IB Math AA HL — Topic 1

Simple Proof & Reasoning

Proof by Deduction · Disproof by Counter Example

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Consecutive integers:** any two consecutive integers can be written as n and $n+1$ (or $2n$ and $2n+1$ for even/odd), where $n \in \mathbb{Z}$.
- **Even/odd forms:** an even integer $= 2k$; an odd integer $= 2k+1$, for some $k \in \mathbb{Z}$.
- **Divisibility test:** $m \mid a$ if and only if $a = mk$ for some $k \in \mathbb{Z}$.
- **Difference of two squares:** $a^2 - b^2 = (a-b)(a+b)$; use this as the engine of divisibility and factorisation proofs.
- **Disproof by counterexample:** to disprove "$P(n)$ is true for all n", exhibit one explicit value $n = n_0$ for which $P(n_0)$ is false, and evaluate both sides.
- **Sum of first n positive integers:** $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.
- **Proof by deduction:** start from general forms (e.g. $n \in \mathbb{Z}$), apply algebraic steps, and end with a sentence concluding what has been proved.

GDC Tips

- **Searching for counterexamples:** use the GDC table feature (`G-Solve` $\to$ `Table`) to evaluate an expression $f(n)$ over a range; the first row that fails the claimed property is your counterexample.
- **Divisibility check:** store candidate values and use `int(`n/m`)` vs. n/m to test whether $m \mid n$; a non-integer quotient confirms failure.
- **Conjecture from a table:** generate $f(n)$ for $n = 1, 2, \ldots, 6$ in the spreadsheet app, observe the pattern, then write a conjecture — state it clearly before attempting a proof.
- **Finding where one expression overtakes another:** graph both on the same axes, use `Intersection` to find the crossover point; prove the inequality holds beyond that point by algebra.
- **Verifying a factorisation:** expand your factored form on the GDC (e.g. with `expand(` in CAS) to confirm it matches the original expression before writing the proof.
- **Evidence $\neq$ proof:** after checking $f(n)$ for 10 values on the GDC, always state "this evidence supports but does not prove the result"; a deductive argument is still required for full marks.

Common Mistakes to Avoid

1. **Assuming the result.** Writing the conclusion at the start of a "show that" proof and working backwards is circular; always begin from the hypothesis and deduce the target.
2. **Non-general starting point.** Using a specific number (e.g. $n = 3$) instead of a general integer in a deduction proof produces a verification, not a proof; define $n \in \mathbb{Z}$ at the outset.
3. **Incomplete counterexample.** Stating "let $n = 5$" without computing both sides of the inequality or showing the claim fails loses the accuracy mark; always evaluate the expression explicitly.
4. **Confusing "even" with "divisible by 4".** Writing $2n$ for an even integer is correct, but $(2n)^2 = 4n^2$; students often claim $4n^2$ is merely even rather than a multiple of 4.
5. **Dropping the conclusion sentence.** Ending algebra with an expression such as $6k$ without writing "which is a multiple of 6, so the result holds" loses the R1 reasoning mark in the mark scheme.
6. **Wrong number set in deduction.** Claiming a result "for all n" when the proof requires $n \in \mathbb{Z}^+$ (e.g. a sum formula) is an error; state the domain of n precisely and check it matches the question.

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Section A — Easy

EASY

1. Let $n \in \mathbb{Z}$. Show that the product of two consecutive integers is always even. *Calculator not allowed* **[3 marks]**

2. Let $f(n) = n^2 + n + 11$, where $n \in \mathbb{Z}^+$. A student claims that $f(n)$ is prime for all positive integers n. Find a counterexample to disprove this claim. *Calculator allowed* **[3 marks]**

n	$f(n) = n^2 + n + 11$
1	13
2	17
3	23
4	31

3. Let $n \in \mathbb{Z}$. Show that $(n+3)^2 - (n-3)^2$ is always a multiple of 12. *Calculator not allowed* **[3 marks]**

4. Let $g(n) = 2^n - 1$, where $n \in \mathbb{Z}^+$. The table below shows values of $g(n)$ for small n. A student claims that $g(n)$ is prime for every positive integer n. Find the smallest counterexample and state the value of n and $g(n)$.
Calculator allowed **[3 marks]**

n	$g(n) = 2^n - 1$
1	1
2	3
3	7
4	15

5. Let p and q be odd integers. Show that $p^2 + q^2$ is always even but never divisible by 4. *Calculator not allowed* **[3 marks]**

6. A student claims: "For all real numbers x, if $x^2 > 9$ then $x > 3$." Decide whether this claim is true or false. Justify your answer with a proof or a counterexample. *Calculator allowed* **[3 marks]**

7. Let $n \in \mathbb{Z}^+$. Show that $n^2 + 3n + 2$ is always the product of two consecutive integers and hence always even.
Calculator allowed **[3 marks]**

8. Let $h(n) = n^2 - n + 1$, where $n \in \mathbb{Z}^+$. The table below shows selected values of $h(n)$. A student conjectures that $h(n)$ is always odd. Find a counterexample to disprove this claim. *Calculator allowed* **[3 marks]**

n	$h(n) = n^2 - n + 1$
1	1
2	3
3	7
4	13

9. Let $a, b \in \mathbb{R}$. By considering $(a-b)^2 \geq 0$, show that $a^2 + b^2 \geq 2ab$. *Calculator allowed* **[3 marks]**

10. Let $n \in \mathbb{Z}$. Show that the difference between the squares of any two consecutive odd integers is divisible by 8. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Let $n \in \mathbb{Z}^+$. Show that the product of any two consecutive odd integers is always odd. *Calculator not allowed* **[4 marks]**

12. Consider the statement: "For all integers $n \geq 1$, the expression $n^2 + n + 41$ is prime." By finding the smallest counterexample, disprove this statement. *Calculator allowed* **[4 marks]**

n	$n^2 + n + 41$
1	43
2	47
3	53
4	61

13. Let $m, n \in \mathbb{Z}$. Prove that if m and n are both odd, then $m^2 + n^2$ is always even but never divisible by 4.
Calculator not allowed **[4 marks]**

14. Consider the statement: "For all real numbers x, $\dfrac{x^2+4}{x^2+1} \geq 3$." Determine whether this statement is true or false. If false, find a counterexample; if true, prove it. *Calculator allowed* **[4 marks]**

15. Let $n \in \mathbb{Z}$. Prove that $n^2 - n$ is always even for any integer n. Hence show that $n^2 - n + 6$ is always even.
Calculator not allowed **[4 marks]**

16. Consider the claim: "For all integers $n \geq 2$, the expression $2^n - 1$ is prime." Using technology to search the range $2 \leq n \leq 12$, find the smallest value of n for which the claim fails and verify that the expression is composite at that value. *Calculator allowed* **[4 marks]**

17. Let $a, b \in \mathbb{Z}$. By writing $a = 2p$ and $b = 2q + 1$ where $p, q \in \mathbb{Z}$, prove that $a^2 + b^2 + 1$ is always odd when a is even and b is odd. *Calculator allowed* **[4 marks]**

18. The table below shows values of $f(n) = n^3 - n$ for $n = 1, 2, 3, 4, 5$, where $n \in \mathbb{Z}^+$. *Calculator allowed* **[4 marks]**

n	$f(n) = n^3 - n$
1	0
2	6
3	24
4	60
5	120

By inspecting the table, state a conjecture about the divisibility of $f(n)$ by 6. Hence prove your conjecture for all $n \in \mathbb{Z}^+$ by factorising $n^3 - n$.

19. Let $x \in \mathbb{R}$. Prove that $x^2 - 6x + 13 > 0$ for all real values of x by completing the square. *Calculator allowed* **[4 marks]**

20. Let $n \in \mathbb{Z}^+$. Consider the expression $5n^2 + 5n + 2$. A student claims this expression is always divisible by 6. Identify the flaw in the following argument and determine the correct statement about divisibility by 2.

"Since $5n^2 + 5n = 5n(n+1)$ and $n(n+1)$ is always even, we have $5n(n+1)$ is divisible by 10, so $5n^2 + 5n + 2$ is divisible by 6."

Calculator allowed **[4 marks]**

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Section C — Hard

HARD

21. Let $n \in \mathbb{Z}^+$. Show that $n^3 - n$ is always divisible by 6. *Calculator not allowed* **[5 marks]**

22. Let $a, b \in \mathbb{R}$. Show that $a^2 + ab + b^2 > 0$ for all a, b not both zero. *Calculator not allowed* **[5 marks]**

23. A student claims: "For every prime p, the expression $p^2 - p + 1$ is also prime."
The table below shows values of p and $p^2 - p + 1$ for the first several primes p. *Calculator allowed* **[5 marks]**

p	$p^2 - p + 1$	Prime?
2	3	Yes
3	7	Yes
5	21	No
7	43	Yes

(a) State the value of p that provides a counterexample to the student's claim, evaluate $p^2 - p + 1$ at that value, and explain why it disproves the claim. **[3 marks]**

(b) Hence write down the smallest prime $q > 5$ for which $q^2 - q + 1$ is prime, and verify your answer. **[2 marks]**

24. Let $n \in \mathbb{Z}$. Prove that if n^2 is odd, then n is odd. *Calculator not allowed* **[5 marks]**

25. The function $f : \mathbb{Z}^+ \to \mathbb{Z}$ is defined by $f(n) = n^2 + n + 41$.
The table below lists values of $f(n)$ for small $n \in \mathbb{Z}^+$. *Calculator allowed* **[5 marks]**

n	$f(n)$	Prime?
1	43	Yes
2	47	Yes
3	53	Yes
4	61	Yes
5	71	Yes

(a) A student conjectures that $f(n)$ is prime for all $n \in \mathbb{Z}^+$. Find the smallest value of n for which $f(n)$ is not prime, and state $f(n)$ at that value. **[3 marks]**

(b) Explain why the evidence in the table, on its own, does not constitute a proof that $f(n)$ is prime for all $n \in \mathbb{Z}^+$. **[2 marks]**

26. Let $n \in \mathbb{Z}$. Prove that the product of any four consecutive integers is always divisible by 24. *Calculator allowed* **[5 marks]**

27. Let $x \in \mathbb{R}$. Show that $\dfrac{x^4+1}{2} \geq x^2 - \dfrac{1}{2}$, and state when equality holds. *Calculator not allowed* **[5 marks]**

28. Let $n \in \mathbb{Z}^+$. It is claimed that $3^n > 2^n + n^3$ for all $n \geq 4$. *Calculator allowed* **[5 marks]**

(a) Verify the claim for $n = 4$ and $n = 5$. **[2 marks]**

(b) Use a graphing display calculator to find the smallest integer $n \geq 4$ for which $3^n - 2^n - n^3$ first exceeds 10^6. State this value of n and the corresponding value of $3^n - 2^n - n^3$. **[2 marks]**

(c) State, with a reason, whether the evidence from parts (a) and (b) alone constitutes a proof of the claim for all $n \geq 4$. **[1 marks]**

29. Let $a, b \in \mathbb{R}^+$ with $a \neq b$. Prove that $\dfrac{a}{b} + \dfrac{b}{a} > 2$. *Calculator allowed* **[5 marks]**

30. Let $n \in \mathbb{Z}$. Prove by deduction that $n^2 + 3n + 2$ is never a prime number when $n \geq 0$, $n \in \mathbb{Z}$. Hence determine whether $n^2 + 3n + 2$ can be prime for $n \in \mathbb{Z}$ with $n < 0$, giving a specific example or a proof as appropriate. *Calculator allowed* **[5 marks]**

Worked Solutions

Q1 — Product of consecutive integers is even [EASY]

Write the two consecutive integers as n and $n+1$, where $n \in \mathbb{Z}$. **M1**

Either n or $n+1$ is even, so their product $n(n+1)$ is divisible by 2. **R1**

Therefore the product of any two consecutive integers is always even. **A1**

Q2 — Counterexample to prime claim [EASY]

Testing $n = 10$: **M1**

$f(10) = 100 + 10 + 11 = 121 = 11 \times 11$ **A1**

Since 121 is not prime, the claim is false. $n = 10$ is a valid counterexample. **R1**

Q3 — Difference of squares divisible by 12 [EASY]

Expand using the difference of two squares:

$$(n+3)^2 - (n-3)^2 = [(n+3) + (n-3)][(n+3) - (n-3)] = (2n)(6) = 12n$$

M1A1

Since $n \in \mathbb{Z}$, $12n$ is a multiple of 12 for all integers n. **A1**

Q4 — Smallest counterexample for Mersenne primes [EASY]

From the table, $g(1) = 1$ (not prime but the claim requires primality), $g(2) = 3$, $g(3) = 7$ are prime. **M1**

Testing $n = 4$: $g(4) = 2^4 - 1 = 15 = 3 \times 5$. **A1**

Since 15 is not prime, the smallest counterexample is $n = 4$, $g(4) = 15$. **A1**

Q5 — Sum of squares of two odd integers [EASY]

Write $p = 2j + 1$ and $q = 2k + 1$ where $j, k \in \mathbb{Z}$. **M1**

Then $p^2 + q^2 = (2j+1)^2 + (2k+1)^2 = 4j^2 + 4j + 1 + 4k^2 + 4k + 1 = 4(j^2 + j + k^2 + k) + 2$. **A1**

This expression equals $2[2(j^2 + j + k^2 + k) + 1]$, which is even. Since the bracket $2(j^2 + j + k^2 + k) + 1$ is odd, $p^2 + q^2$ is not divisible by 4. **A1**

Q6 — Counterexample to implication about $x^2 > 9$ [EASY]

The claim is **false**. **A1**

Counterexample: let $x = -4$. Then $x^2 = 16 > 9$ but $x = -4 < 3$. **M1**

Since the hypothesis $x^2 > 9$ holds but the conclusion $x > 3$ fails, the claim is disproved. **R1**

Q7 — Factorisation shows expression always even [EASY]

Factorise: $n^2 + 3n + 2 = (n+1)(n+2)$. **M1**

$(n+1)$ and $(n+2)$ are consecutive integers. **A1**

One of any two consecutive integers is even, so their product $(n+1)(n+2)$ is always even. **A1**

Q8 — Counterexample to odd claim [EASY]

The values in the table are all odd; testing $n = 5$: **M1**

$h(5) = 25 - 5 + 1 = 21$, still odd. Testing $n = 6$: $h(6) = 36 - 6 + 1 = 31$, still odd.

Testing $n = 10$: $h(10) = 100 - 10 + 1 = 91$, still odd. Testing $n = 0$ (if $n \in \mathbb{Z}^+$ excludes 0); testing $n = 11$: $h(11) = 121 - 11 + 1 = 111$ odd.

Note: The conjecture is in fact true for all $n \in \mathbb{Z}^+$; since $n^2 - n = n(n-1)$ is always even, $h(n) = n(n-1) + 1$ is always odd. The claim is **true**, so no counterexample exists. A correct response states this. **M1**

$n(n-1)$ is the product of consecutive integers, hence even, so $h(n)$ is always odd. The conjecture holds for all $n \in \mathbb{Z}^+$. **A1A1**

Q9 — Inequality from non-negative square [EASY]

Since $(a-b)^2 \geq 0$ for all $a, b \in \mathbb{R}$, **M1**

expand: $a^2 - 2ab + b^2 \geq 0$. **A1**

Adding $2ab$ to both sides gives $a^2 + b^2 \geq 2ab$. **A1**

Q10 — Difference of squares of consecutive odd integers divisible by 8 [EASY]

Write two consecutive odd integers as $2n+1$ and $2n+3$, where $n \in \mathbb{Z}$. **M1**

$(2n+3)^2 - (2n+1)^2 = [(2n+3) + (2n+1)][(2n+3) - (2n+1)] = (4n+4)(2) = 8(n+1)$. **A1**

Since $n + 1 \in \mathbb{Z}$, the difference is always a multiple of 8. **A1**

Q11 — Product of consecutive odd integers [MEDIUM]

Let the two consecutive odd integers be $2k-1$ and $2k+1$ where $k \in \mathbb{Z}^+$. **M1**

Their product is $(2k-1)(2k+1) = 4k^2 - 1$. **A1**

Write $4k^2 - 1 = 2(2k^2-1)+1$, which is of the form $2m+1$ where $m = 2k^2 - 1 \in \mathbb{Z}$. **M1**

Therefore the product of any two consecutive odd integers is always odd. **A1**

Q12 — Counterexample to primality claim [MEDIUM]

Testing values using technology: for $n = 40$, $f(40) = 40^2 + 40 + 41 = 1600 + 40 + 41 = 1681$. **M1**

Note that $1681 = 41^2$, which is not prime. **A1**

Note: The student must identify $n = 40$ *as producing* 41×41.

Since $n = 40$ gives $n^2 + n + 41 = 41^2$, which is composite, the statement is false. **R1**

The smallest such counterexample is $n = 40$. **A1**

Q13 — Sum of squares of two odd integers [MEDIUM]

Write $m = 2a+1$ and $n = 2b+1$ where $a, b \in \mathbb{Z}$. **M1**

Then $m^2 + n^2 = (2a+1)^2 + (2b+1)^2 = 4a^2 + 4a + 1 + 4b^2 + 4b + 1 = 4(a^2 + a + b^2 + b) + 2$. **A1**

Since $4(a^2 + a + b^2 + b) + 2 = 2(2(a^2 + a + b^2 + b) + 1)$, the expression is even. **M1**

The remainder when divided by 4 is $2 \neq 0$, so $m^2 + n^2$ is even but never divisible by 4. **A1**

Q14 — Counterexample to inequality claim [MEDIUM]

The statement is **false**. **A1**

Note: Award A1 for the correct decision.

Testing $x = 0$ (using technology or substitution): $\dfrac{0+4}{0+1} = 4$, which does not satisfy ≥ 3... however $4 \geq 3$.

Testing $x = 2$: $\dfrac{8}{5} = 1.6 < 3$. **M1**

At $x = 2$, $\dfrac{x^2+4}{x^2+1} = \dfrac{8}{5} = 1.6$, and $1.6 < 3$. **A1**

Since the expression equals $1.6 < 3$ at $x = 2$, the statement is disproved. **R1**

Q15 — Parity of $n^2 - n$ and $n^2 - n + 6$ [MEDIUM]

Write $n^2 - n = n(n-1)$, the product of two consecutive integers. **M1**

One of n or $n-1$ must be even, so $n(n-1)$ is always even, i.e. $2 \mid n^2 - n$. **A1**

Hence $n^2 - n + 6 = n(n-1) + 6$. **M1**

Since $n(n-1)$ is even and 6 is even, their sum is even, so $n^2 - n + 6$ is always even. **A1**

Q16 — First failure of Mersenne-type claim [MEDIUM]

Using technology, compute $2^n - 1$ for $n = 2, 3, 4, \ldots$: **M1**

$n = 2$: 3 (prime); $n = 3$: 7 (prime); $n = 4$: 15; $n = 4$ gives $2^4 - 1 = 15$. **A1**

Note: Award A1 for correctly identifying $n = 4$ as the first failure.

$15 = 3 \times 5$, so 15 is composite (has factors other than 1 and itself). **A1**

The smallest value for which the claim fails is $n = 4$, since $2^4 - 1 = 15 = 3 \times 5$ is not prime. **R1**

Q17 — Parity of $a^2 + b^2 + 1$ [MEDIUM]

Substituting $a = 2p$ and $b = 2q + 1$ where $p, q \in \mathbb{Z}$: **M1**

$a^2 + b^2 + 1 = (2p)^2 + (2q+1)^2 + 1 = 4p^2 + 4q^2 + 4q + 1 + 1$. **A1**

$= 4p^2 + 4q^2 + 4q + 2 = 2(2p^2 + 2q^2 + 2q + 1)$. **M1**

Since $2p^2 + 2q^2 + 2q + 1$ is odd (even + odd), the whole expression equals $2 \times$ odd, which is even... *Note: re-examine*: $2(2p^2 + 2q^2 + 2q + 1)$ means the expression is even; however the bracket $2p^2 + 2q^2 + 2q + 1$ is odd, confirming $a^2 + b^2 + 1$ is even but equals $2 \times$ odd, i.e. not divisible by 4. Since $2(2p^2 + 2q^2 + 2q + 1)$ is of the form $2m$ with m odd, the expression is even (but specifically $\equiv 2 \pmod 4$). For the question demand: $a^2 + b^2 + 1$ is always **odd**... *recalculate*: $4p^2 + (4q^2 + 4q + 1) + 1 = 4p^2 + 4q^2 + 4q + 2$; this is $2(2p^2 + 2q^2 + 2q + 1)$, which is even. **A1**

Note: Award final A1 for a correct conclusion stating the expression is always even.

Q18 — Divisibility of $n^3 - n$ by 6 [MEDIUM]

Conjecture: $f(n) = n^3 - n$ is divisible by 6 for all $n \in \mathbb{Z}^+$. **A1**

Factorise: $n^3 - n = n(n^2 - 1) = (n-1)\,n\,(n+1)$. **M1**

This is the product of three consecutive integers. Among any three consecutive integers, one is divisible by 3 and at least one is divisible by 2. **M1**

Therefore $(n-1)n(n+1)$ is divisible by $2 \times 3 = 6$ for all $n \in \mathbb{Z}^+$. **A1**

Q19 — Positivity by completing the square [MEDIUM]

Complete the square: $x^2 - 6x + 13 = (x-3)^2 - 9 + 13$. **M1**

$= (x-3)^2 + 4$. **A1**

Since $(x-3)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $(x-3)^2 + 4 \geq 0 + 4 = 4$. **M1**

Therefore $x^2 - 6x + 13 \geq 4 > 0$ for all $x \in \mathbb{R}$. **A1**

Q20 — Flaw in divisibility argument [MEDIUM]

Flaw: The student incorrectly concludes that $5n(n+1)$ being divisible by 10 implies $5n^2 + 5n + 2$ is divisible by 6; adding 2 to a multiple of 10 does not yield a multiple of 6. **R1**

Note: Award R1 for identifying that the jump from "divisible by 10" to "divisible by 6" is unjustified.

Correct analysis: $5n^2 + 5n = 5n(n+1)$. Since $n(n+1)$ is the product of consecutive integers, it is always even, so $5n(n+1)$ is divisible by 2. **M1**

Therefore $5n^2 + 5n + 2 = 5n(n+1) + 2 \equiv 0 + 0 = 0 \pmod{2}$, so the expression is always even (divisible by 2). **A1**

Testing $n = 1$: $5 + 5 + 2 = 12$, divisible by 2 but also by 6; testing $n = 2$: $20 + 10 + 2 = 32$, not divisible by 6. The expression is always divisible by 2 but **not** always by 6. **A1**

Q21 — Divisibility of $n^3 - n$ by 6 [HARD]

Factorise: $n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = (n-1)n(n+1)$ **M1**

This is the product of three consecutive integers, where $n \in \mathbb{Z}^+$. **A1**

Among any three consecutive integers, at least one is divisible by 2 and exactly one is divisible by 3. **M1**

Therefore the product $(n-1)n(n+1)$ is divisible by $2 \times 3 = 6$. **R1**

Hence $n^3 - n$ is always divisible by 6 for all $n \in \mathbb{Z}^+$. **AG**

Note: Award R1 for a complete justification that the product of three consecutive integers contains a factor of 2 and a factor of 3.

Q22 — Non-negativity of $a^2 + ab + b^2$ [HARD]

Complete the square in a: write $a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 - \frac{b^2}{4} + b^2$ **M1**

$= \left(a + \frac{b}{2}\right)^2 + \frac{3b^2}{4}$ **A1**

Since $\left(a + \frac{b}{2}\right)^2 \geq 0$ and $\frac{3b^2}{4} \geq 0$, the sum is ≥ 0. **R1**

Equality requires $a + \frac{b}{2} = 0$ and $b = 0$ simultaneously, giving $a = b = 0$. **M1**

Since a and b are not both zero, the sum is strictly positive: $a^2 + ab + b^2 > 0$. **A1**

Q23 — Counterexample to primality claim [HARD]

(a) From the table, $p = 5$ gives $p^2 - p + 1 = 25 - 5 + 1 = 21$. **M1**

$21 = 3 \times 7$, so 21 is not prime. **A1**

Since a prime input $p = 5$ produces a non-prime output, the claim is false. **R1**

(b) $q = 7$: $7^2 - 7 + 1 = 49 - 7 + 1 = 43$, which is prime. **A1A1**

Note: Award A1 for stating $q = 7$ and A1 for the correct verification 43 is prime.

Q24 — Contrapositive: n^2 odd implies n odd [HARD]

State the contrapositive: "If n is even then n^2 is even." **M1**

Let $n = 2k$ for some $k \in \mathbb{Z}$. **M1**

Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even. **A1**

The contrapositive has been proved; since a conditional statement and its contrapositive are logically equivalent, the original statement "if n^2 is odd then n is odd" is true. **R1**

Hence for all $n \in \mathbb{Z}$, if n^2 is odd then n is odd. **A1**

Q25 — Euler's polynomial counterexample [HARD]

(a) Using a GDC to evaluate $f(n) = n^2 + n + 41$ and test primality:

$f(40) = 1600 + 40 + 41 = 1681 = 41^2$, which is not prime. **M1A1**

The smallest such value is $n = 40$, giving $f(40) = 1681$. **A1**

(b) The table provides evidence for only five values of n; it does not verify all $n \in \mathbb{Z}^+$. A finite number of cases cannot establish a universal statement — a proof must cover infinitely many cases by a general argument. **R1R1**

Note: Award R1 for identifying finitely many cases tested, R1 for stating that a universal proof requires a general argument.

Q26 — Product of four consecutive integers divisible by 24 [HARD]

Let the four consecutive integers be $n, n+1, n+2, n+3$ for some $n \in \mathbb{Z}$. **M1**

Their product $P = n(n+1)(n+2)(n+3)$ can be written as $P = \dfrac{(n+3)!}{(n-1)!} \cdot 1$. More directly, note that $P = 4!\dbinom{n+3}{4}$ (a combinatorial argument), so $P = 24\dbinom{n+3}{4}$. **M1**

Since $\binom{n+3}{4}$ is always an integer for $n \in \mathbb{Z}$, we have $P = 24k$ for some integer k. **A1**

Alternatively: among four consecutive integers there are at least two even numbers (contributing a factor of at least $2 \times 4 = 8$) and at least one multiple of 3, giving divisibility by $8 \times 3 = 24$. **R1**

Hence the product of any four consecutive integers is always divisible by 24. **A1**

Note: Award full marks for either the combinatorial argument or the factor-counting argument, provided both the factor-of-8 and factor-of-3 contributions are justified.

Q27 — Inequality via non-negative squared term [HARD]

Rearrange the inequality: $\frac{x^4+1}{2} \geq x^2 - \frac{1}{2}$ is equivalent to $x^4 + 1 \geq 2x^2 - 1$. **M1**

Rearranging: $x^4 - 2x^2 + 2 \geq 0$, i.e. $x^4 - 2x^2 + 1 + 1 \geq 0$. **M1**

Write as $(x^2 - 1)^2 + 1 \geq 0$. **A1**

Since $(x^2-1)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $(x^2-1)^2 + 1 \geq 1 > 0$. **R1**

The inequality is strict for all $x \in \mathbb{R}$; equality in the original expression $\frac{x^4+1}{2} = x^2 - \frac{1}{2}$ is never achieved. **A1**

Note: The question asks for when equality holds. Award the final A1 only if the candidate states that equality is never achieved (or equivalently, there is no real x for which equality holds).

Q28 — Growth comparison 3^n vs $2^n + n^3$ [HARD]

(a) $n = 4$: $3^4 = 81$, $2^4 + 4^3 = 16 + 64 = 80$. Since $81 > 80$, the claim holds. **A1**

$n = 5$: $3^5 = 243$, $2^5 + 5^3 = 32 + 125 = 157$. Since $243 > 157$, the claim holds. **A1**

(b) Using a GDC to evaluate $g(n) = 3^n - 2^n - n^3$ for integer $n \geq 4$:

$g(23) = 3^{23} - 2^{23} - 23^3 = 94\,143\,178\,827 - 8\,388\,608 - 12\,167 = 94\,134\,778\,052$ (unrounded).

The smallest n for which $g(n) > 10^6$ is $n = 17$: $g(17) = 129\,140\,163 - 131\,072 - 4\,913 = 129\,004\,178 > 10^6$. **A1A1**

Note: GDC setup: define $g(n) = 3^n - 2^n - n^3$ and tabulate for $n = 4, 5, \ldots$ until $g(n) > 10^6$.

(c) No. Checking finitely many cases is not a proof; it constitutes numerical evidence only and cannot establish the claim for all $n \geq 4$. **R1**

Q29 — AM-GM inequality for positive reals [HARD]

Since $a, b \in \mathbb{R}^+$ with $a \neq b$, consider $\dfrac{a}{b} + \dfrac{b}{a} - 2$. **M1**

Write over a common denominator: $\dfrac{a}{b} + \dfrac{b}{a} - 2 = \dfrac{a^2 + b^2 - 2ab}{ab} = \dfrac{(a-b)^2}{ab}$. **M1A1**

Since $a, b \in \mathbb{R}^+$, we have $ab > 0$. **R1**

Since $a \neq b$, $(a-b)^2 > 0$. Therefore $\dfrac{(a-b)^2}{ab} > 0$, giving $\dfrac{a}{b} + \dfrac{b}{a} > 2$. **A1**

Note: Award R1 for explicitly invoking both $ab > 0$ and $(a-b)^2 > 0$ to justify strict inequality.

Q30 — $n^2 + 3n + 2$ never prime for $n \geq 0$; investigation for $n < 0$ [HARD]

Factorise: $n^2 + 3n + 2 = (n+1)(n+2)$. **M1**

For $n \geq 0, n \in \mathbb{Z}$: we have $n + 1 \geq 1$ and $n + 2 \geq 2$. **A1**

Since $n + 2 = (n+1) + 1 > n + 1 \geq 1$, both factors are integers greater than 1 when $n \geq 1$, so the product is composite. For $n = 0$: $(0+1)(0+2) = 2$, which is prime. **M1**

Note: Correct handling of $n = 0$ giving 2 is required for M1.

Repair: The claim should state $n \geq 1, n \in \mathbb{Z}$, for $n^2 + 3n + 2$ to be never prime. **A1**

For $n < 0$, $n \in \mathbb{Z}$: take $n = -4$: $(n+1)(n+2) = (-3)(-2) = 6$, not prime. Take $n = -3$: $(-2)(-1) = 2$, which is prime. Hence $n^2 + 3n + 2$ can be prime for $n < 0$; a specific example is $n = -3$ giving $f(-3) = 2$. **A1**