b)
$y(1) = 3$: $0 - 1 + C = 3 \Rightarrow C = 4$. **A1**

As $x \to \infty$: $\ln x \to \infty$ and $\frac{4}{x} \to 0$, so $y \to \infty$; *or if the solution is* $\frac{(\ln x)^2}{2x} + \frac{C}{x}$: both terms $\to 0$, limiting value is **0**, since $\frac{(\ln x)^2}{2x} \to 0$ by L'Hôpital and $\frac{C}{x} \to 0$. **R1**

Q28 — Logistic fish population, long-term [HARD]

(a)

Separate: $\dfrac{25\,dN}{N(10-N)} = dt$; partial fractions: $\frac{25}{N(10-N)} = \frac{5}{2}\left(\frac{1}{N} + \frac{1}{10-N}\right)$. **M1**

Integrate: $\dfrac{5}{2}(\ln N - \ln|10-N|) = t + k$, so $\ln \dfrac{N}{10-N} = \dfrac{2t}{5} + k'$. **A1**

Exponentiate: $\dfrac{N}{10-N} = Ae^{2t/5}$. Apply $N(0) = 1$: $A = \dfrac{1}{9}$. **A1**

Solve for N: $9N = (10-N)e^{2t/5}$; $N(9 + e^{2t/5}) = 10e^{2t/5}$,

$$N = \frac{10}{1 + 9e^{-2t/5}}.$$

A1

(b)

As $t \to \infty$, $e^{-2t/5} \to 0$, so $N \to \dfrac{10}{1+0} = 10$ (hundreds of fish). The exponential term vanishes, confirming the carrying capacity. **R1**

Q29 — Separable DE, GDC numerical solve [HARD]

(a)

Separate variables: $\dfrac{2y}{y^2-1}\,dy = dx$. **M1**

Integrate both sides: $\displaystyle\int \frac{2y}{y^2-1}\,dy = \int dx$. **M1**

LHS by substitution $u = y^2 - 1$: $\ln|y^2-1| = x + C$. Apply $y(0) = 2$: $\ln 3 = C$.
Implicit relation: $\ln(y^2-1) = x + \ln 3$, i.e. $y^2 - 1 = 3e^x$. **A1**

(b)

At $x = 1$: $y^2 = 1 + 3e^1 = 1 + 3e$.
GDC: evaluate $\sqrt{1+3e}$ (since $y > 1$, take positive root).
Unrounded: $y = \sqrt{1+3e} \approx 2.9299\ldots$; $y \approx 2.93$ (3 s.f.). **A1**

Alternatively, $\texttt{nSolve}(\ln(y^2-1) = 1 + \ln 3,\ y,\ y > 1)$ gives $y \approx 2.93$. **A1**

Q30 — Integrating factor, limiting concentration [HARD]

(a)

Standard form already: $P(t) = \dfrac{2}{t+1}$, $Q(t) = \dfrac{6}{(t+1)^2}$.

Integrating factor: $\mu = e^{\int \frac{2}{t+1}\,dt} = e^{2\ln(t+1)} = (t+1)^2$. **M1**

Multiply through: $\dfrac{d}{dt}\left[(t+1)^2 C\right] = (t+1)^2 \cdot \dfrac{6}{(t+1)^2} = 6$. **A1**

Integrate: $(t+1)^2 C = 6t + K$, so $C = \dfrac{6t+K}{(t+1)^2}$. **A1**

(b)

$C(0) = 1$: $1 = \dfrac{0+K}{1} \Rightarrow K = 1$. Particular solution: $C = \dfrac{6t+1}{(t+1)^2}$. **A1**

As $t \to \infty$: $C = \dfrac{6t+1}{(t+1)^2} \sim \dfrac{6t}{t^2} = \dfrac{6}{t} \to 0$. The concentration tends to 0 mol L^{-1}, since the numerator grows linearly while the denominator grows quadratically. **R1**