IB Math AA HL — Topic 1

Sequences & Series

Sigma Notation · Arithmetic & Geometric · Applications · Compound Interest & Depreciation

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

- **Arithmetic sequence:** $u_n = u_1 + (n-1)d$; $\quad S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$
- **Geometric sequence:** $u_n = u_1 r^{n-1}$; $\quad S_n = \frac{u_1(r^n - 1)}{r-1}, r \neq 1$
- **Sum to infinity (GP):** $S_\infty = \frac{u_1}{1-r}$, valid only when $|r| < 1$; diverges if $|r| \geq 1$
- **nth term from partial sums:** $u_n = S_n - S_{n-1}$ for $n \geq 2$; $u_1 = S_1$
- **Sigma notation:** $\sum_{k=1}^{n}(a + (k-1)d)$ is arithmetic; $\sum_{k=1}^{n} u_1 r^{k-1}$ is geometric
- **Compound interest:** $FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$, where $k =$ compoundings per year, $n =$ years
- **Depreciation (reducing balance):** $V_n = V_0\left(1 - \frac{d}{100}\right)^n$; this is a GP with ratio $r = 1 - \frac{d}{100}$

GDC Tips

- **TVM Solver (Finance app):** set N, $I\%$, PV, PMT, FV, P/Y, C/Y carefully; sign convention: money paid out is negative, money received is positive.
- **Finding first n exceeding a threshold:** store u_n or S_n as a sequence in a List, then use Table or scroll to find the first entry crossing the target value.
- **Summing many terms:** use `sum(seq(expression, k, 1, n))` on the GDC rather than computing by hand; cross-check with the closed formula.
- **Comparing two investment schemes:** graph both balance functions on the same axes and use Intersection to find the crossover year; round up for "when does scheme A overtake?".
- **Logarithms for time problems:** if the TVM solver is unavailable, solve $A(1+r)^n = B$ by taking ln of both sides; confirm the integer answer satisfies the inequality in context.
- **Inflation-adjusted value:** divide the nominal future value by $(1+i)^n$ where i is the annual inflation rate; evaluate on the GDC and round to 2 decimal places for money.

Common Mistakes to Avoid

1. **Confusing u_n with S_n.** Writing $u_n = S_n$ instead of $u_n = S_n - S_{n-1}$; always check whether the question asks for a term or a sum.
2. **Forgetting to justify convergence.** Stating S_∞ without verifying $|r| < 1$; full marks require an explicit check and the conclusion that the series converges.
3. **Off-by-one error in u_n.** Using $u_n = u_1 + nd$ instead of $u_1 + (n-1)d$; equivalently starting sigma sums at $k = 0$ when the sequence is indexed from $k = 1$.
4. **Wrong compounding frequency.** Dividing the annual rate by 12 but leaving n in years (or vice versa); n must equal the total number of compounding periods, not the number of years.
5. **Rounding n in the wrong direction.** Rounding down when the question asks "after how many years does the value exceed" a target; always round up and verify the inequality holds at that integer.
6. **Sign errors in the common ratio or difference.** Assuming d or r is positive without checking; a decreasing AP has $d < 0$ and a decaying GP has $0 < r < 1$, both of which change inequality directions when solving.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)

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1. EASY No Calculator **[3 marks]**

An arithmetic sequence has first term $u_1 = 7$ and common difference $d = 4$, where $u_1, d \in \mathbb{Z}^+$. Write down the values of u_2, u_3, and u_4.

2. EASY No Calculator **[3 marks]**

The nth term of a sequence is given by $u_n = 3n - 1$, where $n \in \mathbb{Z}^+$. Evaluate $\sum_{n=1}^{4} u_n$.

3. EASY Calculator **[3 marks]**

A geometric sequence has first term $u_1 = 5$ and common ratio $r = 2$, where $u_1, r \in \mathbb{Z}^+$. The table below shows the first four terms of the sequence.

n	1	2	3	4
u_n	5	10	a	b

Find the values of a and b.

4. EASY No Calculator **[3 marks]**

An arithmetic sequence has $u_1 = 3$ and $u_5 = 19$, where $u_1, u_5 \in \mathbb{Z}^+$. Find the common difference d.

5. EASY Calculator **[3 marks]**

A savings account pays compound interest at a rate of 3% per annum. Mia deposits $2000 into the account. Find the value of the account, in dollars, after 4 years. Give your answer correct to 2 decimal places.

6. EASY No Calculator **[3 marks]**

A geometric series has first term $u_1 = 12$ and common ratio $r = \frac{1}{3}$, where $u_1 \in \mathbb{Z}^+$ and $r \in \mathbb{Q}$. Write down why the series has a sum to infinity, and find S_∞.

7. EASY Calculator **[3 marks]**

A car is purchased for $18000. Its value depreciates by 15% each year. Find the value of the car, in dollars, after 3 years. Give your answer correct to 2 decimal places.

8. EASY Calculator **[3 marks]**

A sequence of figures is built from rows of small squares. Figure 1 has 1 row of 3 squares, Figure 2 has 2 rows of 3 squares, and Figure 3 has 3 rows of 3 squares, as shown.

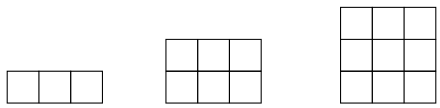

Fig. 1 Fig. 2 Fig. 3

Write down the total number of squares in Figure n. Hence find the total number of squares in Figure 10.

9. EASY Calculator **[3 marks]**

An arithmetic series has first term $u_1 = 6$, common difference $d = 4$, and n terms, where $u_1, d, n \in \mathbb{Z}^+$. Find the sum of the first 20 terms.

10. EASY Calculator **[3 marks]**

A bank account pays compound interest at a nominal annual rate of 6%, compounded monthly. Paulo invests $5000. Find the value of the account, in dollars, after 2 years. Give your answer correct to 2 decimal places.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

The sum of the first n terms of an arithmetic sequence is given by $S_n = 3n^2 - 5n$, where $n \in \mathbb{Z}^+$. Find the first term u_1, the common difference d, and the 15th term u_{15}.

12. MEDIUM No Calculator **[4 marks]**

A geometric sequence has first term $u_1 = 48$ and common ratio r, where $r \in \mathbb{R}$. The sum to infinity of the sequence is 80.

(a) Find the value of r. State clearly why the sum to infinity exists. [2]

(b) Find the value of n such that $u_n < 1$. [2]

13. MEDIUM Calculator **[4 marks]**

A factory purchases a machine for \$24000. Each year its value depreciates by 18% of its value at the start of that year.

(a) Write down the value of the machine at the end of the first year. [1]

(b) Find the value of the machine at the end of year 8, giving your answer to the nearest dollar. [2]

(c) Find the first year in which the value of the machine falls below \$3000. [1]

14. MEDIUM No Calculator **[4 marks]**

The table below shows selected terms of an arithmetic sequence $\{u_n\}$, where $n \in \mathbb{Z}^+$.

n	2	5	8	k
u_n	11	23	35	71

(a) Write down the common difference d. [1]

(b) Find the first term u_1. [1]

(c) Find the value of k. [2]

15. MEDIUM Calculator **[4 marks]**

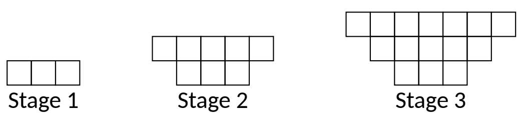
Fatima invests $5500 in an account that pays a nominal annual interest rate of 3.6%, compounded monthly. She makes no further deposits or withdrawals.

(a) Find the value of her investment after 6 years, giving your answer to two decimal places. [2]

(b) Find the number of complete months required for the investment to exceed $7000. [2]

16. MEDIUM Calculator **[4 marks]**

The diagram shows a geometric pattern of shaded squares. Stage 1 has 3 squares in the bottom row, Stage 2 has $3 + 5 = 8$ squares, and each new stage adds a row of squares on top, with the number of squares in each new row following an arithmetic sequence with first term 3 and common difference 2.



(a) Write down the number of squares added at Stage n. [1]

(b) Find the total number of squares at Stage n. [2]

(c) Find the first stage at which the total number of squares exceeds 500. [1]

17. MEDIUM Calculator **[4 marks]**

Two savings plans are compared over a period of years. Plan A pays simple interest: an initial deposit of $2000 earns $90 per year. Plan B pays compound interest: an initial deposit of $2000 grows at 3.8% per annum, compounded annually. Find the first year in which the total value of Plan B exceeds the total value of Plan A.

18. MEDIUM No Calculator **[4 marks]**

Consider the series $\sum_{k=3}^{n}(6k-5)$.

(a) Write down the first term and the last term of this series in terms of n. [1]

(b) Show that $\sum_{k=3}^{n}(6k-5) = 3n^2 - 2n - 8$. [3]

19. MEDIUM Calculator **[4 marks]**

An arithmetic sequence has $u_1 = 7$ and $u_{20} = 64$. A geometric sequence has $v_1 = 7$ and $v_{20} = 64$, with all terms positive.

(a) Find the common difference d of the arithmetic sequence. [1]

(b) Find the common ratio r of the geometric sequence, giving your answer to four significant figures. [2]

(c) Find v_5, giving your answer to three significant figures. [1]

20. MEDIUM Calculator **[4 marks]**

A ball is dropped from a height of 3.2 m. After each bounce it rises to 75% of the height from which it fell. The ball continues to bounce indefinitely.

(a) Show that the total vertical distance travelled by the ball (down and up) after the first drop and all subsequent bounces can be written as $3.2 + 2(3.2)(0.75) + 2(3.2)(0.75)^2 + \cdots$ [1]

(b) Find the total vertical distance travelled, justifying that the sum to infinity exists. [3]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **No Calculator** **[5 marks]**

An arithmetic sequence has first term $u_1 \in \mathbb{R}$ and common difference $d \in \mathbb{R}$, $d \neq 0$. The 1st, 4th and 13th terms of this arithmetic sequence form a geometric sequence. The sum of the first ten terms of the arithmetic sequence is $S_{10} = 120$.

(a) Show that $u_1 = \frac{3}{2}d$. [3]

(b) Find the values of u_1 and d. [2]

22. **HARD** **Calculator** **[5 marks]**

A savings account pays compound interest at a rate of $r\%$ per annum, compounded monthly. Anika deposits $5000 into the account. After 3 years (36 months) the balance is $5807.36.

(a) Find the value of r. [2]

(b) Find the total number of complete months for the balance to first exceed $8000. [3]

23. HARD No Calculator **[5 marks]**

The sum of the first n terms of a series is given by $S_n = 3n^2 - n$, where $n \in \mathbb{Z}^+$.

(a) Write down S_1 and S_2. [1]

(b) Find an expression for the nth term u_n in terms of n, for $n \geq 2$. [2]

(c) Show that the series is arithmetic, stating the first term and common difference. [2]

24. HARD No Calculator **[5 marks]**

A geometric sequence has first term $a \in \mathbb{R}^+$ and common ratio $r \in \mathbb{R}$. Three consecutive terms of the sequence are $k-4$, k, and $3k-16$, where $k \in \mathbb{R}$.

(a) Find the two possible values of k. [3]

(b) For each value of k, determine whether the series has a sum to infinity, justifying your answer. [2]

25. HARD Calculator **[5 marks]**

A factory purchases a machine for $24000. The machine depreciates at a rate of 18% per year on a reducing-balance basis. A straight-line depreciation model would reduce the value by a fixed amount each year so that the machine reaches $0 after exactly 10 years. The table below shows values under both models for selected years.

Year n	Reducing-balance value ($)	Straight-line value ($)
0	24000.00	24000.00
1	19680.00	21600.00
2	16137.60	19200.00
5	8897.76	12000.00
7	5982.85	7200.00
8	4905.94	4800.00
10	3298.75	0.00

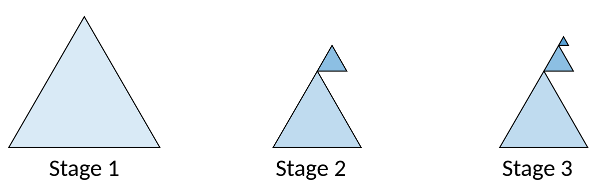
(a) Write down the annual straight-line depreciation amount. [1]

(b) Using the table, verify that the reducing-balance value first falls below the straight-line value in year 1. Find the first subsequent year in which the reducing-balance value rises back above the straight-line value. [2]

(c) Find the total depreciation under the reducing-balance model over the first 10 complete years, giving your answer to the nearest dollar. [2]

26. HARD No Calculator **[5 marks]**

A shaded figure is built in stages. Stage 1 is an equilateral triangle of side length 24 cm, with area A_1. At each subsequent stage, exactly one new equilateral triangle is attached to the figure, with side length equal to one third of the side length used at the previous stage. Let A_n denote the area of the triangle attached at stage n (so A_1 is the area of the original triangle).



(a) Show that $A_1 = 144\sqrt{3}$ cm^2. [1]

(b) Show that the areas $A_1, A_2, A_3, \ldots$ form a geometric sequence, and find the common ratio. [2]

(c) Find the total area of the figure as the number of stages tends to infinity, giving your answer in the form $p\sqrt{3}$ where $p \in \mathbb{Q}$. [2]

27. **HARD** **Calculator** **[5 marks]**

A sequence is defined by $u_1 = 6$ and $u_{n+1} = u_n + 4n + 1$ for $n \in \mathbb{Z}^+$.

(a) Write down the values of u_2, u_3 and u_4. [1]

(b) Show that $u_n = 2n^2 - n + 5$ for $n \geq 1$. [3]

(c) Hence find the value of $\sum_{n=1}^{20} u_n$. [1]

28. **HARD** **Calculator** **[5 marks]**

An arithmetic series has first term $a \in \mathbb{R}$ and common difference $d \in \mathbb{R}$. The sum of the first n terms is S_n. It is given that $u_7 = 6$ and $S_{12} = 90$.

(a) Find a and d. [2]

(b) The terms u_1, u_p, u_q (for integers p, q with $1 < p < q$) form the first three terms of a geometric sequence with sum to infinity equal to 48. Find the values of p and q, justifying that the geometric series converges. [3]

29. HARD Calculator **[5 marks]**

A loan of $15000 is taken out at the start of year 1. Interest is charged at 6.5% per annum, compounded annually. The borrower repays $\$P$ at the end of each year, where $P \in \mathbb{R}^+$.

(a) Show that the amount owed at the end of year n, after the repayment, is

$$A_n = 15000 \times 1.065^n - P\left(\frac{1.065^n - 1}{0.065}\right).$$

[2]

(b) The loan is fully repaid after exactly 8 years (i.e. $A_8 = 0$). Find the value of P, giving your answer to the nearest cent. [2]

(c) Find the total amount of interest paid over the 8 years, giving your answer to the nearest dollar. [1]

30. HARD No Calculator **[5 marks]**

The kth term of a series is given by $u_k = \dfrac{4}{(2k-1)(2k+1)}$ for $k \in \mathbb{Z}^+$.

(a) Show that $\dfrac{4}{(2k-1)(2k+1)} = \dfrac{2}{2k-1} - \dfrac{2}{2k+1}$. [1]

(b) Hence find a closed form for $\displaystyle\sum_{k=1}^{n} u_k$. [3]

(c) Hence find $\displaystyle\sum_{k=1}^{\infty} u_k$, justifying that this sum exists. [1]

Mark Schemes

Q1 — Terms of an Arithmetic Sequence EASY

Using $u_n = u_1 + (n-1)d$ with $u_1 = 7$, $d = 4$: **M1**

$u_2 = 7 + 4 = 11, \quad u_3 = 7 + 8 = 15, \quad u_4 = 7 + 12 = 19$ **A1 A1**

Note: Award A1 for all three values correct; all three must be stated for full marks.

Q2 — Evaluating a Sum EASY

Writing out the terms: $u_1 = 2$, $u_2 = 5$, $u_3 = 8$, $u_4 = 11$ **M1**

$\sum_{n=1}^{4}(3n-1) = 2 + 5 + 8 + 11$ **(M1)**

$= 26$ **A1**

Q3 — Terms of a Geometric Sequence EASY

Using $u_n = u_1 \cdot r^{n-1}$ with $u_1 = 5$, $r = 2$: **M1**

$a = u_3 = 5 \times 2^2 = 20$ **A1**

$b = u_4 = 5 \times 2^3 = 40$ **A1**

Q4 — Common Difference from Two Terms EASY

Using $u_5 = u_1 + 4d$: **M1**

$19 = 3 + 4d$ **(M1)**

$4d = 16 \implies d = 4$ **A1**

Q5 — Compound Interest EASY

Using $FV = PV \times (1 + \frac{r}{100})^n$ with $PV = 2000$, $r = 3$, $n = 4$: **M1**

$FV = 2000 \times (1.03)^4 = 2000 \times 1.12550881\ldots$ **(A1)**

$FV = \$2251.02$ (to 2 d.p.) **A1**

Q6 — Sum to Infinity EASY

Since $|r| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$, the series converges and S_∞ exists. **R1**

Using $S_\infty = \dfrac{u_1}{1-r}$: **M1**

$S_\infty = \dfrac{12}{1-\frac{1}{3}} = \dfrac{12}{\frac{2}{3}} = 18$ **A1**

Q7 — Reducing-Balance Depreciation EASY

Using $V = V_0 \times (1-0.15)^n$ with $V_0 = 18000$, $n = 3$: **M1**

$V = 18000 \times (0.85)^3 = 18000 \times 0.614125$ **(A1)**

$V = \$11054.25$ (to 2 d.p.) **A1**

Q8 — Figurate Counting Pattern EASY

Each figure n has n rows, each with 3 squares, so the total number of squares in Figure n is $3n$. **A1**

Substituting $n = 10$: **M1**

Total $= 3 \times 10 = 30$ **A1**

Q9 — Sum of an Arithmetic Series EASY

Using $S_n = \dfrac{n}{2}(2u_1 + (n-1)d)$ with $u_1 = 6$, $d = 4$, $n = 20$: **M1**

$S_{20} = \dfrac{20}{2}(2(6) + 19(4)) = 10(12 + 76)$ **(A1)**

$S_{20} = 10 \times 88 = 880$ **A1**

Q10 — Compound Interest Monthly EASY

Using $FV = PV \times (1 + \frac{r}{100k})^{kn}$ with $PV = 5000$, $r = 6$, $k = 12$, $n = 2$: **M1**

$FV = 5000 \times (1.005)^{24} = 5000 \times 1.12715977\ldots$ **(A1)**

$FV = \$5635.80$ (to 2 d.p.) **A1**

Q11 — Arithmetic Sequence from Sn MEDIUM

Using $u_1 = S_1$: $u_1 = 3(1)^2 - 5(1) = -2$ **A1**

Using $u_2 = S_2 - S_1$: $S_2 = 3(4) - 5(2) = 2$, so $u_2 = 2 - (-2) = 4$ **M1**

$d = u_2 - u_1 = 4 - (-2) = 6$ **A1**

$u_{15} = u_1 + 14d = -2 + 14(6) = 82$ **A1**

Q12 — Sum to Infinity and Threshold Term MEDIUM

(a) $\dfrac{u_1}{1-r} = 80 \implies \dfrac{48}{1-r} = 80$ **M1**

$1 - r = \dfrac{48}{80} = 0.6 \implies r = 0.4$; since $|r| = 0.4 < 1$, the sum to infinity exists. **A1**

(b) $u_n = 48(0.4)^{n-1} < 1 \implies (0.4)^{n-1} < \dfrac{1}{48}$ **M1**

Taking logarithms: $n - 1 > \dfrac{\ln(1/48)}{\ln(0.4)} \approx 4.21 \implies n > 5.21 \implies n = 6$ **A1**

Q13 — Depreciation Reducing Balance MEDIUM

(a) Value after year 1 $= 24000 \times 0.82 = \$19680$ **A1**

(b) Value after year $n = 24000 \times (0.82)^n$ **M1**

GDC: $24000 \times (0.82)^8 = 4905.938\ldots$, so value $= \$4906$ (to the nearest dollar) **A1**

(c) GDC: $24000 \times (0.82)^{10} = 3298.75\ldots > 3000$; $24000 \times (0.82)^{11} = 2704.98\ldots < 3000$.
The value first falls below \$3000 in year 11. **A1**

Q14 — Arithmetic Sequence from a Table MEDIUM

(a) From $u_2 = 11$ to $u_5 = 23$: difference $= 12$ over 3 steps, so $d = 4$. **A1**

(b) $u_1 = u_2 - d = 11 - 4 = 7$ **A1**

(c) $u_k = u_1 + (k-1)d \implies 71 = 7 + (k-1)(4)$ **M1**

$64 = 4(k-1) \implies k - 1 = 16 \implies k = 17$ **A1**

Q15 — Compound Interest Monthly MEDIUM

(a) Monthly rate $= \dfrac{3.6\%}{12} = 0.3\%$; number of periods $= 72$. **M1**

GDC: $A = 5500 \times (1.003)^{72} = 5500 \times 1.240701\ldots = \$6823.86$ **A1**

(b) Solve $5500 \times (1.003)^m > 7000$: $(1.003)^m > \dfrac{7000}{5500} = 1.2\overline{72}$ **M1**

GDC: $m > \dfrac{\ln(7000/5500)}{\ln(1.003)} = 80.508\ldots$
Check: at $m = 80$, $A = \$6989.36 < 7000$; at $m = 81$, $A = \$7010.33 > 7000$.
So $m = 81$ complete months. **A1**

Q16 — Stacked Squares Pattern MEDIUM

(a) The number of squares added at Stage n is $2n+1$. **A1**

(b) Total at Stage $n = \sum_{k=1}^{n}(2k+1) = 2 \cdot \dfrac{n(n+1)}{2} + n = n(n+1) + n$ **M1**

$= n^2 + n + n = n^2 + 2n$ **A1**

Note: Accept $n(n+2)$.

(c) Solve $n^2 + 2n > 500$: GDC or completing the square gives $n > 21.4\ldots$, so $n = 22$. **A1**

Q17 — Comparing Investment Plans MEDIUM

Plan A after t years: $V_A = 2000 + 90t$. Plan B after t years: $V_B = 2000 \times (1.038)^t$. **M1**

Require first integer t with $V_B > V_A$: set up a table on the GDC. **M1**

At $t = 9$: $V_A = 2810.00$, $V_B = 2797.73$ ($V_B < V_A$).
At $t = 10$: $V_A = 2900.00$, $V_B = 2904.05$ ($V_B > V_A$). **A1**

Plan B first exceeds Plan A in year 10. **A1**

Q18 — Sigma Notation Closed Form MEDIUM

(a) First term ($k = 3$): $6(3) - 5 = 13$. Last term ($k = n$): $6n - 5$. **A1**

(b) Number of terms $= n - 3 + 1 = n - 2$. **M1**

$\sum_{k=3}^{n}(6k-5) = \dfrac{(n-2)}{2}(13 + (6n-5)) = \dfrac{(n-2)(6n+8)}{2}$ **M1**

$= (n-2)(3n+4) = 3n^2 + 4n - 6n - 8 = 3n^2 - 2n - 8$ **A1 AG**

Q19 — Arithmetic and Geometric Endpoints MEDIUM

(a) $d = \frac{u_{20} - u_1}{19} = \frac{64 - 7}{19} = \frac{57}{19} = 3$ **A1**

(b) $v_{20} = v_1 \cdot r^{19} \implies 64 = 7r^{19}$ **M1**

GDC: $r = \left(\frac{64}{7}\right)^{1/19} = 1.123526\ldots$, so $r = 1.124$ (4 s.f.) **A1**

(c) $v_5 = 7 \times (1.123526\ldots)^4 = 7 \times 1.593430\ldots = 11.1540\ldots$, so $v_5 = 11.2$ (3 s.f.) **A1**

Q20 — Bouncing Ball Sum to Infinity MEDIUM

(a) The ball falls 3.2 m, then rises 3.2×0.75 m and falls 3.2×0.75 m, then rises 3.2×0.75^2 m and falls 3.2×0.75^2 m, etc.
Total $= 3.2 + 2(3.2)(0.75) + 2(3.2)(0.75)^2 + \cdots$ **AG**

(b) The series $2(3.2)(0.75) + 2(3.2)(0.75)^2 + \cdots$ is geometric with first term $a = 2(3.2)(0.75) = 4.8$ and common ratio $r = 0.75$. **M1**

Since $|r| = 0.75 < 1$, the sum to infinity exists. **R1**

Total $= 3.2 + \frac{4.8}{1 - 0.75} = 3.2 + 19.2 = 22.4$ m **A1**

Q21 — AP and GP Combined HARD

(a) The 1st, 4th and 13th terms are u_1, $u_4 = u_1 + 3d$, $u_{13} = u_1 + 12d$. **(M1)**

For a geometric sequence: $(u_4)^2 = u_1 \cdot u_{13}$
$(u_1 + 3d)^2 = u_1(u_1 + 12d)$ **M1**

$u_1^2 + 6u_1d + 9d^2 = u_1^2 + 12u_1d$
$9d^2 = 6u_1d$; since $d \neq 0$, divide by $3d$: $u_1 = \frac{3}{2}d$ **A1 AG**

(b) $S_{10} = \frac{10}{2}(2u_1 + 9d) = 120 \implies 2u_1 + 9d = 24$ **M1**

Substitute $u_1 = \frac{3}{2}d$: $3d + 9d = 24 \implies 12d = 24 \implies d = 2$
$u_1 = \frac{3}{2}(2) = 3$ **A1**
Note: Both values required for A1. Check: $u_1 = 3$, $u_4 = 9$, $u_{13} = 27$; $9^2 = 81 = 3 \times 27$ ✓

Q22 — Compound Interest Threshold HARD

(a) Let monthly rate $= \frac{r}{1200}$. After 36 months: $5000\left(1 + \frac{r}{1200}\right)^{36} = 5807.36$ **M1**

GDC solver: $\left(1 + \frac{r}{1200}\right)^{36} = 1.161472\ldots \implies \frac{r}{1200} = 1.161472\ldots^{1/36} - 1 = 0.0041\overline{6}\ldots$

$r = 5.00$ (to 3 s.f.) **A1**

(b) Need smallest integer n (months) such that $5000\left(1+\frac{5}{1200}\right)^n > 8000$ **M1**

$(1.004167)^n > 1.6 \implies n > \frac{\ln 1.6}{\ln 1.004167} = 112.87\ldots$ **A1**

Check: at $n = 113$, balance $= \$7998.81 < 8000$; at $n = 114$, balance $= \$8032.14 > 8000$.
The balance first exceeds \$8000 after $n = 114$ complete months. **A1**

Q23 — Sn Polynomial and Un HARD

(a) $S_1 = 3(1)^2 - 1 = 2$; $S_2 = 3(4) - 2 = 10$ **A1**

(b) For $n \geq 2$: $u_n = S_n - S_{n-1}$ **M1**

$u_n = (3n^2 - n) - [3(n-1)^2 - (n-1)] = 3n^2 - n - 3n^2 + 6n - 3 + n - 1 = 6n - 4$ **A1**

(c) Check $n = 1$: $u_1 = S_1 = 2 = 6(1) - 4$, so $u_n = 6n - 4$ for all $n \geq 1$. **M1**

$u_n - u_{n-1} = (6n-4) - (6(n-1) - 4) = 6$, a constant.
The series is arithmetic with first term $u_1 = 2$ and common difference $d = 6$. **A1 AG**

Q24 — GP from Three Terms HARD

(a) For a geometric sequence: $\frac{k}{k-4} = \frac{3k-16}{k}$ **M1**

$k^2 = (k-4)(3k-16) = 3k^2 - 28k + 64$
$2k^2 - 28k + 64 = 0 \implies k^2 - 14k + 32 = 0$ **M1**

This does not factorise over the integers; using the quadratic formula:
$k = \frac{14 \pm \sqrt{196 - 128}}{2} = 7 \pm \sqrt{17}$
$k = 7 - \sqrt{17}$ or $k = 7 + \sqrt{17}$ **A1**

(b) For $k = 7 - \sqrt{17}$: terms are $k - 4 = -1.1231\ldots$, $k = 2.8769\ldots$, $3k - 16 = -7.3693\ldots$;
ratio $r = k/(k-4) = -2.5616\ldots$; check $(3k-16)/k = -2.5616\ldots$ ✓ (valid GP). $|r| > 1$, so S_∞ does not exist. **A1**

For $k = 7 + \sqrt{17}$: terms are $7.1231\ldots$, $11.1231\ldots$, $17.3693\ldots$;
ratio $r = 1.562\ldots$; check both ratios agree ✓ (valid GP). $|r| > 1$, so S_∞ does not exist. **R1**

Note: Neither value of k produces a convergent geometric series, since $|r| > 1$ in both cases.

Q25 — Depreciation Comparison HARD

(a) Annual straight-line depreciation $= 24000 \div 10 = \$2400$ **A1**

(b) From the table: at year 1, $V_1 = \$19680 < L_1 = \$21600$, so the reducing-balance value has already fallen below the straight-line value by year 1. **A1**

Continuing the table: at year 7, $V_7 = \$5982.85 < L_7 = \$7200.00$ (still below); at year 8, $V_8 = \$4905.94 > L_8 = \$4800.00$ (back above).
The reducing-balance value rises back above the straight-line value in year 8. **A1**

(c) Total depreciation $= 24000 - 24000(0.82)^{10}$ **M1**

$= 24000(1 - 0.82^{10}) = 24000 \times 0.862552\ldots = 20701.25\ldots$
Total depreciation $= \$20701$ (to the nearest dollar). **A1**

Q26 — Fractal Area Sum to Infinity **HARD**

(a) Area of an equilateral triangle with side s: $\dfrac{s^2\sqrt{3}}{4}$. With $s = 24$:

$$A_1 = \frac{24^2\sqrt{3}}{4} = \frac{576\sqrt{3}}{4} = 144\sqrt{3}\text{ cm}^2$$ **A1 AG**

(b) At stage n, the side length used is $24/3^{n-1}$, so the area is

$$A_n = \frac{(24/3^{n-1})^2\sqrt{3}}{4} = 144\sqrt{3} \times \frac{1}{9^{n-1}}$$ **M1**

Since exactly one new triangle is attached at every stage (side scaled by $\frac{1}{3}$, so area scaled by $\frac{1}{9}$), $\dfrac{A_n}{A_{n-1}} = \dfrac{1}{9}$ for all $n \geq 2$ – a constant ratio, so $A_1, A_2, A_3, \ldots$ form a geometric sequence with common ratio $r = \dfrac{1}{9}$.
A1

(c) Since $|r| = \frac{1}{9} < 1$, the sum to infinity exists. **R1**

$$S_\infty = \frac{A_1}{1-r} = \frac{144\sqrt{3}}{1-\frac{1}{9}} = \frac{144\sqrt{3}}{\frac{8}{9}} = 144\sqrt{3} \times \frac{9}{8} = 162\sqrt{3}$$

Total area $= 162\sqrt{3}$ cm^2, so $p = 162$. **A1**

Q27 — Recurrence Closed Form **HARD**

(a) $u_2 = 6 + 4(1) + 1 = 11$; $u_3 = 11 + 4(2) + 1 = 20$; $u_4 = 20 + 4(3) + 1 = 33$ **A1**

(b) Summing the recurrence from $k = 1$ to $n - 1$: **M1**

$$u_n - u_1 = \sum_{k=1}^{n-1}(4k+1) = 4 \cdot \frac{(n-1)n}{2} + (n-1) = 2n(n-1) + (n-1) = (n-1)(2n+1)$$ **M1**

$u_n = 6 + (n-1)(2n+1) = 6 + 2n^2 - n - 1 = 2n^2 - n + 5$ **A1 AG**

Note: Check $n = 1$: $2 - 1 + 5 = 6\,\checkmark$; $n = 2$: $8 - 2 + 5 = 11\,\checkmark$.

(c) $$\sum_{n=1}^{20} u_n = \sum_{n=1}^{20}(2n^2 - n + 5) = 2 \cdot \frac{20 \cdot 21 \cdot 41}{6} - \frac{20 \cdot 21}{2} + 5(20)$$ **M1**

$= 2(2870) - 210 + 100 = 5740 - 210 + 100 = 5630$ **A1**

Q28 — AP Parameters and Embedded GP HARD

(a) $u_7 = a + 6d = 6$ and $S_{12} = \frac{12}{2}(2a + 11d) = 90 \implies 2a + 11d = 15$ **M1**

From the first equation $a = 6 - 6d$; substitute: $2(6 - 6d) + 11d = 15 \implies 12 - d = 15 \implies d = -3$.
$a = 6 - 6(-3) = 24$ **A1**

Check: $S_{12} = 6(2(24) + 11(-3)) = 6(48 - 33) = 6(15) = 90$ ✓

(b) $u_1 = 24$, $u_p = 24 - 3(p - 1)$, $u_q = 24 - 3(q - 1)$. **(M1)**

For a GP: common ratio $R = \frac{u_p}{u_1}$. Sum to infinity $= \frac{u_1}{1 - R} = 48$.

$\frac{24}{1 - R} = 48 \implies 1 - R = \frac{1}{2} \implies R = \frac{1}{2}$; since $|R| < 1$, the geometric series converges. **M1**

$u_p = u_1 R = 12 \implies 24 - 3(p - 1) = 12 \implies p - 1 = 4 \implies p = 5$
$u_q = u_1 R^2 = 6 \implies 24 - 3(q - 1) = 6 \implies q - 1 = 6 \implies q = 7$ **A1**

Check: $u_1 = 24$, $u_5 = 12$, $u_7 = 6$: $12^2 = 144 = 24 \times 6$ ✓

Q29 — Loan Repayment HARD

(a) At end of year 1, after repayment: $A_1 = 15000(1.065) - P$. **M1**

At end of year 2: $A_2 = A_1(1.065) - P = 15000(1.065)^2 - P(1.065) - P$.
By induction (summing the geometric series of repayments):

$A_n = 15000(1.065)^n - P\left[1 + 1.065 + \cdots + 1.065^{n-1}\right] = 15000 \times 1.065^n - P \cdot \frac{1.065^n - 1}{0.065}$ **A1 AG**

(b) Set $A_8 = 0$: **M1**

GDC: $1.065^8 = 1.6549956\ldots$; $\frac{1.065^8 - 1}{0.065} = 10.076856\ldots$

$P = \frac{15000 \times 1.6549956\ldots}{10.076856\ldots} = \frac{24824.935\ldots}{10.076856\ldots} = 2463.56\ldots$
$P = \$2463.56$ **A1**

(c) Total paid $= 8P = 8 \times 2463.56 = 19708.48\ldots$
Total interest $= 19708.48 - 15000 = \$4708$ (to the nearest dollar). **A1**

Q30 — Telescoping Series HARD

(a) $\frac{2}{2k - 1} - \frac{2}{2k + 1} = \frac{2(2k + 1) - 2(2k - 1)}{(2k - 1)(2k + 1)} = \frac{4k + 2 - 4k + 2}{(2k - 1)(2k + 1)} = \frac{4}{(2k - 1)(2k + 1)}$ **A1 AG**

(b) $\sum_{k=1}^{n} u_k = \sum_{k=1}^{n} \left(\frac{2}{2k - 1} - \frac{2}{2k + 1}\right)$ **M1**

This is a telescoping sum: **M1**

$= \left(\frac{2}{1} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{5}\right) + \cdots + \left(\frac{2}{2n - 1} - \frac{2}{2n + 1}\right) = 2 - \frac{2}{2n + 1} = \frac{4n}{2n + 1}$, i.e. $u_1 + \cdots + u_n =$

$4n/(2n+1)$ **A1**

(c) As $n \to \infty$: $\dfrac{4n}{2n+1} \to \dfrac{4}{2} = 2.$ **A1**

The series converges (the partial sums tend to a finite limit), and $\displaystyle\sum_{k=1}^{\infty} u_k = 2.$ **R1**

Note: The R1 requires a justification sentence, e.g. "the partial sums form a sequence converging to 2".