IB Math AA HL — Topic 1

Simple Proof & Reasoning

Proof by Deduction · Disproof by Counterexample

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:____________________________ Date:______________

Score:_________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

- **Consecutive integers:** any two consecutive integers can be written as n and $n+1$ (or $2n$ and $2n+1$ for even/odd), where $n \in \mathbb{Z}$.
- **Even/odd forms:** an even integer $= 2k$; an odd integer $= 2k+1$, for some $k \in \mathbb{Z}$.
- **Divisibility test:** $m \mid a$ if and only if $a = mk$ for some $k \in \mathbb{Z}$.
- **Difference of two squares:** $a^2 - b^2 = (a-b)(a+b)$; use this as the engine of divisibility and factorisation proofs.
- **Disproof by counterexample:** to disprove "$P(n)$ is true for all n", exhibit one explicit value $n = n_0$ for which $P(n_0)$ is false, and evaluate both sides.
- **Sum of first n positive integers:** $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.
- **Proof by deduction:** start from general forms (e.g. $n \in \mathbb{Z}$), apply algebraic steps, and end with a sentence concluding what has been proved.

GDC Tips

- **Searching for counterexamples:** use the GDC table feature (G-Solve $\rightarrow$ Table) to evaluate an expression $f(n)$ over a range; the first row that fails the claimed property is your counterexample.
- **Divisibility check:** store candidate values and use int(n/m) vs. n/m to test whether $m \mid n$; a non-integer quotient confirms failure.
- **Conjecture from a table:** generate $f(n)$ for $n = 1, 2, \ldots, 6$ in the spreadsheet app, observe the pattern, then write a conjecture — state it clearly before attempting a proof.
- **Finding where one expression overtakes another:** graph both on the same axes, use Intersection to find the crossover point; prove the inequality holds beyond that point by algebra.
- **Verifying a factorisation:** expand your factored form on the GDC (e.g. with `expand(` in CAS) to confirm it matches the original expression before writing the proof.
- **Evidence $\neq$ proof:** after checking $f(n)$ for 10 values on the GDC, always state "this evidence supports but does not prove the result"; a deductive argument is still required for full marks.

Common Mistakes to Avoid

1. **Assuming the result.** Writing the conclusion at the start of a "show that" proof and working backwards is circular; always begin from the hypothesis and deduce the target.
2. **Non-general starting point.** Using a specific number (e.g. $n = 3$) instead of a general integer in a deduction proof produces a verification, not a proof; define $n \in \mathbb{Z}$ at the outset.
3. **Incomplete counterexample.** Stating "let $n = 5$" without computing both sides of the inequality or showing the claim fails loses the accuracy mark; always evaluate the expression explicitly.
4. **Confusing "even" with "divisible by 4".** Writing $2n$ for an even integer is correct, but $(2n)^2 = 4n^2$; students often claim $4n^2$ is merely even rather than a multiple of 4.
5. **Dropping the conclusion sentence.** Ending algebra with an expression such as $6k$ without writing "which is a multiple of 6, so the result holds" loses the R1 reasoning mark in the mark scheme.
6. **Wrong number set in deduction.** Claiming a result "for all n" when the proof requires $n \in \mathbb{Z}^+$ (e.g. a sum formula) is an error; state the domain of n precisely and check it matches the question.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. **EASY** **No Calculator** **[3 marks]**

Let $n \in \mathbb{Z}$. Show that the product of two consecutive integers is always even.

2. **EASY** **Calculator** **[3 marks]**

Let $f(n) = n^2 + n + 11$, where $n \in \mathbb{Z}^+$. A student claims that $f(n)$ is prime for all positive integers n. Find a counterexample to disprove this claim.

n	$f(n) = n^2 + n + 11$
1	13
2	17
3	23
4	31

3. **EASY** **No Calculator** **[3 marks]**

Let $n \in \mathbb{Z}$. Show that $(n+3)^2-(n-3)^2$ is always a multiple of 12.

4. EASY Calculator **[3 marks]**

Let $g(n) = 2^n - 1$, where $n \in \mathbb{Z}^+$. The table below shows values of $g(n)$ for small n. A student claims that $g(n)$ is prime for every positive integer n. Find the smallest counterexample and state the value of n and $g(n)$.

n	$g(n) = 2^n - 1$
1	1
2	3
3	7
4	15

5. EASY No Calculator **[3 marks]**

Let p and q be odd integers. Show that p^2+q^2 is always even but never divisible by 4.

6. EASY Calculator **[3 marks]**

A student claims: "For all real numbers x, if $x^2 > 9$ then $x > 3$." Decide whether this claim is true or false. Justify your answer with a proof or a counterexample.

7. EASY Calculator **[3 marks]**

Let $n \in \mathbb{Z}^+$. Show that $n^2 + 3n + 2$ is always the product of two consecutive integers and hence always even.

8. EASY Calculator **[3 marks]**

Let $k(n) = n^2 + n + 1$, where $n \in \mathbb{Z}^+$. The table below shows values of $k(n)$ for small n. A student conjectures that $k(n)$ is prime for every positive integer n. Find a counterexample to disprove this claim.

n	$k(n) = n^2 + n + 1$
1	3
2	7
3	13
4	21

9. EASY Calculator **[3 marks]**

Let $a, b \in \mathbb{R}$. By considering $(a-b)^2 \geq 0$, show that $a^2+b^2 \geq 2ab$.

10. EASY Calculator **[3 marks]**

Let $n \in \mathbb{Z}$. Show that the difference between the squares of any two consecutive odd integers is divisible by 8.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

Let $n \in \mathbb{Z}^+$. Show that the product of any two consecutive odd integers is always odd.

12. MEDIUM Calculator **[4 marks]**

Consider the statement: "For all integers $n \geq 1$, the expression $n^2 + n + 41$ is prime." By finding the smallest counterexample, disprove this statement.

n	$n^2 + n + 41$
1	43
2	47
3	53
4	61

13. MEDIUM No Calculator **[4 marks]**

Let $m, n \in \mathbb{Z}$. Prove that if m and n are both odd, then $m^2 + n^2$ is always even but never divisible by 4.

14. MEDIUM Calculator **[4 marks]**

Consider the statement: "For all real numbers x, $\dfrac{x^2+4}{x^2+1} \geq 3$." Determine whether this statement is true or false.

If false, find a counterexample; if true, prove it.

15. MEDIUM No Calculator **[4 marks]**

Let $n \in \mathbb{Z}$. Prove that $n^2 - n$ is always even for any integer n. Hence show that $n^2 - n + 6$ is always even.

16. MEDIUM Calculator **[4 marks]**

Consider the claim: "For all integers $n \geq 2$, the expression $2^n - 1$ is prime." Using technology to search the range $2 \leq n \leq 12$, find the smallest value of n for which the claim fails and verify that the expression is composite at that

value.

17. MEDIUM Calculator **[4 marks]**

Let $a, b \in \mathbb{Z}$. By writing $a = 2p$ and $b = 2q+1$ where $p, q \in \mathbb{Z}$, prove that a^2+b^2+1 is always even when a is even and b is odd.

18. MEDIUM Calculator **[4 marks]**

The table below shows values of $f(n) = n^3 - n$ for $n = 1, 2, 3, 4, 5$, where $n \in \mathbb{Z}^+$.

n	$f(n) = n^3 - n$
1	0
2	6
3	24
4	60
5	120

By inspecting the table, state a conjecture about the divisibility of $f(n)$ by 6. Hence prove your conjecture for all $n \in \mathbb{Z}^+$ by factorising n^3-n.

19. MEDIUM Calculator **[4 marks]**

Let $x \in \mathbb{R}$. Prove that $x^2-6x+13 > 0$ for all real values of x by completing the square.

20. MEDIUM Calculator **[4 marks]**

Let $n \in \mathbb{Z}^+$. Consider the expression $5n^2 + 5n + 2$. A student claims this expression is always divisible by 6. Identify the flaw in the following argument and determine the correct statement about divisibility by 2.

> "Since $5n^2 + 5n = 5n(n+1)$ and $n(n+1)$ is always even, we have $5n(n+1)$ is divisible by 10, so $5n^2 + 5n + 2$ is divisible by 6."

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **No Calculator** **[5 marks]**

Let $n \in \mathbb{Z}^+$. Show that $n^3 - n$ is always divisible by 6.

22. **HARD** **No Calculator** **[5 marks]**

Let $a, b \in \mathbb{R}$. Show that $a^2 + ab + b^2 > 0$ for all a, b not both zero.

23. **HARD** **Calculator** **[5 marks]**

A student claims: "For every prime p, the expression $p^2 - p + 1$ is also prime." The table below shows values of p and $p^2 - p + 1$ for the first several primes p.

p	$p^2 - p + 1$	Prime?
2	3	Yes
3	7	Yes
5	21	No
7	43	Yes

(a) State the value of p that provides a counterexample to the student's claim, evaluate $p^2 - p + 1$ at that value, and explain why it disproves the claim. [3]

(b) Hence write down the smallest prime $q > 5$ for which $q^2 - q + 1$ is prime, and verify your answer. [2]

24. HARD No Calculator **[5 marks]**

Let $n \in \mathbb{Z}$. Prove that if n^2 is odd, then n is odd.

25. HARD Calculator **[5 marks]**

The function $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by $f(n) = n^2+n+41$. The table below lists values of $f(n)$ for small $n \in \mathbb{Z}^+$.

n	$f(n)$	Prime?
1	43	Yes
2	47	Yes
3	53	Yes
4	61	Yes
5	71	Yes

(a) A student conjectures that $f(n)$ is prime for all $n \in \mathbb{Z}^+$. Find the smallest value of n for which $f(n)$ is not prime, and state $f(n)$ at that value. [3]

(b) Explain why the evidence in the table, on its own, does not constitute a proof that $f(n)$ is prime for all $n \in \mathbb{Z}^+$. [2]

26. HARD Calculator **[5 marks]**

Let $n \in \mathbb{Z}$. Prove that the product of any four consecutive integers is always divisible by 24.

27. HARD No Calculator **[5 marks]**

Let $x \in \mathbb{R}$. Show that $\dfrac{x^4+1}{2} \geq x^2 - \dfrac{1}{2}$, and state when equality holds.

28. HARD Calculator **[5 marks]**

Let $n \in \mathbb{Z}^+$. It is claimed that $3^n > 2^n + n^3$ for all $n \geq 4$.

(a) Verify the claim for $n = 4$ and $n = 5$. [2]

(b) Use a graphing display calculator to find the smallest integer $n \geq 4$ for which $3^n - 2^n - n^3$ first exceeds 10^6. State this value of n and the corresponding value of $3^n - 2^n - n^3$. [2]

(c) State, with a reason, whether the evidence from parts (a) and (b) alone constitutes a proof of the claim for all $n \geq 4$. [1]

29. HARD Calculator **[5 marks]**

Let $a, b \in \mathbb{R}^+$ with $a \neq b$. Prove that $\frac{a}{b} + \frac{b}{a} > 2$.

30. HARD Calculator **[5 marks]**

Let $n \in \mathbb{Z}$. Prove by deduction that $n^2 + 3n + 2$ is never a prime number when $n \geq 1, n \in \mathbb{Z}$. Hence determine whether $n^2 + 3n + 2$ can be prime for $n \in \mathbb{Z}$ with $n < 0$, giving a specific example or a proof as appropriate.

Mark Schemes

Q1 — Product of Consecutive Integers is Even EASY

Write the two consecutive integers as n and $n+1$, where $n \in \mathbb{Z}$. **M1**

Either n or $n+1$ is even, so their product $n(n+1)$ is divisible by 2. **R1**

Therefore the product of any two consecutive integers is always even. **A1**

Q2 — Counterexample to a Primality Claim EASY

Testing $n = 10$: **M1**

$f(10) = 100 + 10 + 11 = 121 = 11 \times 11$ **A1**

Since 121 is not prime, the claim is false. $n = 10$ is a valid counterexample. **R1**

Q3 — Difference of Squares Divisible by 12 EASY

Expand using the difference of two squares:

$$(n+3)^2 - (n-3)^2 = [(n+3) + (n-3)][(n+3) - (n-3)] = (2n)(6) = 12n$$

M1A1

Since $n \in \mathbb{Z}$, $12n$ is a multiple of 12 for all integers n. **A1**

Q4 — Smallest Counterexample for Mersenne-style Primes EASY

Testing $n = 1$: $g(1) = 2^1 - 1 = 1$. **M1**

By definition, a prime number has exactly two distinct positive divisors; 1 has only one divisor, so 1 is not prime. **A1**

Since $g(1) = 1$ is not prime, the claim already fails at $n = 1$. The smallest counterexample is $n = 1$, $g(1) = 1$. **A1**

Note: $g(2) = 3$, $g(3) = 7$ are prime, but this is irrelevant once the smallest positive integer $n = 1$ is already shown to fail.

Q5 — Sum of Squares of Two Odd Integers EASY

Write $p = 2j+1$ and $q = 2k+1$ where $j, k \in \mathbb{Z}$. **M1**

Then $p^2 + q^2 = (2j+1)^2 + (2k+1)^2 = 4j^2 + 4j + 1 + 4k^2 + 4k + 1 = 4(j^2 + j + k^2 + k) + 2$. **A1**

This expression equals $2[2(j^2+j+k^2+k)+1]$, which is even. Since the bracket $2(j^2+j+k^2+k)+1$ is odd, p^2+q^2 is not divisible by 4. **A1**

Q6 — Counterexample to an Implication EASY

The claim is false. **A1**

Counterexample: let $x=-4$. Then $x^2=16>9$ but $x=-4<3$. **M1**

Since the hypothesis $x^2>9$ holds but the conclusion $x>3$ fails, the claim is disproved. **R1**

Q7 — Factorisation Shows Expression is Even EASY

Factorise: $n^2+3n+2=(n+1)(n+2)$. **M1**

$(n+1)$ and $(n+2)$ are consecutive integers. **A1**

One of any two consecutive integers is even, so their product $(n+1)(n+2)$ is always even. **A1**

Q8 — Counterexample to a Primality Claim EASY

Testing $n=1,2,3$: $k(1)=3$, $k(2)=7$, $k(3)=13$, all prime — the table evidence looks consistent with the claim. **M1**

Testing $n=4$: $k(4)=16+4+1=21=3\times 7$. **A1**

Since 21 is not prime, the claim is false. $n=4$ is a valid counterexample. **R1**

Q9 — Inequality from a Non-negative Square EASY

Since $(a-b)^2\geq 0$ for all $a,b\in\mathbb{R}$, **M1**

expand: $a^2-2ab+b^2\geq 0$. **A1**

Adding $2ab$ to both sides gives $a^2+b^2\geq 2ab$. **A1**

Q10 — Difference of Squares of Consecutive Odd Integers EASY

Write two consecutive odd integers as $2n+1$ and $2n+3$, where $n\in\mathbb{Z}$. **M1**

$(2n+3)^2-(2n+1)^2=[(2n+3)+(2n+1)][(2n+3)-(2n+1)]=(4n+4)(2)=8(n+1)$. **A1**

Since $n+1\in\mathbb{Z}$, the difference is always a multiple of 8. **A1**

Q11 — Product of Consecutive Odd Integers MEDIUM

Let the two consecutive odd integers be $2k-1$ and $2k+1$ where $k \in \mathbb{Z}^+$. **M1**

Their product is $(2k-1)(2k+1) = 4k^2 - 1$. **A1**

Write $4k^2 - 1 = 2(2k^2-1)+1$, which is of the form $2m+1$ where $m = 2k^2 - 1 \in \mathbb{Z}$. **M1**

Therefore the product of any two consecutive odd integers is always odd. **A1**

Q12 — Counterexample to a Primality Claim MEDIUM

Testing values using technology: for $n = 40$, $f(40) = 40^2 + 40 + 41 = 1600 + 40 + 41 = 1681$. **M1**

Note that $1681 = 41^2$, which is not prime. **A1**

Since $n = 40$ gives $n^2 + n + 41 = 41^2$, which is composite, the statement is false. **R1**

The smallest such counterexample is $n = 40$. **A1**

Q13 — Sum of Squares of Two Odd Integers MEDIUM

Write $m = 2a+1$ and $n = 2b+1$ where $a, b \in \mathbb{Z}$. **M1**

Then $m^2 + n^2 = (2a+1)^2 + (2b+1)^2 = 4a^2 + 4a + 1 + 4b^2 + 4b + 1 = 4(a^2+a+b^2+b)+2$. **A1**

Since $4(a^2+a+b^2+b)+2 = 2(2(a^2+a+b^2+b)+1)$, the expression is even. **M1**

The remainder when divided by 4 is $2 \neq 0$, so m^2+n^2 is even but never divisible by 4. **A1**

Q14 — Counterexample to an Inequality Claim MEDIUM

The statement is false. **A1**

Testing $x = 2$: **M1**

At $x = 2$, $\dfrac{x^2+4}{x^2+1} = \dfrac{8}{5}$ (i.e. $8/5) = 1.6$, and $1.6 < 3$. **A1**

Since the expression equals $1.6 < 3$ at $x = 2$, the statement is disproved. **R1**

Q15 — Parity of a Quadratic Expression MEDIUM

Write $n^2 - n = n(n-1)$, the product of two consecutive integers. **M1**

One of n or $n-1$ must be even, so $n(n-1)$ is always even, i.e. $2 \mid n^2 - n$. **A1**

Hence $n^2 - n + 6 = n(n-1) + 6$. **M1**

Since $n(n-1)$ is even and 6 is even, their sum is even, so $n^2 - n + 6$ is always even. **A1**

Q16 — First Failure of a Mersenne-style Claim MEDIUM

Using technology, compute $2^n - 1$ for $n = 2, 3, 4, \ldots$: **M1**

$n = 2$: 3 (prime); $n = 3$: 7 (prime); $n = 4$: $2^4 - 1 = 15$. **A1**

$15 = 3 \times 5$, so 15 is composite (has factors other than 1 and itself). **A1**

The smallest value for which the claim fails is $n = 4$, since $2^4 - 1 = 15 = 3 \times 5$ is not prime. **R1**

Q17 — Parity of a Sum of Squares Plus One MEDIUM

Substituting $a = 2p$ and $b = 2q + 1$ where $p, q \in \mathbb{Z}$: **M1**

$a^2 + b^2 + 1 = (2p)^2 + (2q + 1)^2 + 1 = 4p^2 + 4q^2 + 4q + 1 + 1.$ **A1**

$= 4p^2 + 4q^2 + 4q + 2 = 2(2p^2 + 2q^2 + 2q + 1).$ **M1**

Since $2p^2 + 2q^2 + 2q + 1 \in \mathbb{Z}$, the expression $a^2 + b^2 + 1$ is of the form $2 \times (\text{integer})$, so it is always even when a is even and b is odd. **A1**

Q18 — Divisibility by Six from a Table MEDIUM

Conjecture: $f(n) = n^3 - n$ is divisible by 6 for all $n \in \mathbb{Z}^+$. **A1**

Factorise: $n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$. **M1**

This is the product of three consecutive integers. Among any three consecutive integers, one is divisible by 3 and at least one is divisible by 2. **M1**

Therefore $(n - 1)n(n + 1)$ is divisible by $2 \times 3 = 6$ for all $n \in \mathbb{Z}^+$. **A1**

Q19 — Positivity by Completing the Square MEDIUM

Complete the square: $x^2 - 6x + 13 = (x - 3)^2 - 9 + 13.$ **M1**

$= (x - 3)^2 + 4.$ **A1**

Since $(x - 3)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $(x - 3)^2 + 4 \geq 0 + 4 = 4$. **M1**

Therefore $x^2 - 6x + 13 \geq 4 > 0$ for all $x \in \mathbb{R}$. **A1**

Q20 — Flaw in a Divisibility Argument MEDIUM

Flaw: The student incorrectly concludes that $5n(n + 1)$ being divisible by 10 implies $5n^2 + 5n + 2$ is divisible by 6; adding 2 to a multiple of 10 does not yield a multiple of 6. **R1**

Correct analysis: $5n^2 + 5n = 5n(n + 1)$. Since $n(n + 1)$ is the product of consecutive integers, it is always even, so $5n(n + 1)$ is divisible by 2. **M1**

Therefore $5n^2+5n+2 = 5n(n+1)+2 \equiv 0+0 = 0 \pmod 2$, so the expression is always even (divisible by 2). **A1**

Testing $n = 1$: $5+5+2 = 12$, divisible by 2 but also by 6; testing $n = 2$: $20+10+2 = 32$, not divisible by 6. The expression is always divisible by 2 but not always by 6. **A1**

Q21 — Divisibility of n Cubed Minus n by Six HARD

Factorise: $n^3 - n = n(n^2-1) = n(n-1)(n+1) = (n-1)n(n+1)$ **M1**

This is the product of three consecutive integers, where $n \in \mathbb{Z}^+$. **A1**

Among any three consecutive integers, at least one is divisible by 2 and exactly one is divisible by 3. **M1**

Therefore the product $(n-1)n(n+1)$ is divisible by $2 \times 3 = 6$. **R1**

Hence $n^3 - n$ is always divisible by 6 for all $n \in \mathbb{Z}^+$. **AG**

Q22 — Non-negativity of a Quadratic Form HARD

Complete the square in a: write $a^2+ab+b^2 = \left(a+\frac{b}{2}\right)^2 - \frac{b^2}{4} + b^2$ **M1**

$= \left(a+\frac{b}{2}\right)^2 + \frac{3b^2}{4}$ **A1**

Since $\left(a+\frac{b}{2}\right)^2 \geq 0$ and $\frac{3b^2}{4} \geq 0$, the sum is ≥ 0. **R1**

Equality requires $a + \frac{b}{2} = 0$ and $b = 0$ simultaneously, giving $a = b = 0$. **M1**

Since a and b are not both zero, the sum is strictly positive: $a^2+ab+b^2 > 0$. **A1**

Q23 — Counterexample to a Primality Claim HARD

(a)

From the table, $p = 5$ gives $p^2 - p + 1 = 25 - 5 + 1 = 21$. **M1**

$21 = 3 \times 7$, so 21 is not prime. **A1**

Since a prime input $p = 5$ produces a non-prime output, the claim is false. **R1**

(b)

$q = 7$: $7^2 - 7 + 1 = 49 - 7 + 1 = 43$, which is prime. **A1A1**

Q24 — Contrapositive Proof **HARD**

State the contrapositive: "If n is even then n^2 is even." **M1**

Let $n = 2k$ for some $k \in \mathbb{Z}$. **M1**

Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even. **A1**

The contrapositive has been proved; since a conditional statement and its contrapositive are logically equivalent, the original statement "if n^2 is odd then n is odd" is true. **R1**

Hence for all $n \in \mathbb{Z}$, if n^2 is odd then n is odd. **A1**

Q25 — Eulers Polynomial Counterexample **HARD**

(a)
Using a GDC to evaluate $f(n) = n^2 + n + 41$ and test primality:
$f(40) = 1600 + 40 + 41 = 1681 = 41^2$, which is not prime. **M1A1**

The smallest such value is $n = 40$, giving $f(40) = 1681$. **A1**

(b)
The table provides evidence for only five values of n; it does not verify all $n \in \mathbb{Z}^+$. A finite number of cases cannot establish a universal statement — a proof must cover infinitely many cases by a general argument.
R1R1

Q26 — Product of Four Consecutive Integers **HARD**

Let the four consecutive integers be $n, n+1, n+2, n+3$ for some $n \in \mathbb{Z}$. **M1**

Their product $P = n(n+1)(n+2)(n+3)$ can be written as $P = 4!\binom{n+3}{4}$, so $P = 24\binom{n+3}{4}$. **M1**

Since $\binom{n+3}{4}$ is always an integer for $n \in \mathbb{Z}$, we have $P = 24k$ for some integer k. **A1**

Alternatively: among four consecutive integers there are at least two even numbers (contributing a factor of at least $2 \times 4 = 8$) and at least one multiple of 3, giving divisibility by $8 \times 3 = 24$. **R1**

Hence the product of any four consecutive integers is always divisible by 24. **A1**

Q27 — Inequality via a Non-negative Squared Term **HARD**

Rearrange the inequality: $\dfrac{x^4+1}{2} \geq x^2 - \dfrac{1}{2}$ is equivalent to $x^4 + 1 \geq 2x^2 - 1$. **M1**

Rearranging: $x^4 - 2x^2 + 2 \geq 0$, i.e. $x^4 - 2x^2 + 1 + 1 \geq 0$. **M1**

Write as $(x^2-1)^2 + 1 \geq 0$. **A1**

Since $(x^2-1)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $(x^2-1)^2 + 1 \geq 1 > 0$. **R1**

The inequality is strict for all $x \in \mathbb{R}$; equality in the original expression $\dfrac{x^4+1}{2} = x^2 - \dfrac{1}{2}$ is never achieved.
A1

Q28 — Growth Comparison of Exponential and Polynomial Terms HARD

(a)
$n = 4$: $3^4 = 81$, $2^4 + 4^3 = 16 + 64 = 80$. Since $81 > 80$, the claim holds. **A1**

$n = 5$: $3^5 = 243$, $2^5 + 5^3 = 32 + 125 = 157$. Since $243 > 157$, the claim holds. **A1**

(b)
Using a GDC to evaluate $g(n) = 3^n - 2^n - n^3$ for integer $n \geq 4$:
$g(12) = 3^{12} - 2^{12} - 12^3 = 531441 - 4096 - 1728 = 525617$, which does not exceed 10^6.
$g(13) = 3^{13} - 2^{13} - 13^3 = 1594323 - 8192 - 2197 = 1583934$, which exceeds 10^6. **A1A1**

The smallest n for which $g(n) > 10^6$ is $n = 13$: $g(13) = 1583934 > 10^6$.

(c)
No. Checking finitely many cases is not a proof; it constitutes numerical evidence only and cannot establish the claim for all $n \geq 4$. **R1**

Q29 — AM-GM Inequality for Positive Reals HARD

Since $a, b \in \mathbb{R}^+$ with $a \neq b$, consider $\dfrac{a}{b} + \dfrac{b}{a} - 2$. **M1**

Write over a common denominator: $\dfrac{a}{b} + \dfrac{b}{a} - 2 = \dfrac{a^2 + b^2 - 2ab}{ab} = \dfrac{(a-b)^2}{ab}$. **M1A1**

Since $a, b \in \mathbb{R}^+$, we have $ab > 0$ (i.e. $a * b > 0$). **R1**

Since $a \neq b$, $(a-b)^2 > 0$. Therefore $\dfrac{(a-b)^2}{ab} > 0$, giving $\dfrac{a}{b} + \dfrac{b}{a} > 2$. **A1**

Q30 — Never Prime for n at Least 1 HARD

Factorise: $n^2 + 3n + 2 = (n+1)(n+2)$. **M1**

For $n \geq 1, n \in \mathbb{Z}$: we have $n + 1 \geq 2$ and $n + 2 \geq 3$. **M1**

Both factors are integers strictly greater than 1, so the product $(n+1)(n+2)$ has a nontrivial factorisation and is therefore composite for every $n \geq 1$. **A1**

Hence $n^2 + 3n + 2$ is never prime for $n \geq 1, n \in \mathbb{Z}$. **A1**

For $n < 0, n \in \mathbb{Z}$: take $n = -4$: $(n+1)(n+2) = (-3)(-2) = 6$, not prime. Take $n = -3$: $(-2)(-1) = 2$, which is prime. Hence $n^2 + 3n + 2$ can be prime for $n < 0$; a specific example is $n = -3$ giving $f(-3) = 2$. **A1**