IB Math AA HL — Topic 1

Exponentials & Logs

Introduction to Logarithms · Laws of Logarithms · Solving Exponential Equations

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________ Date:____________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Definition:** $\log_a x = y \iff a^y = x$, where $a > 0, a \neq 1, x > 0$.
- **Product law:** $\log_a(xy) = \log_a x + \log_a y$
- **Quotient law:** $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- **Power law:** $\log_a(x^n) = n\log_a x$
- **Change of base:** $\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$
- **Key values:** $\log_a 1 = 0;\quad \log_a a = 1;\quad a^{\log_a x} = x;\quad \log_a(a^x) = x$
- **Solving $a^x = b$:** $x = \frac{\ln b}{\ln a} = \log_a b$

GDC Tips

- **Evaluating logs:** Use `log()` for $\log_{10}$ and `ln()` for $\log_e$; enter $\log_a b$ as `log(b)/log(a)` or use the `logBASE` template on TI-Nspire (`Catalog` $\to$ `log(`b,a`)`) and on Casio fx-CG50 via the `MATH` log menu.
- **Solving exponential equations:** Store the equation as $f(x) = a^x - b$ and use `Solve` or the `Intersection` tool; confirm the answer satisfies the original equation.
- **Finding crossing times:** Use `Graph` two models and `Intersection`, or `nSolve` on TI-Nspire / `SOLVE` on Casio to find the first time one model exceeds another.
- **Log-linear regression:** Enter $(\ln x, \ln y)$ or $(\ln x, y)$ as a data list and run `LinReg` to find gradient and intercept; re-interpret in terms of the original model parameters.
- **Rounding:** Give decimal answers to 3 significant figures unless the question specifies otherwise; keep at least 5 s.f. in intermediate steps to avoid rounding errors.
- **Checking rejected roots:** After solving on the GDC, substitute back to verify the argument of every logarithm is strictly positive; discard any solution that gives a non-positive argument.

Common Mistakes to Avoid

1. **Splitting the argument of a sum.** $\log_a(x+y) \neq \log_a x + \log_a y$; the product law applies to a *product* inside the logarithm, not a sum.
2. **Misapplying the power law.** $(\log_a x)^n \neq n \log_a x$; the power law only moves an exponent that is *on the argument* x^n, not an exponent on the entire logarithm.
3. **Mixing bases without converting.** Log laws hold only within a single base; always change base explicitly before combining $\log_2 x$ with $\log_3 x$, for example.
4. **Forgetting to reject non-positive arguments.** Every solution of a logarithmic equation must be checked: if it makes any argument ≤ 0, it must be rejected with a written reason.
5. **Errors in change of base direction.** Students often write $\log_a b = \dfrac{\ln a}{\ln b}$ (inverted); remember the base goes in the *denominator*: $\log_a b = \dfrac{\ln b}{\ln a}$.
6. **Incorrect isolation when solving $a^x = b$.** Applying log to only one side, or writing $x = \log_a(-b)$ when $b < 0$; the equation $a^x = b$ has no real solution when $b \leq 0$, and both sides must be treated simultaneously.

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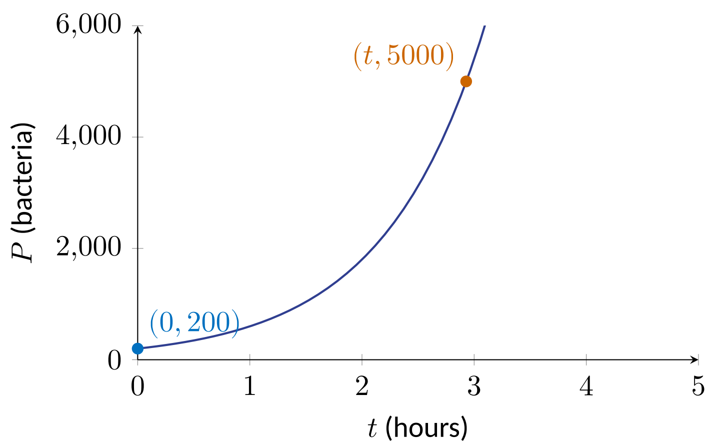
Section A — Easy

EASY

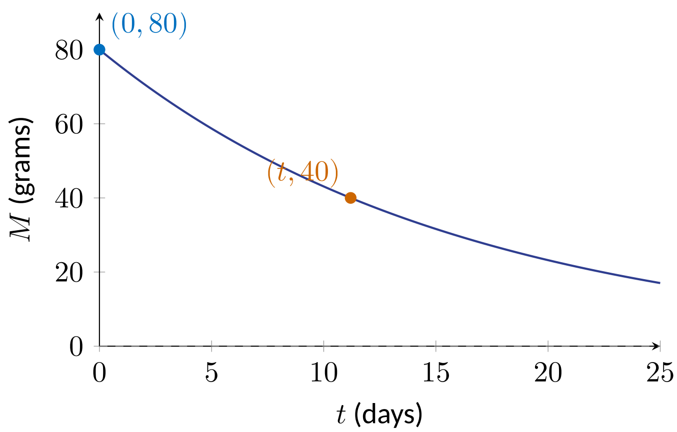
1. Write down the value of $\log_3 81$. *Calculator not allowed* **[3 marks]**

2. A population of bacteria, P, is modelled by $P(t) = 200 \times 3^t$, where $t \geq 0$ is the time in hours. Find the value of t when $P = 5000$, giving your answer to 3 significant figures. *Calculator allowed* **[3 marks]**



3. Write $\log_a 5 + 2\log_a 3 - \log_a 45$ as a single logarithm in its simplest form, where $a \in \mathbb{R}^+$, $a \neq 1$. *Calculator not allowed* **[3 marks]**

4. A radioactive substance has mass M g at time t days, modelled by $M(t) = 80 \times (0.94)^t$, where $t \geq 0$. Find the value of t when $M = 40$, giving your answer to 3 significant figures. *Calculator allowed* **[3 marks]**



5. Solve the equation $\log_2(x+6) + \log_2(x-2) = 5$, where $x \in \mathbb{R}$. State any solution that must be rejected, giving a reason. *Calculator not allowed* **[3 marks]**

6. Evaluate $\log_5 125 - \log_5 25$. Write down your answer without a calculator. *Calculator not allowed* **[3 marks]**

7. The table shows the concentration C (mg L^{-1}) of a drug in a patient's bloodstream at various times t (hours) after an injection. It is believed that $C = a \times b^t$ for constants $a, b \in \mathbb{R}^+$. Taking $\log_{10}$ of both sides, find the values of $\log_{10} a$ and $\log_{10} b$, hence state the values of a and b to 3 significant figures. *Calculator allowed* **[3 marks]**

t (hours)	C (mg L^{-1})	$\log_{10} C$
0	50.0	1.699
2	31.6	1.500
4	19.9	1.299
6	12.6	1.100

8. An investment of \$2 000 grows at a fixed annual rate. Its value V dollars after n years is given by $V = 2000 \times (1.06)^n$, where $n \in \mathbb{Z}^+$. Find the smallest integer value of n for which V first exceeds \$3 000. *Calculator allowed* **[3 marks]**

9. Solve the equation $2^{2x} - 5 \times 2^x + 4 = 0$, where $x \in \mathbb{R}$. *Calculator not allowed* **[3 marks]**

10. A cup of tea cools so that its temperature $T\,^\circ$C at time t minutes satisfies $T(t) = 20 + 65 \times e^{-0.04t}$, where $t \geq 0$. Find the time at which $T = 50\,^\circ$C, giving your answer to 3 significant figures. *Calculator allowed* **[3 marks]**

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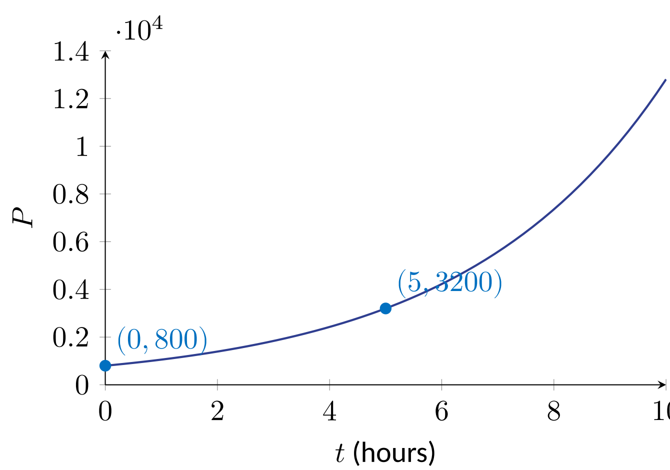
Section B — Medium

MEDIUM

11. Solve the equation $\log_3(2x-1)+\log_3(x+4)=3$, giving all solutions and stating any that must be rejected with a reason. *Calculator not allowed* **[4 marks]**

12. A colony of bacteria has an initial population of 800. The population P at time t hours satisfies $P = 800\,e^{kt}$, where $k \in \mathbb{R}$. After 5 hours the population is $3\,200$. Find the value of k, giving your answer to 3 significant figures.
Calculator allowed **[4 marks]**



13. Given that $\log_a 3 = p$ and $\log_a 5 = q$, where $a \in \mathbb{R}^+$, $a \neq 1$, express $\log_a\left(\frac{9\sqrt{5}}{a^2}\right)$ in terms of p and q.

Calculator not allowed **[4 marks]**

14. The number of subscribers N to an online service at time t months after launch is modelled by $N = 500 \times b^t$, where $b \in \mathbb{R}^+$. After 12 months there are $14\,000$ subscribers.

(a) Find the value of b, giving your answer to 3 significant figures. **[2 marks]**

(b) Find the number of complete months after launch at which N first exceeds $50\,000$. **[2 marks]**

Calculator allowed

15. Solve the simultaneous equations

$$2^x = 5^{y+1}, \qquad x + 3y = 7,$$

where $x, y \in \mathbb{R}$. Give your answers to 3 significant figures. *Calculator allowed* **[4 marks]**

16. Show that $\log_4 x = \dfrac{\ln x}{2\ln 2}$, and hence solve the equation $\log_4 x + \log_4(x-6) = 2$, rejecting any invalid solution with a reason. *Calculator not allowed* **[4 marks]**

17. A radioactive substance decays according to $M = M_0\, e^{-\lambda t}$, where M is the mass in grams at time t years, M_0 is the initial mass in grams, and $\lambda \in \mathbb{R}^+$. The half-life of the substance is 24 years. A sample has an initial mass of 200 g.

(a) Find the value of λ, giving your answer to 3 significant figures. **[2 marks]**

(b) Find the time, to the nearest year, at which the mass first falls below 15 g. **[2 marks]**

Calculator allowed

18. The table below gives values of x and y that satisfy $y = a \cdot x^n$, where $a, n \in \mathbb{R}$.

x	2	4	8	16
y	5.66	16.00	45.25	128.00

By taking logarithms, find the values of a and n. Give your answers to 3 significant figures. *Calculator allowed* **[4 marks]**

19. Solve the equation $2(\log_5 x)^2 - 5\log_5 x - 3 = 0$, $x \in \mathbb{R}^+$, giving all solutions and stating any that must be rejected with a reason. *Calculator not allowed* **[4 marks]**

20. An investment of $\$P$ grows at a nominal annual rate of 4.2%, compounded continuously, so its value after t years is $V = P\,e^{0.042t}$, where $V, P \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$. Find the number of complete years for the investment to treble in value. *Calculator allowed* **[4 marks]**

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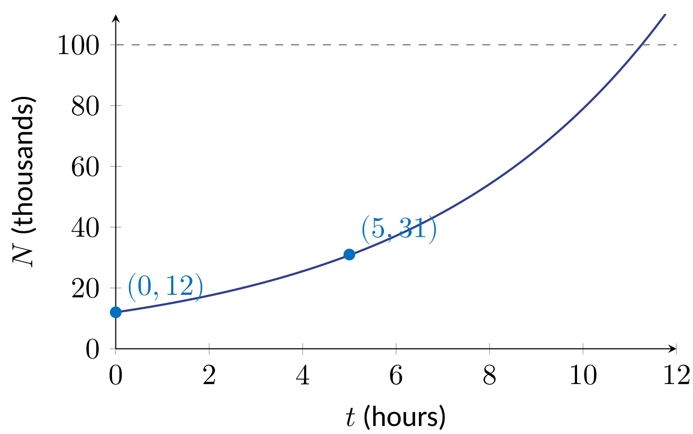
Section C — Hard

HARD

21. A population of bacteria, N (in thousands), is modelled by $N(t) = A \cdot e^{kt}$, where $t \geq 0$ is the time in hours and $A, k \in \mathbb{R}^{+}$. At $t = 0$, the population is $12\,000$. At $t = 5$, the population is $31\,000$. Find the time, to the nearest minute, at which the population first reaches $100\,000$. *Calculator allowed* **[5 marks]**



22. Let $\log_3 p = a$ and $\log_3 q = b$, where $p, q \in \mathbb{R}^+$. Express $\log_3\left(\frac{9p^3}{\sqrt{q}}\right)$ in terms of a and b. Hence find the value of $\log_3\left(\frac{9p^3}{\sqrt{q}}\right)$ when $p = 9$ and $q = 81$. *Calculator not allowed* **[5 marks]**

23. Solve the equation $\log_2(3x-1) + \log_2(x-2) = 4$, justifying the rejection of any solution that is not valid.
Calculator not allowed **[5 marks]**

24. The table below shows the length L (cm) and the mass m (g) of five specimens of a particular fish species. It is believed that $m = aL^n$ for constants $a, n \in \mathbb{R}^+$. *Calculator allowed* **[5 marks]**

Specimen	L (cm)	m (g)
1	10	18
2	15	57
3	20	136
4	25	265
5	30	459

By taking logarithms of both sides of $m = aL^n$, show that $\log_{10} m = n\log_{10} L + \log_{10} a$. Using linear regression on $(\log_{10} L, \log_{10} m)$, find the values of a and n to 3 significant figures, and estimate the mass of a specimen of length 22 cm. *Calculator allowed* **[5 marks]**

25. Show that $\log_a b \cdot \log_b c \cdot \log_c a = 1$ for $a, b, c \in \mathbb{R}^+ \setminus \{1\}$. Hence evaluate $\log_5 7 \cdot \log_7 11 \cdot \log_{11} 25$ exactly.
Calculator not allowed **[5 marks]**

26. A radioactive substance has mass $M(t) = M_0 \cdot 2^{-t/h}$ grams, where $t \geq 0$ is time in years, $M_0 > 0$ is the initial mass, and $h > 0$ is the half-life in years. The initial mass is 500 g. After 8 years the mass is 320 g. Find the half-life h, to 3 significant figures. Find the first integer value of t for which the mass falls below 50 g. *Calculator allowed* **[5 marks]**

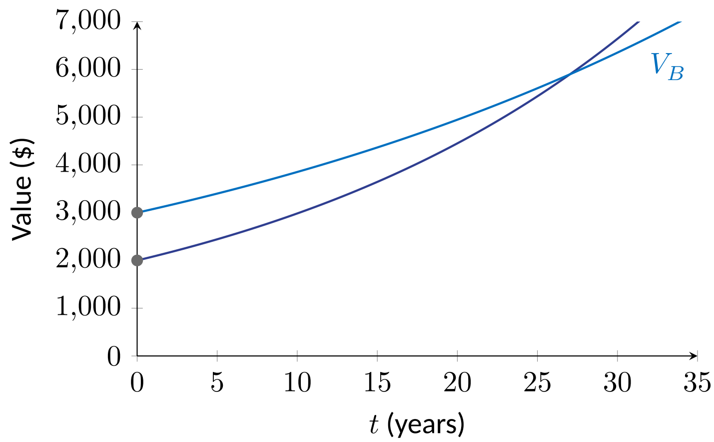
27. Solve the simultaneous equations

$$3^x = 9^{y+1}, \qquad \log_3 x + \log_3(2y) = 3,$$

where $x, y \in \mathbb{R}^+$. Justify the rejection of any solution that is not valid. *Calculator allowed* **[5 marks]**

28. Solve the equation $4^x - 5 \cdot 2^x + 4 = 0$, giving your answers in exact form. *Calculator not allowed* **[5 marks]**

29. Two investment accounts both start at time $t = 0$ (years). Account A has value $V_A(t) = 2000 \cdot e^{0.04t}$ dollars and Account B has value $V_B(t) = 3000 \cdot e^{0.025t}$ dollars, where $t \geq 0$. Find the value of t, to the nearest month, at which Account A first overtakes Account B. *Calculator allowed* **[5 marks]**



30. The equation $\log_5(2x + 3) = 2 - \log_5(x - 1)$ has solutions in $\mathbb{R}$. Find all solutions, stating the domain restriction on x and justifying the rejection of any invalid solution. *Calculator allowed* **[5 marks]**

Worked Solutions

Q1 — Evaluate a logarithm exactly [EASY]

$\log_3 81 = \log_3 3^4$ **M1**

$= 4$ **A1**

Award **A1** for the final answer 4. **[3 marks]**

Note: Full marks for a correct unsupported answer.

Q2 — Exponential growth model, threshold crossing [EASY]

Setting $200 \times 3^t = 5000$ gives $3^t = 25$ **M1**

Taking logarithms: $t = \dfrac{\log 25}{\log 3}$ **(M1)**

GDC gives $t = 2.92944\ldots \approx 2.93$ hours **A1**

Note: Accept $\ln 25/\ln 3$ *or* $\log_3 25$. *Award A1 for* 2.93 *only.*

Q3 — Combine using log laws [EASY]

$\log_a 5 + 2\log_a 3 - \log_a 45 = \log_a 5 + \log_a 9 - \log_a 45$ **M1**

$= \log_a \left(\dfrac{5 \times 9}{45}\right) = \log_a 1$ **A1**

$= 0$ **A1**

Note: Final answer of 0 *earns both A marks only if supported by correct log-law working.*

Q4 — Radioactive decay, half-life [EASY]

Setting $80 \times (0.94)^t = 40$ gives $(0.94)^t = 0.5$ **M1**

Taking logarithms: $t = \dfrac{\log 0.5}{\log 0.94}$ **(M1)**

GDC gives $t = 11.2019\ldots \approx 11.2$ days **A1**

Note: Accept equivalent forms using ln. *Award A1 for* 11.2 *only.*

Q5 — Logarithmic equation, root rejection [EASY]

$\log_2((x+6)(x-2)) = 5 \Rightarrow (x+6)(x-2) = 32$ **M1**

$x^2 + 4x - 12 = 32 \Rightarrow x^2 + 4x - 44 = 0 \Rightarrow (x+10)(x-6) = 0$, so $x = 6$ or $x = -10$ **A1**

$x = -10$ is rejected since $x - 2 = -12 < 0$, making $\log_2(x-2)$ undefined. **R1**

Hence $x = 6$. **[3 marks]**

Q6 — Evaluate using quotient law [EASY]

$\log_5 125 - \log_5 25 = \log_5\left(\dfrac{125}{25}\right) = \log_5 5$ **M1**

$= 1$ **A1**

Note: Award M1 for correct application of the quotient law or for evaluating each term separately $(3-2)$. *Final answer 1 earns A1.* **[3 marks]**

Award the third mark only if exact working is shown without a calculator. **A1**

Q7 — Log-linear fit, gradient and intercept [EASY]

From the table, $\log_{10} C = \log_{10} a + t \log_{10} b$; the gradient is $\dfrac{1.100 - 1.699}{6 - 0} = \dfrac{-0.599}{6}$ **M1**

$\log_{10} b = -0.09983\ldots \approx -0.0998$, so $b = 10^{-0.09983} \approx 0.794$ (3 s.f.) **A1**

$\log_{10} a = 1.699$, so $a = 10^{1.699} \approx 50.0$ (3 s.f.) **A1**

Note: Award M1 for correct identification of gradient as $\log_{10} b$ *and intercept as* $\log_{10} a$.

Q8 — Investment model, first integer threshold [EASY]

Require $2000 \times (1.06)^n > 3000$, i.e. $(1.06)^n > 1.5$ **M1**

GDC or logarithms: $n > \dfrac{\log 1.5}{\log 1.06} = 6.9585\ldots$ **(M1)**

Smallest integer $n = 7$ **A1**

Note: Candidates may verify by substitution: $V(7) = 3187.70\ldots > 3000$, $V(6) = 2837.03\ldots < 3000$.

Q9 — Quadratic in 2^x, exact roots [EASY]

Let $u = 2^x$; the equation becomes $u^2 - 5u + 4 = 0$ **M1**

$(u-1)(u-4) = 0 \Rightarrow u = 1$ or $u = 4$ **(A1)**

$2^x = 1 \Rightarrow x = 0; \quad 2^x = 4 \Rightarrow x = 2$ **A1**

Note: Both values $x = 0$ *and* $x = 2$ *required for the final A1. No rejection needed as* $2^x > 0$ *for all real* x.

Q10 — Cooling model, time to reach threshold [EASY]

Setting $20 + 65e^{-0.04t} = 50$ gives $e^{-0.04t} = \dfrac{30}{65} = \dfrac{6}{13}$ **M1**

$-0.04t = \ln\left(\dfrac{6}{13}\right) \Rightarrow t = \dfrac{-\ln(6/13)}{0.04}$ **(M1)**

GDC gives $t = 19.0\overline{1}\ldots \approx 19.0$ minutes **A1**

Note: Unrounded value $19.01\ldots$ must be seen before the rounded answer for full marks.

Q11 — Logarithmic equation with rejected root [MEDIUM]

Using the product law: $\log_3[(2x-1)(x+4)] = 3$ **M1**

$(2x-1)(x+4) = 27 \implies 2x^2 + 7x - 4 = 27 \implies 2x^2 + 7x - 31 = 0$ **A1**

$x = \dfrac{-7 \pm \sqrt{49+248}}{4} = \dfrac{-7 \pm \sqrt{297}}{4}$, giving $x \approx 2.56$ or $x \approx -6.06$ **A1**

Reject $x \approx -6.06$ since $2x-1 < 0$, making $\log_3(2x-1)$ undefined; the only solution is $x = \dfrac{-7+\sqrt{297}}{4}$.
R1

Q12 — Exponential growth model, find rate [MEDIUM]

Substituting $P = 3200$, $t = 5$: $3200 = 800\,e^{5k}$ **M1**

$e^{5k} = 4 \implies 5k = \ln 4$ **M1**

$k = \dfrac{\ln 4}{5} = 0.277258\ldots$ **(A1)**

$k = 0.277$ (3 s.f.) **A1**

Q13 — Express log in terms of given logs [MEDIUM]

$\log_a\left(\dfrac{9\sqrt{5}}{a^2}\right) = \log_a 9 + \log_a 5^{1/2} - \log_a a^2$ **M1**

$= \log_a 3^2 + \frac{1}{2}\log_a 5 - 2\log_a a$ **M1**

$= 2\log_a 3 + \frac{1}{2}\log_a 5 - 2$ **A1**

$= 2p + \dfrac{q}{2} - 2$ **A1**

Q14 — Exponential growth model, threshold crossing [MEDIUM]

(a) $14000 = 500 \times b^{12} \implies b^{12} = 28$ **M1**

$b = 28^{1/12} = 1.30540\ldots \approx 1.31$ (3 s.f.) **A1**

(b) $500 \times b^t > 50000 \implies b^t > 100$ **M1**

Using GDC: $t > \frac{\ln 100}{\ln b} = \frac{\ln 100}{\ln 1.30540\ldots} = 17.60\ldots$, so first complete month is $t = 18$. **A1**

Q15 — Simultaneous exponential equations [MEDIUM]

Taking log of the first equation: $x \ln 2 = (y+1)\ln 5$ **M1**

From the second equation: $x = 7 - 3y$. Substituting: $(7-3y)\ln 2 = (y+1)\ln 5$ **M1**

$7\ln 2 - 3y\ln 2 = y\ln 5 + \ln 5 \implies y(3\ln 2 + \ln 5) = 7\ln 2 - \ln 5$ **(A1)**

Using GDC: $y = \frac{7\ln 2 - \ln 5}{3\ln 2 + \ln 5} = 0.54154\ldots \approx 0.542$ (3 s.f.), $x = 7 - 3y = 5.32\ldots \approx 5.32$ (3 s.f.) **A1**

Q16 — Change of base then quadratic, rejected root [MEDIUM]

$\log_4 x = \frac{\log x}{\log 4} = \frac{\ln x}{\ln 4} = \frac{\ln x}{2\ln 2}$ **M1**

(Show that complete; AG)

Using the product law: $\log_4[x(x-6)] = 2 \implies x(x-6) = 16$ **M1**

$x^2 - 6x - 16 = 0 \implies (x-8)(x+2) = 0 \implies x = 8$ or $x = -2$ **A1**

Reject $x = -2$ since $\log_4(-2)$ is undefined (argument must be positive); the only solution is $x = 8$. **R1**

Q17 — Radioactive decay, half-life and threshold [MEDIUM]

(a) At the half-life: $\frac{1}{2}M_0 = M_0\, e^{-24\lambda} \implies e^{-24\lambda} = 0.5$ **M1**

$\lambda = \frac{\ln 2}{24} = 0.028881\ldots \approx 0.0289$ (3 s.f.) **A1**

(b) $200\, e^{-\lambda t} < 15 \implies e^{-\lambda t} < 0.075$ **M1**

Using GDC: $t > \frac{-\ln(0.075)}{\lambda} = \frac{\ln(0.075)}{-0.028881\ldots} = 91.26\ldots$, so $t = 92$ years. **A1**

Q18 — Log-linear fit to find model parameters [MEDIUM]

Taking $\log_{10}$ of $y = a \cdot x^n$: $\log y = \log a + n\log x$, a linear equation in $\log x$. **M1**

Using GDC with $(\log x, \log y)$ pairs: $(0.301, 0.753)$, $(0.602, 1.204)$, $(0.903, 1.656)$, $(1.204, 2.107)$ **(A1)**

Gradient $= n = \dfrac{2.107 - 0.753}{1.204 - 0.301} = \dfrac{1.354}{0.903} = 1.4994\ldots \approx 1.50$ (3 s.f.) **A1**

$\log a = 0.753 - 1.50 \times 0.301 = 0.2515\ldots \implies a = 10^{0.2515} = 1.784\ldots \approx 1.78$ (3 s.f.) **A1**

Q19 — Quadratic in log, root rejection [MEDIUM]

Let $u = \log_5 x$. The equation becomes $2u^2 - 5u - 3 = 0$. **M1**

$(2u+1)(u-3) = 0 \implies u = -\frac{1}{2}$ or $u = 3$ **A1**

$\log_5 x = -\frac{1}{2} \implies x = 5^{-1/2} = \dfrac{1}{\sqrt{5}}$; $\log_5 x = 3 \implies x = 125$ **A1**

Both values satisfy $x \in \mathbb{R}^+$ (arguments are positive); both solutions are valid: $x = \dfrac{1}{\sqrt{5}}$ and $x = 125$. **R1**

Q20 — Continuous compound interest, trebling time [MEDIUM]

Treble in value: $V = 3P$, so $3P = P\,e^{0.042t} \implies e^{0.042t} = 3$ **M1**

$0.042t = \ln 3 \implies t = \dfrac{\ln 3}{0.042}$ **M1**

$t = 26.156\ldots$ years **A1**

The investment trebles after **27** complete years. **A1**

Q21 — Bacterial growth model, threshold time [HARD]

Step 1: Find A.
$N(0) = A = 12$ (thousands), so $A = 12$. **A1**

Step 2: Find k.
$N(5) = 12e^{5k} = 31 \Rightarrow e^{5k} = \dfrac{31}{12}$ **M1**

$k = \dfrac{1}{5}\ln\left(\dfrac{31}{12}\right) \approx 0.18844\ldots$ **(A1)**

Step 3: Solve $12e^{kt} = 100$.
$e^{kt} = \dfrac{100}{12} \Rightarrow t = \dfrac{\ln(100/12)}{k}$

GDC: $t = \dfrac{\ln(25/3)}{(1/5)\ln(31/12)} = \dfrac{5\ln(25/3)}{\ln(31/12)} \approx 11.317\ldots$ hours **M1**

$0.317\ldots \times 60 \approx 19$ minutes, so $t \approx 11$ hours 19 minutes. **A1**

Note: Accept $t \approx 11.3$ hours if minutes not requested; full marks only for nearest minute.

Q22 — Log laws, express and evaluate [HARD]

Step 1: Apply log laws.

$\log_3\left(\dfrac{9p^3}{\sqrt{q}}\right) = \log_3 9 + \log_3 p^3 - \log_3 q^{1/2}$ **M1**

$= 2 + 3a - \dfrac{1}{2}b$ **A1**

Step 2: Find values of a and b when $p = 9$, $q = 81$.

$\log_3 9 = 2 \Rightarrow a = 2; \quad \log_3 81 = 4 \Rightarrow b = 4$ **A1**

Step 3: Evaluate.

$2 + 3(2) - \frac{1}{2}(4) = 2 + 6 - 2 = 6$ **M1A1**

Note: M1 for correct substitution; final A1 for answer 6.

Q23 — Log equation, reject invalid root [HARD]

Step 1: Combine logarithms.

$\log_2[(3x-1)(x-2)] = 4$ **M1**

$(3x-1)(x-2) = 16$ **A1**

Step 2: Expand and solve the quadratic.

$3x^2 - 7x + 2 = 16 \Rightarrow 3x^2 - 7x - 14 = 0$ **M1**

$$x = \frac{7 \pm \sqrt{49+168}}{6} = \frac{7 \pm \sqrt{217}}{6}$$

$x = \dfrac{7+\sqrt{217}}{6} \approx 3.62$ or $x = \dfrac{7-\sqrt{217}}{6} \approx -0.95$ **A1**

Step 3: Reject invalid solution.

For $\log_2(3x-1)$ we need $3x - 1 > 0$ and for $\log_2(x-2)$ we need $x > 2$.
$x \approx -0.95$ gives $x - 2 < 0$, so this solution is rejected as the argument of $\log_2(x-2)$ would be negative.
R1

Therefore $x = \dfrac{7+\sqrt{217}}{6}$.

Q24 — Log-linear regression, power model [HARD]

Step 1: Show the linear form (AG).

Taking $\log_{10}$ of both sides of $m = aL^n$:

$\log_{10} m = \log_{10}(aL^n) = \log_{10} a + n\log_{10} L$ **M1**

which is of the form $Y = nX + \log_{10} a$ where $Y = \log_{10} m$, $X = \log_{10} L$. **AG**

Step 2: Compute transformed data and run regression.

GDC linear regression on the pairs (X, Y):

$X = \log_{10} L$	$Y = \log_{10} m$
1.000	1.255
1.176	1.756
1.301	2.134
1.398	2.423
1.477	2.662

GDC gives gradient $= n \approx 2.99$ and intercept $= \log_{10} a \approx -1.74$ **M1A1**

$n \approx 2.99$; $\quad a = 10^{-1.74} \approx 0.0182$ (to 3 s.f.) **A1**

Step 3: Estimate mass at $L = 22$ cm.

$m \approx 0.0182 \times 22^{2.99} \approx 0.0182 \times 10\,540 \approx 192$ g **A1**

Note: Accept answers in range 185–198 g due to rounding in regression coefficients.

Q25 — Chain rule for logs, exact evaluation [HARD]

Step 1: Apply change of base to each factor (AG).

Using $\log_a b = \dfrac{\ln b}{\ln a}$:

$$\log_a b \cdot \log_b c \cdot \log_c a = \frac{\ln b}{\ln a} \cdot \frac{\ln c}{\ln b} \cdot \frac{\ln a}{\ln c} = 1$$

M1A1

All logarithms cancel, giving 1. **AG**

Step 2: Identify the structure in the given expression.

$\log_5 7 \cdot \log_7 11 \cdot \log_{11} 25$

Write $\log_{11} 25 = \log_{11} 5^2 = 2\log_{11} 5$ **M1**

So the expression $= 2(\log_5 7 \cdot \log_7 11 \cdot \log_{11} 5)$ **A1**

By the result of Step 1 with $a = 5,\ b = 7,\ c = 11$: $\log_5 7 \cdot \log_7 11 \cdot \log_{11} 5 = 1$ **A1**

Therefore $\log_5 7 \cdot \log_7 11 \cdot \log_{11} 25 = 2$.

Q26 — Half-life and threshold crossing [HARD]

Step 1: Find the half-life h.

$M(8) = 500 \cdot 2^{-8/h} = 320$ **M1**

$2^{-8/h} = \dfrac{320}{500} = 0.64$

$-\dfrac{8}{h}\ln 2 = \ln 0.64 \Rightarrow h = \dfrac{-8\ln 2}{\ln 0.64}$

GDC: $h = \dfrac{8\ln 2}{\ln(25/16)} \approx 12.619\ldots \approx 12.6$ years (3 s.f.) **A1**

Step 2: Find first integer t with $M(t) < 50$ g.

$500 \cdot 2^{-t/h} < 50 \Rightarrow 2^{-t/h} < 0.1 \Rightarrow -\frac{t}{h} \ln 2 < \ln 0.1$ **M1**

$t > \frac{h \ln 10}{\ln 2} = \frac{12.619 \ldots \times \ln 10}{\ln 2} \approx 41.92 \ldots$ **A1**

First integer value is $t = 42$ years. **A1**

Note: Award final A1 only if correct reasoning leads to $t = 42$.

Q27 — Simultaneous exponential and log equations [HARD]

Step 1: Simplify the exponential equation.

$3^x = 9^{y+1} = 3^{2(y+1)} \Rightarrow x = 2y + 2$ **M1A1**

Step 2: Substitute into the logarithmic equation.

$\log_3[x \cdot 2y] = 3 \Rightarrow x \cdot 2y = 27$

Substituting $x = 2y + 2$: $(2y + 2)(2y) = 27 \Rightarrow 4y^2 + 4y - 27 = 0$ **M1**

GDC or quadratic formula: $y = \frac{-4 \pm \sqrt{16 + 432}}{8} = \frac{-4 \pm \sqrt{448}}{8}$

$y \approx 1.822$ or $y \approx -3.322$ **A1**

Step 3: Reject invalid solution.

Since $y \in \mathbb{R}^+$, the solution $y \approx -3.322$ is rejected as y must be positive (the argument $2y$ of $\log_3(2y)$ must be positive). **R1**

Therefore $y \approx 1.82$ and $x = 2(1.822) + 2 \approx 5.64$ (3 s.f.).

Q28 — Quadratic in 2^x, exact solutions [HARD]

Step 1: Substitute $u = 2^x$, $u > 0$.

$4^x = (2^x)^2 = u^2$, so the equation becomes $u^2 - 5u + 4 = 0$ **M1**

Step 2: Factorise.

$(u - 1)(u - 4) = 0 \Rightarrow u = 1$ or $u = 4$ **A1**

Step 3: Solve for x.

$2^x = 1 \Rightarrow x = 0$ **A1**

$2^x = 4 \Rightarrow x = 2$ **A1**

Both values $u = 1 > 0$ and $u = 4 > 0$ are valid; no rejection necessary. **A1**

Therefore $x = 0$ or $x = 2$.

Note: Award first A1 only if both roots of the quadratic stated.

Q29 — When one account overtakes another [HARD]

Step 1: Set up the equation $V_A = V_B$.

$2000e^{0.04t} = 3000e^{0.025t}$ **M1**

$e^{0.04t-0.025t} = \frac{3000}{2000} = 1.5$

$e^{0.015t} = 1.5$ **A1**

Step 2: Solve for t.

$0.015t = \ln 1.5 \Rightarrow t = \frac{\ln 1.5}{0.015}$ **M1**

GDC: $t = \frac{\ln 1.5}{0.015} \approx 27.031\ldots$ years **A1**

Step 3: Convert to nearest month.

$0.031\ldots \times 12 \approx 0.38$ months, so $t \approx 27$ years 0 months.

Account A first overtakes Account B after approximately **27** years and **0** months (i.e. at $t \approx 27.0$ years). **A1**

Note: Award final A1 for 27 years 0 months or equivalent 27.0 years.

Q30 — Log equation, domain check, reject invalid root [HARD]

Step 1: State domain restriction.

For $\log_5(2x+3)$: $2x+3>0 \Rightarrow x > -\frac{3}{2}$.
For $\log_5(x-1)$: $x-1>0 \Rightarrow x>1$.
Combined domain: $x>1$. **A1**

Step 2: Combine into a single logarithmic equation.

$\log_5(2x+3) + \log_5(x-1) = 2$
$\log_5[(2x+3)(x-1)] = 2$ **M1**

$(2x+3)(x-1) = 25$
$2x^2 + x - 3 = 25 \Rightarrow 2x^2 + x - 28 = 0$ **A1**

Step 3: Solve the quadratic.

GDC or formula: $x = \frac{-1 \pm \sqrt{1+224}}{4} = \frac{-1 \pm 15}{4}$

$x = \frac{14}{4} = \frac{7}{2}$ or $x = \frac{-16}{4} = -4$ **A1**

Step 4: Reject invalid solution.

$x=-4$ does not satisfy $x>1$, so it is rejected as both logarithm arguments would be negative. **R1**

Therefore $x = \frac{7}{2}$.