

IB Math AA HL — Topic 1 Exponentials & Logs

Introduction to Logarithms · Laws of Logarithms · Solving Exponential Equations

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name: _____ Date: _____
Score: _____/120



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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.



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Key Formulae

- Definition: $\log_a x = y \iff a^y = x$, where $a > 0$, $a \neq 1$, $x > 0$.
- Product law: $\log_a(xy) = \log_a x + \log_a y$
- Quotient law: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- Power law: $\log_a(x^n) = n \log_a x$
- Change of base: $\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$
- Key values: $\log_a 1 = 0$; $\log_a a = 1$; $a^{\log_a x} = x$; $\log_a(a^x) = x$
- Solving $a^x = b$: $x = \frac{\ln b}{\ln a} = \log_a b$

GDC Tips

- **Evaluating logs.** Use $\log()$ for $\log_{10}$ and $\ln()$ for $\log_e$; enter $\log_a b$ as $\log(b)/\log(a)$ or use the logBASE template on TI-Nspire (Catalog $\rightarrow$ log(b,a)) and on Casio fx-CG50 via the MATH log menu.
- **Solving exponential equations.** Store the equation as $f(x) = a^x - b$ and use Solve or the Intersection tool; confirm the answer satisfies the original equation.
- **Finding crossing times.** Use Graph two models and Intersection, or nSolve on TI-Nspire / SOLVE on Casio to find the first time one model exceeds another.
- **Log-linear regression.** Enter $(\ln x, \ln y)$ or $(\ln x, y)$ as a data list and run LinReg to find gradient and intercept; re-interpret in terms of the original model parameters.
- **Rounding.** Give decimal answers to 3 significant figures unless the question specifies otherwise; keep at least 5 s.f. in intermediate steps to avoid rounding errors.
- **Checking rejected roots.** After solving on the GDC, substitute back to verify the argument of every logarithm is strictly positive; discard any solution that gives a non-positive argument.

Common Mistakes to Avoid

1. **Splitting the argument of a sum.** $\log_a(x + y) \neq \log_a x + \log_a y$; the product law applies to a product inside the logarithm, not a sum.
2. **Misapplying the power law.** $(\log_a x)^n \neq n \log_a x$; the power law only moves an exponent that is on the argument x^n , not an exponent on the entire logarithm.
3. **Mixing bases without converting.** Log laws hold only within a single base; always change base explicitly before combining $\log_2 x$ with $\log_3 x$, for example.
4. **Forgetting to reject non-positive arguments.** Every solution of a logarithmic equation must be checked: if it makes any argument ≤ 0 , it must be rejected with a written reason.
5. **Errors in change of base direction.** Students often write $\log_a b = \frac{\ln a}{\ln b}$ (inverted); remember the base goes in the denominator: $\log_a b = \frac{\ln b}{\ln a}$.
6. **Incorrect isolation when solving $a^x = b$.** Applying log to only one side, or writing $x = \log_a(-b)$ when $b < 0$; the equation $a^x = b$ has no real solution when $b \leq 0$, and both sides must be treated simultaneously.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)



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1. **EASY** No Calculator

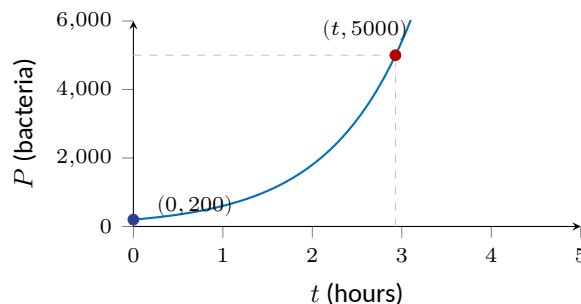
[3 marks]

Write down the value of $\log_3 81$.

2. **EASY** Calculator

[3 marks]

A population of bacteria, P , is modelled by $P(t) = 200 \times 3^t$, where $t \geq 0$ is the time in hours. Find the value of t when $P = 5000$, giving your answer to 3 significant figures.



3. **EASY** No Calculator

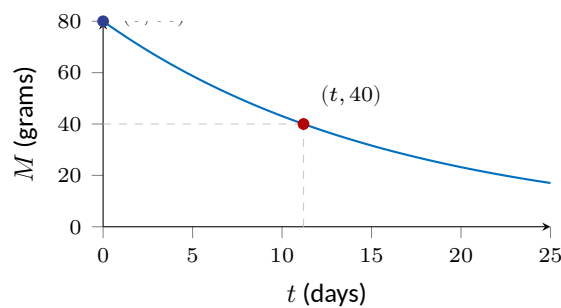
[3 marks]

Write $\log_a 5 + 2 \log_a 3 - \log_a 45$ as a single logarithm in its simplest form, where $a \in \mathbb{R}^+$, $a \neq 1$.

4. **EASY** Calculator

[3 marks]

A radioactive substance has mass M g at time t days, modelled by $M(t) = 80 \times (0.94)^t$, where $t \geq 0$. Find the value of t when $M = 40$, giving your answer to 3 significant figures.



5. **EASY** No Calculator

[3 marks]

Solve the equation $\log_2(x+6) + \log_2(x-2) = 5$, where $x \in \mathbb{R}$. State any solution that must be rejected, giving a reason.

6. **EASY** No Calculator

[3 marks]

Evaluate $\log_5 125 - \log_5 25$. Write down your answer without a calculator.

7. **EASY** Calculator

[3 marks]

The table shows the concentration C (mg L^{-1}) of a drug in a patient's bloodstream at various times t (hours) after an injection. It is believed that $C = a \times b^t$ for constants $a, b \in \mathbb{R}^+$. Taking $\log_{10}$ of both sides, find the values of $\log_{10} a$ and $\log_{10} b$, hence state the values of a and b to 3 significant figures.

t (hours)	C (mg L^{-1})	$\log_{10} C$
0	50.0	1.699
2	31.6	1.500
4	19.9	1.299
6	12.6	1.100

8. **EASY** Calculator

[3 marks]

An investment of \$2 000 grows at a fixed annual rate. Its value V dollars after n years is given by $V = 2000 \times (1.06)^n$, where $n \in \mathbb{Z}^+$. Find the smallest integer value of n for which V first exceeds \$3 000.

9. **EASY** No Calculator

[3 marks]

Solve the equation $2^{2x} - 5 \times 2^x + 4 = 0$, where $x \in \mathbb{R}$.

10. **EASY** Calculator

[3 marks]

A cup of tea cools so that its temperature T °C at time t minutes satisfies $T(t) = 20 + 65 \times e^{-0.04t}$, where $t \geq 0$. Find the time at which $T = 50$ °C, giving your answer to 3 significant figures.

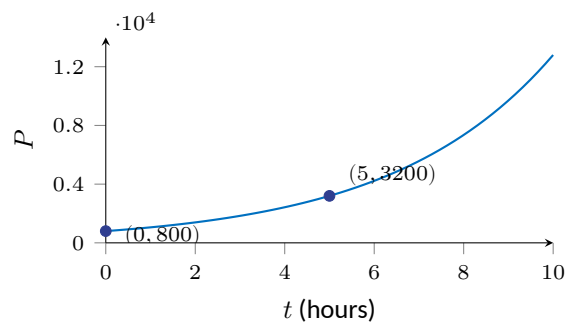
Section B — Medium **MEDIUM** (10 questions $\times$ 4 marks = 40 marks)



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11. **MEDIUM** **No Calculator** [4 marks]
Solve the equation $\log_3(2x - 1) + \log_3(x + 4) = 3$, giving all solutions and stating any that must be rejected with a reason.

12. **MEDIUM** **Calculator** [4 marks]
A colony of bacteria has an initial population of 800. The population P at time t hours satisfies $P = 800e^{kt}$, where $k \in \mathbb{R}$. After 5 hours the population is 3 200. Find the value of k , giving your answer to 3 significant figures.



13. MEDIUM No Calculator

[4 marks]

Given that $\log_a 3 = p$ and $\log_a 5 = q$, where $a \in \mathbb{R}^+$, $a \neq 1$, express $\log_a \left(\frac{9\sqrt{5}}{a^2} \right)$ in terms of p and q .

14. MEDIUM Calculator

[4 marks]

The number of subscribers N to an online service at time t months after launch is modelled by $N = 500 \times b^t$, where $b \in \mathbb{R}^+$. After 12 months there are 14 000 subscribers.

(a) Find the value of b , giving your answer to 3 significant figures. [2 marks]

(b) Find the number of complete months after launch at which N first exceeds 50 000. [2 marks]

15. MEDIUM Calculator

[4 marks]

Solve the simultaneous equations

$$2^x = 5^{y+1}, \quad x + 3y = 7,$$

where $x, y \in \mathbb{R}$. Give your answers to 3 significant figures.

16. **MEDIUM** No Calculator

[4 marks]

Show that $\log_4 x = \frac{\ln x}{2 \ln 2}$, and hence solve the equation $\log_4 x + \log_4(x - 6) = 2$, rejecting any invalid solution with a reason.

17. **MEDIUM** Calculator

[4 marks]

A radioactive substance decays according to $M = M_0 e^{-\lambda t}$, where M is the mass in grams at time t years, M_0 is the initial mass in grams, and $\lambda \in \mathbb{R}^+$. The half-life of the substance is 24 years. A sample has an initial mass of 200 g.

(a) Find the value of λ , giving your answer to 3 significant figures. [2 marks]

(b) Find the time, to the nearest year, at which the mass first falls below 15 g. [2 marks]

18. **MEDIUM** Calculator

[4 marks]

The table below gives values of x and y that satisfy $y = a \cdot x^n$, where $a, n \in \mathbb{R}$.

x	2	4	8	16
y	5.66	16.00	45.25	128.00

By taking logarithms, find the values of a and n . Give your answers to 3 significant figures.

19. **MEDIUM** **No Calculator**

[4 marks]

Solve the equation $2(\log_5 x)^2 - 5\log_5 x - 3 = 0$, $x \in \mathbb{R}^+$, giving all solutions and stating any that must be rejected with a reason.

20. **MEDIUM** **Calculator**

[4 marks]

An investment of \$P grows at a nominal annual rate of 4.2%, compounded continuously, so its value after t years is $V = Pe^{0.042t}$, where $V, P \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$. Find the number of complete years for the investment to treble in value.

Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)

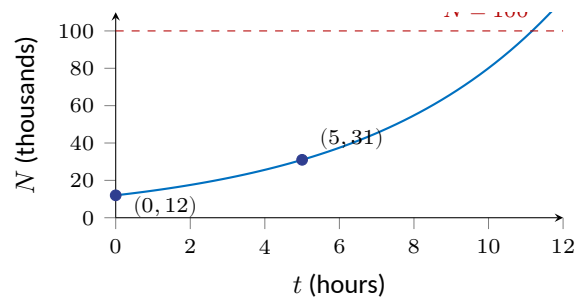


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21. **HARD** **Calculator**

[5 marks]

A population of bacteria, N (in thousands), is modelled by $N(t) = A \cdot e^{kt}$, where $t \geq 0$ is the time in hours and $A, k \in \mathbb{R}^+$. At $t = 0$, the population is 12 000. At $t = 5$, the population is 31 000. Find the time, to the nearest minute, at which the population first reaches 100 000.



22. **HARD** **No Calculator**

[5 marks]

Let $\log_3 p = a$ and $\log_3 q = b$, where $p, q \in \mathbb{R}^+$. Express $\log_3 \left(\frac{9p^3}{\sqrt{q}} \right)$ in terms of a and b . Hence find the value of $\log_3 \left(\frac{9p^3}{\sqrt{q}} \right)$ when $p = 9$ and $q = 81$.

23. **HARD** No Calculator

[5 marks]

Solve the equation $\log_2(3x - 1) + \log_2(x - 2) = 4$, justifying the rejection of any solution that is not valid.

24. **HARD** Calculator

[5 marks]

The table below shows the length L (cm) and the mass m (g) of five specimens of a particular fish species. It is believed that $m = aL^n$ for constants $a, n \in \mathbb{R}^+$.

Specimen	L (cm)	m (g)
1	10	18
2	15	57
3	20	136
4	25	265
5	30	459

By taking logarithms of both sides of $m = aL^n$, show that $\log_{10} m = n \log_{10} L + \log_{10} a$. Using linear regression on $(\log_{10} L, \log_{10} m)$, find the values of a and n to 3 significant figures, and estimate the mass of a specimen of length 22 cm.

25. **HARD** No Calculator

[5 marks]

Show that $\log_a b \cdot \log_b c \cdot \log_c a = 1$ for $a, b, c \in \mathbb{R}^+ \setminus \{1\}$. Hence evaluate $\log_5 7 \cdot \log_7 11 \cdot \log_{11} 25$ exactly.

26. **HARD** Calculator

[5 marks]

A radioactive substance has mass $M(t) = M_0 \cdot 2^{-t/h}$ grams, where $t \geq 0$ is time in years, $M_0 > 0$ is the initial mass, and $h > 0$ is the half-life in years. The initial mass is 500 g. After 8 years the mass is 320 g. Find the half-life h , to 3 significant figures. Find the first integer value of t for which the mass falls below 50 g.

27. **HARD** Calculator

[5 marks]

Solve the simultaneous equations

$$3^x = 9^{y+1}, \quad \log_3 x + \log_3(2y) = 3,$$

where $x, y \in \mathbb{R}^+$. Justify the rejection of any solution that is not valid.

28. **HARD** No Calculator

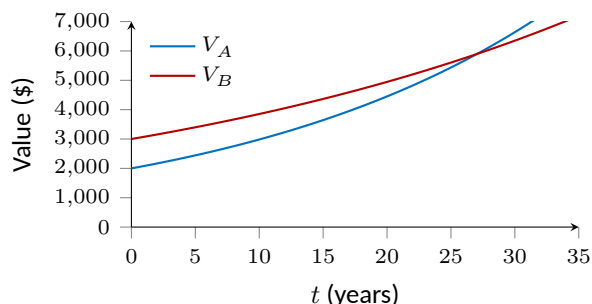
[5 marks]

Solve the equation $4^x - 5 \cdot 2^x + 4 = 0$, giving your answers in exact form.

29. **HARD** Calculator

[5 marks]

Two investment accounts both start at time $t = 0$ (years). Account A has value $V_A(t) = 2000 \cdot e^{0.04t}$ dollars and Account B has value $V_B(t) = 3000 \cdot e^{0.025t}$ dollars, where $t \geq 0$. Find the value of t , to the nearest month, at which Account A first overtakes Account B.



30. **HARD** Calculator

[5 marks]

The equation $\log_5(2x + 3) = 2 - \log_5(x - 1)$ has solutions in $\mathbb{R}$. Find all solutions, stating the domain restriction on x and justifying the rejection of any invalid solution.

Mark Schemes

Q1 — Evaluate a Logarithm Exactly **EASY**

$$\log_3 81 = \log_3 3^4 \quad \text{M1}$$

$$= 4 \quad \text{A1}$$

Note: Full marks for a correct unsupported answer. Award A1 for the final answer 4.

Q2 — Exponential Growth Model Threshold Crossing **EASY**

$$\text{Setting } 200 \times 3^t = 5000 \text{ gives } 3^t = 25 \quad \text{M1}$$

$$\text{Taking logarithms: } t = \frac{\log 25}{\log 3} \quad \text{(M1)}$$

$$\text{GDC gives } t = 2.92944 \dots \approx 2.93 \text{ hours} \quad \text{A1}$$

Note: Accept $\ln 25 / \ln 3$ or $\log_3 25$. Award A1 for 2.93 only.

Q3 — Combine Using Log Laws **EASY**

$$\log_a 5 + 2 \log_a 3 - \log_a 45 = \log_a 5 + \log_a 9 - \log_a 45 \quad \text{M1}$$

$$= \log_a \left(\frac{5 \times 9}{45} \right) = \log_a 1 \quad \text{A1}$$

$$= 0 \quad \text{A1}$$

Note: Final answer of 0 earns both A marks only if supported by correct log-law working.

Q4 — Radioactive Decay Half-Life **EASY**

$$\text{Setting } 80 \times (0.94)^t = 40 \text{ gives } (0.94)^t = 0.5 \quad \text{M1}$$

$$\text{Taking logarithms: } t = \frac{\log 0.5}{\log 0.94} \quad \text{(M1)}$$

$$\text{GDC gives } t = 11.2019 \dots \approx 11.2 \text{ days} \quad \text{A1}$$

Note: Accept equivalent forms using $\ln$. Award A1 for 11.2 only.

Q5 — Logarithmic Equation Root Rejection **EASY**

$$\log_2 ((x+6)(x-2)) = 5 \Rightarrow (x+6)(x-2) = 32 \quad \text{M1}$$

$$x^2 + 4x - 12 = 32 \Rightarrow x^2 + 4x - 44 = 0 \quad \text{A1}$$

This quadratic does not factor over the integers, so use the quadratic formula:

$$x = \frac{-4 \pm \sqrt{16 + 176}}{2} = \frac{-4 \pm \sqrt{192}}{2} = -2 \pm 4\sqrt{3} \quad \text{M1}$$

$$\text{so } x = 4\sqrt{3} - 2 \approx 4.93 \text{ or } x = -4\sqrt{3} - 2 \approx -8.93 \quad \text{A1}$$

$$x = -4\sqrt{3} - 2 \text{ is rejected since } x - 2 < 0, \text{ making } \log_2(x - 2) \text{ undefined.} \quad \text{R1}$$

$$\text{Hence } x = 4\sqrt{3} - 2 \approx 4.93.$$

Note: Back-substitution confirms $\log_2(4\sqrt{3} + 4) + \log_2(4\sqrt{3} - 4) = 5$.

Q6 — Evaluate Using Quotient Law EASY

$$\log_5 125 - \log_5 25 = \log_5 \left(\frac{125}{25} \right) = \log_5 5 \quad \text{M1}$$

$$= 1 \quad \text{A1}$$

Note: Award M1 for correct application of the quotient law or for evaluating each term separately ($3 - 2$).
Final answer 1 earns A1. Award the third mark only if exact working is shown without a calculator.

Q7 — Log-Linear Fit Gradient and Intercept EASY

$$\text{From the table, } \log_{10} C = \log_{10} a + t \log_{10} b; \text{ the gradient is } \frac{1.100 - 1.699}{6 - 0} = \frac{-0.599}{6} \quad \text{M1}$$

$$\log_{10} b = -0.09983 \dots \approx -0.0998, \text{ so } b = 10^{-0.09983} \approx 0.795 \text{ (3 s.f.)} \quad \text{A1}$$

$$\log_{10} a = 1.699, \text{ so } a = 10^{1.699} \approx 50.0 \text{ (3 s.f.)} \quad \text{A1}$$

Note: Award M1 for correct identification of gradient as $\log_{10} b$ and intercept as $\log_{10} a$.

Q8 — Investment Model First Integer Threshold EASY

$$\text{Require } 2000 \times (1.06)^n > 3000, \text{ i.e. } (1.06)^n > 1.5 \quad \text{M1}$$

$$\text{GDC or logarithms: } n > \frac{\log 1.5}{\log 1.06} = 6.9585 \dots \quad \text{(M1)}$$

$$\text{Smallest integer } n = 7 \quad \text{A1}$$

Note: Candidates may verify by substitution: $V(7) = 3187.70 \dots > 3000$, $V(6) = 2837.03 \dots < 3000$.

Q9 — Quadratic in Base 2 Exact Roots EASY

$$\text{Let } u = 2^x; \text{ the equation becomes } u^2 - 5u + 4 = 0 \quad \text{M1}$$

$$(u - 1)(u - 4) = 0 \Rightarrow u = 1 \text{ or } u = 4 \quad \text{(A1)}$$

$$2^x = 1 \Rightarrow x = 0; \quad 2^x = 4 \Rightarrow x = 2 \quad \text{A1}$$

Note: Both values $x = 0$ and $x = 2$ required for the final A1. No rejection needed as $2^x > 0$ for all real x .

Q10 — Cooling Model Time to Reach Threshold **EASY**

$$\text{Setting } 20 + 65e^{-0.04t} = 50 \text{ gives } e^{-0.04t} = \frac{30}{65} = \frac{6}{13} \quad \text{M1}$$

$$-0.04t = \ln\left(\frac{6}{13}\right) \Rightarrow t = \frac{-\ln(6/13)}{0.04} \quad \text{(M1)}$$

$$\text{GDC gives } t = 19.3297 \dots \approx 19.3 \text{ minutes} \quad \text{A1}$$

Note: Unrounded value 19.3297 ... must be seen before the rounded answer for full marks.

Q11 — Logarithmic Equation with Rejected Root **MEDIUM**

$$\text{Using the product law: } \log_3[(2x - 1)(x + 4)] = 3 \quad \text{M1}$$

$$(2x - 1)(x + 4) = 27 \Rightarrow 2x^2 + 7x - 4 = 27 \Rightarrow 2x^2 + 7x - 31 = 0 \quad \text{A1}$$

$$x = \frac{-7 \pm \sqrt{49 + 248}}{4} = \frac{-7 \pm \sqrt{297}}{4}, \text{ giving } x \approx 2.56 \text{ or } x \approx -6.06 \quad \text{A1}$$

$$\text{Reject } x \approx -6.06 \text{ since } 2x - 1 < 0, \text{ making } \log_3(2x - 1) \text{ undefined; the only solution is } x = \frac{-7 + \sqrt{297}}{4} \approx 2.56. \quad \text{R1}$$

Q12 — Exponential Growth Model Find Rate **MEDIUM**

$$\text{Substituting } P = 3200, t = 5: 3200 = 800e^{5k} \quad \text{M1}$$

$$e^{5k} = 4 \Rightarrow 5k = \ln 4 \quad \text{M1}$$

$$k = \frac{\ln 4}{5} = 0.277258 \dots \quad \text{(A1)}$$

$$k = 0.277 \text{ (3 s.f.)} \quad \text{A1}$$

Q13 — Express Log in Terms of Given Logs **MEDIUM**

$$\log_a \left(\frac{9\sqrt{5}}{a^2} \right) = \log_a 9 + \log_a 5^{1/2} - \log_a a^2 \quad \text{M1}$$

$$= \log_a 3^2 + \frac{1}{2} \log_a 5 - 2 \log_a a \quad \text{M1}$$

$$= 2 \log_a 3 + \frac{1}{2} \log_a 5 - 2 \quad \text{A1}$$

$$= 2p + q/2 - 2 \quad \text{A1}$$

Q14 — Exponential Growth Model Threshold Crossing MEDIUM

- (a) $14000 = 500 \times b^{12} \Rightarrow b^{12} = 28$ **M1**
 $b = 28^{1/12} = 1.320687 \dots \approx 1.32$ (3 s.f.) **A1**
 (b) $500 \times b^t > 50000 \Rightarrow b^t > 100$ **M1**
 Using GDC: $t > \frac{\ln 100}{\ln b} = \frac{\ln 100}{\ln 1.320687 \dots} = 16.584 \dots$, so first complete month is $t = 17$. **A1**
 Note: $N(17) = 56119 \dots > 50000$, $N(16) = 42512 \dots < 50000$.

Q15 — Simultaneous Exponential Equations MEDIUM

- Taking log of the first equation: $x \ln 2 = (y + 1) \ln 5$ **M1**
 From the second equation: $x = 7 - 3y$. Substituting: $(7 - 3y) \ln 2 = (y + 1) \ln 5$ **M1**
 $7 \ln 2 - 3y \ln 2 = y \ln 5 + \ln 5 \Rightarrow y(3 \ln 2 + \ln 5) = 7 \ln 2 - \ln 5$ **(A1)**
 Using GDC: $y = \frac{7 \ln 2 - \ln 5}{3 \ln 2 + \ln 5} = 0.879018 \dots \approx 0.879$ (3 s.f.), $x = 7 - 3y = 4.36295 \dots \approx 4.36$ (3 s.f.) **A1**
 Note: Back-substitution: $2^{4.36} = 20.58 \dots$, $5^{1.879} = 20.58 \dots \checkmark$.

Q16 — Change of Base then Quadratic Rejected Root MEDIUM

- $\log_4 x = \frac{\log x}{\log 4} = \frac{\ln x}{\ln 4} = \frac{\ln x}{2 \ln 2}$ **M1**
 (Show that complete; AG)
 Using the product law: $\log_4 [x(x - 6)] = 2 \Rightarrow x(x - 6) = 16$ **M1**
 $x^2 - 6x - 16 = 0 \Rightarrow (x - 8)(x + 2) = 0 \Rightarrow x = 8 \text{ or } x = -2$ **A1**
 Reject $x = -2$ since $\log_4(-2)$ is undefined (argument must be positive); the only solution is $x = 8$. **R1**

Q17 — Radioactive Decay Half-Life and Threshold MEDIUM

- (a) At the half-life: $\frac{1}{2}M_0 = M_0 e^{-24\lambda} \Rightarrow e^{-24\lambda} = 0.5$ **M1**
 $\lambda = \frac{\ln 2}{24} = 0.028881 \dots \approx 0.0289$ (3 s.f.) **A1**
 (b) $200e^{-\lambda t} < 15 \Rightarrow e^{-\lambda t} < 0.075$ **M1**
 Using GDC: $t > \frac{-\ln(0.075)}{\lambda} = \frac{\ln(0.075)}{-0.028881 \dots} = 89.687 \dots$, so $t = 90$ years. **A1**
 Note: $M(89) = 15.30 \dots g > 15$, $M(90) = 14.87 \dots g < 15$.

Q18 — Log-Linear Fit to Find Model Parameters MEDIUM

Taking $\log_{10}$ of $y = a \cdot x^n$: $\log y = \log a + n \log x$, a linear equation in $\log x$. **M1**

Using GDC with $(\log x, \log y)$ pairs: $(0.301, 0.753)$, $(0.602, 1.204)$, $(0.903, 1.656)$, $(1.204, 2.107)$ **(A1)**

Gradient $= n = \frac{2.107 - 0.753}{1.204 - 0.301} = \frac{1.354}{0.903} = 1.4994 \dots \approx 1.50$ (3 s.f.) **A1**

$\log a = 0.753 - 1.50 \times 0.301 = 0.3015 \dots \Rightarrow a = 10^{0.3015} = 2.00$ (3 s.f.) **A1**

Q19 — Quadratic in a Log Root Rejection MEDIUM

Let $u = \log_5 x$. The equation becomes $2u^2 - 5u - 3 = 0$. **M1**

$(2u + 1)(u - 3) = 0 \Rightarrow u = -\frac{1}{2}$ or $u = 3$ **A1**

$\log_5 x = -\frac{1}{2} \Rightarrow x = 5^{-1/2} = 1/\sqrt{5}$; $\log_5 x = 3 \Rightarrow x = 125$ **A1**

Both values satisfy $x \in \mathbb{R}^+$ (arguments are positive); both solutions are valid: $x = \frac{1}{\sqrt{5}}$ and $x = 125$. **R1**

Q20 — Continuous Compound Interest Trebling Time MEDIUM

Treble in value: $V = 3P$, so $3P = Pe^{0.042t} \Rightarrow e^{0.042t} = 3$ **M1**

$0.042t = \ln 3 \Rightarrow t = \frac{\ln 3}{0.042}$ **M1**

$t = 26.156 \dots$ years **A1**

The investment trebles after 27 complete years. **A1**

Q21 — Bacterial Growth Model Threshold Time HARD

Step 1: Find A .

$N(0) = A = 12$ (thousands), so $A = 12$. **A1**

Step 2: Find k .

$N(5) = 12e^{5k} = 31 \Rightarrow e^{5k} = \frac{31}{12}$ **M1**

$k = \frac{1}{5} \ln \left(\frac{31}{12} \right) \approx 0.18982 \dots$ **(A1)**

Step 3: Solve $12e^{kt} = 100$.

$e^{kt} = \frac{100}{12} \Rightarrow t = \frac{\ln(100/12)}{k}$

GDC: $t = \frac{5 \ln(25/3)}{\ln(31/12)} \approx 11.1701 \dots$ hours **M1**

$0.1701 \dots \times 60 \approx 10$ minutes, so $t \approx 11.17$ hours, i.e. 11 hours 10 minutes. **A1**

Note: Accept $t \approx 11.2$ hours if minutes not requested; full marks only for nearest minute.

Q22 — Log Laws Express and Evaluate **HARD**

Step 1: Apply log laws.

$$\log_3 \left(\frac{9p^3}{\sqrt{q}} \right) = \log_3 9 + \log_3 p^3 - \log_3 q^{1/2} \quad \text{M1}$$

$$= 2 + 3a - \frac{1}{2}b \quad \text{A1}$$

Step 2: Find values of a and b when $p = 9, q = 81$.

$$\log_3 9 = 2 \Rightarrow a = 2; \quad \log_3 81 = 4 \Rightarrow b = 4 \quad \text{A1}$$

Step 3: Evaluate.

$$2 + 3(2) - \frac{1}{2}(4) = 2 + 6 - 2 = 6 \quad \text{M1A1}$$

Note: M1 for correct substitution; final A1 for answer 6.

Q23 — Log Equation Reject Invalid Root **HARD**

Step 1: Combine logarithms.

$$\log_2 [(3x - 1)(x - 2)] = 4 \quad \text{M1}$$

$$(3x - 1)(x - 2) = 16 \quad \text{A1}$$

Step 2: Expand and solve the quadratic.

$$3x^2 - 7x + 2 = 16 \Rightarrow 3x^2 - 7x - 14 = 0 \quad \text{M1}$$

$$x = \frac{7 \pm \sqrt{49 + 168}}{6} = \frac{7 \pm \sqrt{217}}{6}$$

$$x = \frac{7 + \sqrt{217}}{6} \approx 3.62 \text{ or } x = \frac{7 - \sqrt{217}}{6} \approx -1.29 \quad \text{A1}$$

Step 3: Reject invalid solution.

For $\log_2(3x - 1)$ we need $3x - 1 > 0$ and for $\log_2(x - 2)$ we need $x > 2$.

$x \approx -1.29$ gives $x - 2 < 0$, so this solution is rejected as the argument of $\log_2(x - 2)$ would be negative.

R1

$$\text{Therefore } x = \frac{7 + \sqrt{217}}{6}.$$

Q24 — Log-Linear Regression Power Model **HARD**

Step 1: Show the linear form (AG).

Taking $\log_{10}$ of both sides of $m = aL^n$:

$$\log_{10} m = \log_{10}(aL^n) = \log_{10} a + n \log_{10} L \quad \text{M1}$$

which is of the form $Y = nX + \log_{10} a$ where $Y = \log_{10} m, X = \log_{10} L$. **AG**

Step 2: Compute transformed data and run regression.

GDC linear regression on the pairs (X, Y) :

$X = \log_{10} L$	$Y = \log_{10} m$
1.000	1.255
1.176	1.756
1.301	2.134
1.398	2.423
1.477	2.662

GDC gives gradient $= n \approx 2.95$ and intercept $= \log_{10} a \approx -1.71$

M1A1

$n \approx 2.95$; $a = 10^{-1.71} \approx 0.0197$ (to 3 s.f.)

A1

Step 3: Estimate mass at $L = 22$ cm.

$m \approx 0.0197 \times 22^{2.95} \approx 181$ g

A1

Q25 — Chain Rule for Logs Exact Evaluation **HARD**

Step 1: Apply change of base to each factor (AG).

Using $\log_a b = \frac{\ln b}{\ln a}$:

$$\log_a b \cdot \log_b c \cdot \log_c a = \frac{\ln b}{\ln a} \cdot \frac{\ln c}{\ln b} \cdot \frac{\ln a}{\ln c} = 1$$

M1A1

All logarithms cancel, giving 1.

AG

Step 2: Identify the structure in the given expression.

$$\log_5 7 \cdot \log_7 11 \cdot \log_{11} 25$$

$$\text{Write } \log_{11} 25 = \log_{11} 5^2 = 2 \log_{11} 5$$

M1

$$\text{So the expression} = 2(\log_5 7 \cdot \log_7 11 \cdot \log_{11} 5)$$

A1

$$\text{By the result of Step 1 with } a = 5, b = 7, c = 11: \log_5 7 \cdot \log_7 11 \cdot \log_{11} 5 = 1$$

A1

$$\text{Therefore } \log_5 7 \cdot \log_7 11 \cdot \log_{11} 25 = 2.$$

Q26 — Half-Life and Threshold Crossing **HARD**

Step 1: Find the half-life h .

$$M(8) = 500 \cdot 2^{-8/h} = 320$$

M1

$$2^{-8/h} = \frac{320}{500} = 0.64$$

$$-\frac{8}{h} \ln 2 = \ln 0.64 \Rightarrow h = \frac{-8 \ln 2}{\ln 0.64}$$

$$\text{GDC: } h = \frac{8 \ln 2}{\ln(25/16)} \approx 12.4251 \dots \approx 12.4 \text{ years (3 s.f.)}$$

A1

Step 2: Find first integer t with $M(t) < 50$ g.

$$500 \cdot 2^{-t/h} < 50 \Rightarrow 2^{-t/h} < 0.1 \Rightarrow -\frac{t}{h} \ln 2 < \ln 0.1$$

M1

$$t > \frac{h \ln 10}{\ln 2} = \frac{12.4251 \dots \times \ln 10}{\ln 2} \approx 41.28 \dots$$

A1

First integer value is $t = 42$ years.

A1

Note: Award final A1 only if correct reasoning leads to $t = 42$.

Q27 — Simultaneous Exponential and Log Equations **HARD**

Step 1: Simplify the exponential equation.

$$3^x = 9^{y+1} = 3^{2(y+1)} \Rightarrow x = 2y + 2$$

M1A1

Step 2: Substitute into the logarithmic equation.

$$\log_3[x \cdot 2y] = 3 \Rightarrow x \cdot 2y = 27$$

$$\text{Substituting } x = 2y + 2: (2y + 2)(2y) = 27 \Rightarrow 4y^2 + 4y - 27 = 0$$

M1

$$\text{GDC or quadratic formula: } y = \frac{-4 \pm \sqrt{16 + 432}}{8} = \frac{-4 \pm \sqrt{448}}{8} = -\frac{1}{2} \pm \sqrt{7}$$

$$y \approx 2.146 \text{ or } y \approx -3.146$$

A1

Step 3: Reject invalid solution.

Since $y \in \mathbb{R}^+$, the solution $y \approx -3.146$ is rejected as y must be positive (the argument $2y$ of $\log_3(2y)$ must be positive).

R1

Therefore $y \approx 2.15$ and $x = 2(2.1458 \dots) + 2 \approx 6.29$ (3 s.f.).

Q28 — Quadratic in Base 2 Exact Solutions **HARD**

Step 1: Substitute $u = 2^x$, $u > 0$.

$$4^x = (2^x)^2 = u^2, \text{ so the equation becomes } u^2 - 5u + 4 = 0$$

M1

Step 2: Factorise.

$$(u - 1)(u - 4) = 0 \Rightarrow u = 1 \text{ or } u = 4$$

A1

Step 3: Solve for x .

$$2^x = 1 \Rightarrow x = 0$$

A1

$$2^x = 4 \Rightarrow x = 2$$

A1

Both values $u = 1 > 0$ and $u = 4 > 0$ are valid; no rejection necessary.

A1

Therefore $x = 0$ or $x = 2$.

Note: Award first A1 only if both roots of the quadratic stated.

Q29 — When One Account Overtakes Another **HARD**

Step 1: Set up the equation $V_A = V_B$.

$$2000e^{0.04t} = 3000e^{0.025t}$$

M1

$$e^{0.04t-0.025t} = \frac{3000}{2000} = 1.5$$

$$e^{0.015t} = 1.5$$

A1

Step 2: Solve for t .

$$0.015t = \ln 1.5 \Rightarrow t = \frac{\ln 1.5}{0.015}$$

M1

$$\text{GDC: } t = \frac{\ln 1.5}{0.015} \approx 27.031 \dots \text{ years}$$

A1

Step 3: Convert to nearest month.

$0.031 \dots \times 12 \approx 0.38$ months, so $t \approx 27$ years 0 months.

Account A first overtakes Account B after approximately 27 years and 0 months (i.e. at $t \approx 27.0$ years). **A1**

Note: Award final A1 for 27 years 0 months or equivalent 27.0 years.

Q30 — Log Equation Domain Check Reject Invalid Root **HARD**

Step 1: State domain restriction.

$$\text{For } \log_5(2x+3): 2x+3 > 0 \Rightarrow x > -\frac{3}{2}.$$

$$\text{For } \log_5(x-1): x-1 > 0 \Rightarrow x > 1.$$

Combined domain: $x > 1$.

A1

Step 2: Combine into a single logarithmic equation.

$$\log_5(2x+3) + \log_5(x-1) = 2$$

$$\log_5[(2x+3)(x-1)] = 2$$

M1

$$(2x+3)(x-1) = 25$$

$$2x^2 + x - 3 = 25 \Rightarrow 2x^2 + x - 28 = 0$$

A1

Step 3: Solve the quadratic.

$$\text{GDC or formula: } x = \frac{-1 \pm \sqrt{1+224}}{4} = \frac{-1 \pm 15}{4}$$

$$x = \frac{14}{4} = \frac{7}{2} \text{ or } x = \frac{-16}{4} = -4$$

A1

Step 4: Reject invalid solution.

$x = -4$ does not satisfy $x > 1$, so it is rejected as both logarithm arguments would be negative.

R1

Therefore $x = 7/2 = 3.5$.