IB Math AA HL — Topic 1

Binomial Theorem

Binomial Expansions · Pascal's Triangle · Extension of the Binomial Theorem

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

Binomial Theorem (finite): $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r, \quad n \in \mathbb{Z}^+.$

General term: $T_{r+1} = \binom{n}{r} a^{n-r} b^r$; identify r before equating powers.

Binomial coefficient: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$; also $\binom{n}{r} = \binom{n}{n-r}$ (symmetry).

Pascal's identity: $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$; row sum $\sum_{r=0}^{n} \binom{n}{r} = 2^n.$

Extended binomial theorem: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots, \quad n \in \mathbb{Q},$ valid for $|x| < 1.$

Extended form with coefficient: $(1+ax)^n$ valid for $|x| < \frac{1}{|a|}$; always state this range explicitly.

Constant term: set the power of x in T_{r+1} equal to zero and solve for r; r must be a non-negative integer.

GDC Tips

Evaluating large $\binom{n}{r}$: use nCr via Math $>$ PRB on the TI-84, or Combinatorial in the Casio ClassPad Catalog.

Expanding & collecting coefficients: enter `expand((ax+b)^n)` in the CAS (ClassPad Main menu) to verify a coefficient answer instantly.

Percentage error of approximation: store the truncated value and compute $\left|\frac{\text{approx} - \text{exact}}{\text{exact}}\right| \times 100$ directly; compare with the required accuracy.

Solving coefficient equations numerically: graph $f(k) =$ coefficient expression $-$ target and use G-Solve $>$ Root (Casio) or 2nd TRACE Zero (TI-84) to find k; check integrality or sign constraints before accepting.

Checking the validity interval: substitute the boundary $x = \pm 1/|a|$ into both sides numerically to confirm the series diverges there; present the open interval $|x| < 1/|a|$ in the answer.

Number of terms for stated accuracy: sum successive terms with Sequence Sum or a For loop, stopping when $|T_{r+1}| <$ tolerance; record the first r satisfying the condition.

Common Mistakes to Avoid

1. **Wrong index on a in the general term.** Students write $T_{r+1} = \binom{n}{r} a^r b^{n-r}$ instead of $a^{n-r} b^r$, swapping the two powers and producing every coefficient incorrectly.
2. **Forgetting the range of validity for extended expansions.** Omitting $|x| < 1/|a|$ loses a dedicated mark; the series is meaningless outside this interval and must be stated every time.
3. **Accepting a non-integer or negative value of r.** When solving for the constant term or a specified coefficient, r must satisfy $r \in \{0, 1, \ldots, n\}$; a fractional or negative r means no such term exists and must be explicitly rejected.
4. **Sign errors with negative terms.** In $(a - b)^n$, writing b^r instead of $(-b)^r$ loses the alternating sign; the correct substitution is $T_{r+1} = \binom{n}{r} a^{n-r} (-b)^r$.
5. **Confusing T_r and T_{r+1}.** The r-th term is $T_r = \binom{n}{r-1} a^{n-r+1} b^{r-1}$, but the standard formula indexes from T_1; always define which convention is used and be consistent throughout.
6. **Applying the extended theorem outside its range.** Substituting $|x| \geq 1/|a|$ to approximate a surd gives a divergent series; verify the substituted value lies strictly inside the validity interval before using the approximation.

Section A — Easy EASY (10 questions $\times$ 3 marks = 30 marks)

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1. EASY No Calculator **[3 marks]**

Write down the expansion of $(1+x)^4$ in ascending powers of x.

2. EASY No Calculator **[3 marks]**

Find the term independent of x in the expansion of $\left(x+\dfrac{2}{x}\right)^6$, where $x \neq 0$.

3. EASY Calculator **[3 marks]**

The coefficient of x^2 in the expansion of $(3+kx)^5$, where $k \in \mathbb{R}$, is 1080. Find the value of k.

4. EASY No Calculator **[3 marks]**

Find the first three terms, in ascending powers of x, of the binomial expansion of $(1+3x)^{-2}$, where $|x| < \dfrac{1}{3}$. State the range of validity.

5. EASY No Calculator **[3 marks]**

The diagram shows Pascal's triangle up to row 5. Use the identity $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ to verify that $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$, and state the values of $\binom{4}{1}$, $\binom{4}{2}$, and $\binom{5}{2}$.

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5   10  10  5   1
```

6. EASY Calculator **[3 marks]**

Find the coefficient of x^3 in the expansion of $(2-x)^7$.

7. EASY Calculator **[3 marks]**

Using the expansion of $(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$ with $x = 0.04$, find an approximate value of $\sqrt{1.04}$. The table gives the truncated-expansion value alongside the true value. State the range of validity of this expansion.

x	Expansion value	True value $\sqrt{1+x}$
0.04	1.0198	$1.019804\ldots$

8. EASY Calculator **[3 marks]**

Write down the value of $\binom{8}{3}$ and hence find the number of ways of choosing a committee of 3 people from a group of 8.

9. EASY Calculator **[3 marks]**

Find the coefficient of x^2 in the expansion of $(1+x)^5(1+2x)$.

10. EASY Calculator **[3 marks]**

The general term of the expansion of $\left(x^2+\frac{1}{x}\right)^6$ is $T_{r+1}=\binom{6}{r}(x^2)^{6-r}\left(\frac{1}{x}\right)^r$, where $r \in \{0,1,2,3,4,5,6\}$.

Find the term in x^6.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

The coefficient of x^3 in the expansion of $(2 + kx)^7$, where $k \in \mathbb{R}$, is equal to 2240. Find the value of k.

12. MEDIUM No Calculator **[4 marks]**

Consider the expansion of $\left(3x^2 - \dfrac{1}{x}\right)^9$, where $x \in \mathbb{R}$, $x \neq 0$.

(a) Write down the general term T_{r+1} of the expansion. [1]

(b) Find the term independent of x. [3]

13. MEDIUM No Calculator **[4 marks]**

The diagram shows rows 0 to 5 of Pascal's triangle.

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}$$

(a) Using the circled values, verify the Pascal identity $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$. [1]

(b) Hence show that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ for all $n, r \in \mathbb{Z}^+$ with $r < n$, by expressing each binomial coefficient in terms of factorials. [3]

14. MEDIUM Calculator **[4 marks]**

A biologist models the probability that exactly k out of 20 plants show a particular gene mutation, where $k \in \{0, 1, 2, \dots, 20\}$, using the binomial distribution with $n = 20$ and $p = 0.35$. The probability is proportional to $\binom{20}{k}(0.35)^k(0.65)^{20-k}$.

Using technology, find the value of k for which this probability is greatest, and state that maximum probability correct to four significant figures.

15. MEDIUM No Calculator **[4 marks]**

Find the first three terms, in ascending powers of x, of the binomial expansion of $\dfrac{1}{(1+3x)^4}$, where $|x| < \dfrac{1}{3}$.

State the range of validity.

16. MEDIUM Calculator **[4 marks]**

Consider the expansion of $(1+x)^{12}$.

(a) Write down $\binom{12}{5}$. [1]

(b) The table below gives values of $(1+x)^{12}$, the 4-term truncation $S_3(x) = 1 + 12x + 66x^2 + 220x^3$, and the percentage error $E = \left|\frac{(1+x)^{12} - S_3(x)}{(1+x)^{12}}\right| \times 100.$

x	$(1+x)^{12}$	$S_3(x)$	E (%)
0.05	1.7959	1.7930	0.1614
0.10	3.1384	3.0820	1.796
0.20	8.9161	7.8000	A

Find the value of A, correct to three significant figures. [3]

17. MEDIUM Calculator **[4 marks]**

In the expansion of $(1+x)^n(1+x)^m = (1+x)^{n+m}$, by comparing the coefficients of x^2 on both sides, show that

$$\binom{n}{2} + nm + \binom{m}{2} = \binom{n+m}{2}.$$

State the values of $n, m \in \mathbb{Z}^+$ for which each binomial coefficient is defined.

18. MEDIUM Calculator **[4 marks]**

The first three terms of the expansion of $(1+ax)^n$ in ascending powers of x are $1 - 12x + 60x^2$, where $a \in \mathbb{R}$ and $n \in \mathbb{Q}$.

(a) Find the values of a and n. [3]

(b) State the range of values of x for which the expansion is valid. [1]

19. MEDIUM Calculator **[4 marks]**

Using technology, find the largest value of $N \in \mathbb{Z}^+$ such that the 3-term approximation

$$S_2(x) = 1 + 2x + 3x^2$$

of $(1-x)^{-2}$ satisfies $|S_2(x) - (1-x)^{-2}| < 0.01$ for all $x \in \left[0, \dfrac{N}{100}\right]$.

20. MEDIUM Calculator **[4 marks]**

The coefficient of x^r in the expansion of $(1+x)^{2r}$, where $r \in \mathbb{Z}^+$, is denoted by C_r.

(a) Show that $C_r = \dbinom{2r}{r}$. [1]

(b) Using technology, find the smallest value of r such that $C_r > 100\,000$. [3]

Section C — Hard **HARD** (10 questions $\times$ 5 marks = 50 marks)

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21. **HARD** **No Calculator** **[5 marks]**

Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{R}$. The expansion of $(kx+2)^n$ contains a term in x^3 whose coefficient is $1\,280$ and a term in x^2 whose coefficient is 960. Find the values of n and k.

22. **HARD** **No Calculator** **[5 marks]**

Consider the expansion of $\left(3x^2 - \dfrac{1}{x}\right)^9$.

(a) Write down the general term T_{r+1} of the expansion. [1]

(b) Find the term independent of x. [2]

(c) Hence find the coefficient of x^{-9} in $\left(1 + \dfrac{1}{x^3}\right)\left(3x^2 - \dfrac{1}{x}\right)^9$. [2]

23. HARD Calculator **[5 marks]**

Show that $\binom{2n}{2} - 2\binom{n}{2} = n^2$ for all $n \in \mathbb{Z}^+$, $n \geq 2$. Hence find the value of n for which

$$\binom{2n}{2} - 2\binom{n}{2} = 5\,625, \qquad n \geq 2.$$

24. HARD Calculator **[5 marks]**

The table below compares the true value of $\sqrt[3]{1+x}$ with the approximation obtained by truncating its binomial expansion to three terms, $P(x) = 1 + \frac{x}{3} - \frac{x^2}{9}$, at selected values of x, where $|x| < 1$.

x	$P(x)$	$\sqrt[3]{1+x}$ (true)	Percentage error (%)
0.1	1.0322	$1.03228\ldots$	A
0.3	1.0900	$1.09139\ldots$	B
0.6	1.1600	$1.16961\ldots$	C

(a) Write down the range of validity of this expansion. [1]

(b) Show that the three-term expansion is $P(x) = 1 + \frac{x}{3} - \frac{x^2}{9}$. [2]

(c) Calculate the values A, B, and C in the table, giving your answers correct to 3 significant figures. State, with a reason, at which of the three x-values the approximation is least accurate. [2]

25. HARD Calculator **[5 marks]**

The binomial coefficient $\binom{n}{r}$ is defined for $n, r \in \mathbb{Z}_0^+$, $r \leq n$. Find the smallest value of $n \in \mathbb{Z}^+$ such that $\binom{n}{4} > 10\,000$.

26. HARD Calculator **[5 marks]**

Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{R}^+$. In the expansion of $\left(x^2 + \frac{k}{x}\right)^n$, the coefficient of x^{11} is 960 and the coefficient of x^5 is $8\,064$. Find the values of n and k.

27. **HARD** **No Calculator** **[5 marks]**

Consider the identity $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$. The Pascal's triangle diagram below highlights two entries in row 4 and their sum in row 5.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

(a) State the values of n and r for which the circled entries illustrate the identity $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$. [1]

(b) Using the identity, show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. [2]

(c) Hence show that $\sum_{k=0}^{n} \binom{n}{k}^2 \leq 4^n$ and state when equality holds. [2]

28. **HARD** **Calculator** **[5 marks]**

A fair six-sided die is rolled 12 times. Using the binomial theorem, write down and simplify the term in the expansion of $\left(\frac{1}{6} + \frac{5}{6}\right)^{12}$ that gives the probability of obtaining exactly 4 sixes. Calculate this probability, giving your answer correct to 4 significant figures. Hence find the probability of obtaining at most 4 sixes, giving your answer correct to 4 significant figures.

29. HARD Calculator **[5 marks]**

Consider $f(x) = (1 - 3x)^{-2}$, $x \in \mathbb{R}$, $|x| < \dfrac{1}{3}$.

(a) Write down the range of validity of the expansion of $f(x)$ as a power series. [1]

(b) Show that the general term of the expansion is $T_{r+1} = (r + 1)3^r x^r$. [2]

(c) Find the value of x, correct to 4 decimal places, for which the sum of the first six terms of the expansion equals 1.5. State whether this value of x lies within the range of validity. [2]

30. HARD No Calculator **[5 marks]**

Let $p, q \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, $n \geq 3$. In the expansion of $(p + qx)^n$, the first three terms in ascending powers of x are $32 - 40x + cx^2$.

(a) Find p, q, and n. [3]

(b) Find the value of c. [1]

(c) Hence find the coefficient of x^3 in the expansion. [1]

Mark Schemes

Q1 — Binomial Expansion EASY

Using the Binomial Theorem with $n = 4$: **M1**

$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$ **A1 A1**

Note: Award A1 for the first two terms correct and A1 for all remaining terms correct.

Q2 — Term Independent of x EASY

General term: $T_{r+1} = \binom{6}{r} x^{6-r} \left(\frac{2}{x}\right)^r = \binom{6}{r} 2^r x^{6-2r}$ **M1**

For the term independent of x: $6 - 2r = 0 \Rightarrow r = 3$ **A1**

$T_4 = \binom{6}{3} \cdot 2^3 = 20 \times 8 = 160$ **A1**

Q3 — Coefficient Equation EASY

General term: $T_{r+1} = \binom{5}{r} 3^{5-r}(kx)^r$; for x^2 set $r = 2$: **M1**

Coefficient of x^2: $\binom{5}{2} \cdot 3^3 \cdot k^2 = 10 \times 27 \times k^2 = 270k^2$ **A1**

$270k^2 = 1080 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$ **A1**

Q4 — Fractional Index Expansion EASY

$(1 + 3x)^{-2} = 1 + (-2)(3x) + \frac{(-2)(-3)}{2!}(3x)^2 + \cdots$ **M1**

$= 1 - 6x + 27x^2 + \cdots$ **A1**

Valid for $|3x| < 1$, i.e. $|x| < 1/3$ **A1**

Q5 — Pascal Identity Verification EASY

From the triangle: $\binom{4}{1} = 4$ and $\binom{4}{2} = 6$ **A1**

$\binom{4}{1} + \binom{4}{2} = 4 + 6 = 10$ **M1**

$\binom{5}{2} = 10$, so the identity holds. **A1**

Q6 — Coefficient of a Power Term EASY

General term: $T_{r+1} = \binom{7}{r} 2^{7-r}(-x)^r = \binom{7}{r}(-1)^r 2^{7-r} x^r$; set $r = 3$: **M1**

$T_4 = \binom{7}{3}(-1)^3 \cdot 2^4 = 35 \times (-1) \times 16$ **A1**

Coefficient of x^3 is -560 **A1**

Q7 — Approximating a Square Root EASY

Substituting $x = 0.04$: $1 + \frac{0.04}{2} - \frac{(0.04)^2}{8} = 1 + 0.02 - 0.0002 = 1.0198$ **M1 A1**

GDC check: $\sqrt{1.04} = 1.019803\,902\ldots$; expansion gives 1.0198 (4 d.p.).

Valid for $|x| < 1$ **A1**

Q8 — Binomial Coefficient and Combinations EASY

GDC: nCr$(8, 3)$
$\binom{8}{3} = 56$ **A1**

The number of committees equals the number of combinations, which is $\binom{8}{3}$ **M1**

Number of ways $= 56$ **A1**

Q9 — Coefficient in a Product Expansion EASY

From $(1 + x)^5$: coefficient of x^2 is $\binom{5}{2} = 10$; coefficient of x is $\binom{5}{1} = 5$ **M1**

Coefficient of x^2 in the product: $10 \times 1 + 5 \times 2 = 10 + 10$ **A1**

$= 20$ **A1**

Q10 — General Term of an Expansion EASY

$T_{r+1} = \binom{6}{r}(x^2)^{6-r}(x^{-1})^r = \binom{6}{r}x^{12-2r-r} = \binom{6}{r}x^{12-3r}$ **M1**

Set $12 - 3r = 6 \Rightarrow r = 2$ **A1**

$T_3 = \binom{6}{2}x^6 = 15x^6$ **A1**

Q11 — Coefficient Equation with Unknown k MEDIUM

General term: $T_{r+1} = \binom{7}{r}(2)^{7-r}(kx)^r$ **M1**

For x^3: $r = 3$, so $T_4 = \binom{7}{3}(2)^4k^3x^3 = 35 \cdot 16 \cdot k^3x^3 = 560k^3x^3$ **A1**

Setting coefficient equal to 2240: $560k^3 = 2240 \Rightarrow k^3 = 4$ **M1**

$k = \sqrt[3]{4}$ (GDC form: `4^(1/3)`) **A1**

Q12 — Constant Term in a Binomial Expansion MEDIUM

(a) $T_{r+1} = \binom{9}{r}(3x^2)^{9-r}\left(-\frac{1}{x}\right)^r = \binom{9}{r}3^{9-r}(-1)^r x^{18-2r-r} = \binom{9}{r}3^{9-r}(-1)^r x^{18-3r}$ **A1**

(b) For the term independent of x: $18 - 3r = 0 \Rightarrow r = 6$ **M1**

$T_7 = \binom{9}{6}3^3(-1)^6 = 84 \cdot 27 \cdot 1$ **A1**

$= 2268$ **A1**

Q13 — Pascal Identity Proof MEDIUM

(a) From the triangle: $4 + 6 = 10$, i.e. $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$ **A1**

(b) $\binom{n}{r} + \binom{n}{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!}$ **M1**

Factor out $\frac{n!}{(r+1)!(n-r)!}$: $= \frac{n!(r+1)}{(r+1)!(n-r)!} + \frac{n!(n-r)}{(r+1)!(n-r)!} = \frac{n!(r+1+n-r)}{(r+1)!(n-r)!}$ **M1**

$= \frac{(n+1)!}{(r+1)!((n+1)-(r+1))!} = \binom{n+1}{r+1}$ **A1 AG**

Q14 — Maximum Binomial Probability MEDIUM

Define $f(k) = \binom{20}{k}(0.35)^k(0.65)^{20-k}$ for $k = 0, 1, \ldots, 20$. **M1**

Using GDC (list/table of $f(k)$ values or binomialpdf$(20, 0.35, k)$): **(M1)**

Unrounded maximum: $f(7) \approx 0.18440\ldots$ **A1**

The probability is greatest at $k = 7$, with maximum probability 0.1844 (4 s.f.). **A1**

Q15 — Negative Index Expansion MEDIUM

Write $(1+3x)^{-4}$. General term with $n = -4$: **M1**

$(1+3x)^{-4} = 1 + (-4)(3x) + \dfrac{(-4)(-5)}{2!}(3x)^2 + \cdots$ **M1**

$= 1 - 12x + \dfrac{20 \cdot 9}{2}x^2 + \cdots = 1 - 12x + 90x^2 + \cdots$ **A1**

Valid for $|3x| < 1$, i.e. $|x| < 1/3$. **A1**

Q16 — Percentage Error of a Truncation MEDIUM

(a) $\binom{12}{5} = 792$ **A1**

(b) True value: $(1+0.20)^{12} = (1.20)^{12}$. Using GDC: $(1.20)^{12} = 8.91610\ldots$ **M1**

$S_3(0.20) = 1 + 12(0.20) + 66(0.04) + 220(0.008) = 1 + 2.4 + 2.64 + 1.76 = 7.8000$ **(A1)**

$A = \left|\dfrac{8.91610 - 7.8000}{8.91610}\right| \times 100 = \dfrac{1.11610}{8.91610} \times 100 = 12.5\%$ (3 s.f.) **A1**

Q17 — Coefficient Comparison Identity MEDIUM

Required: $n \geq 2, m \geq 2$ so that $\binom{n}{2}$ and $\binom{m}{2}$ are defined; $n, m \in \mathbb{Z}^+$. **A1**

Coefficient of x^2 in $(1+x)^n$: $\binom{n}{2}$. Coefficient of x^2 in $(1+x)^m$: $\binom{m}{2}$. **M1**

Coefficient of x^2 in $(1+x)^n(1+x)^m$: collect $x^0 \cdot x^2$ and $x^1 \cdot x^1$ and $x^2 \cdot x^0$ terms: $\binom{n}{2} \cdot 1 + \binom{n}{1}\binom{m}{1} + 1 \cdot \binom{m}{2} = \binom{n}{2} + nm + \binom{m}{2}$. **M1**

Coefficient of x^2 in $(1+x)^{n+m}$ is $\binom{n+m}{2}$. Since the two sides are equal as polynomials, $\binom{n}{2} + nm +$

$\binom{m}{2} = \binom{n+m}{2}$. **A1 AG**

Q18 — Fractional Index from Given Terms MEDIUM

(a) $(1+ax)^n = 1 + n(ax) + \frac{n(n-1)}{2}(ax)^2 + \cdots$ **M1**

Comparing x^1: $na = -12$. Comparing x^2: $\frac{n(n-1)}{2}a^2 = 60$.

From $na = -12$: $a = -12/n$. Substitute: $\frac{n(n-1)}{2} \cdot \frac{144}{n^2} = 60 \Rightarrow \frac{144(n-1)}{2n} = 60 \Rightarrow 144n - 144 = 120n \Rightarrow n = 6$. **M1**

Then $a = -12/6 = -2$; so $n = 6$, $a = -2$. **A1**

(b) Valid for $|-2x| < 1$, i.e. $|x| < \frac{1}{2}$. **A1**

Q19 — Accuracy of a Series Approximation MEDIUM

Define $g(x) = |S_2(x) - (1-x)^{-2}| = |1 + 2x + 3x^2 - (1-x)^{-2}|$. **M1**

Using GDC, graph or table $g(x)$ for $x \in [0,1)$, checking $g(N/100) < 0.01$: **M1**

$g(0.12) \approx 0.00812 < 0.01$; $g(0.13) \approx 0.01048 > 0.01$ (unrounded values from GDC). **A1**

The largest such N is $N = 12$. **A1**

Q20 — Binomial Coefficient Growth MEDIUM

(a) The general term T_{r+1} of $(1+x)^{2r}$ is $\binom{2r}{r}x^r$, so the coefficient of x^r is $\binom{2r}{r}$, i.e. $C_r = \binom{2r}{r}$. **A1 AG**

(b) Using GDC, compute $\binom{2r}{r}$ for successive $r \in \mathbb{Z}^+$: **M1**

$r = 9$: $\binom{18}{9} = 48620 < 100000$; $r = 10$: $\binom{20}{10} = 184756 > 100000$. **A1**

The smallest value is $r = 10$. **A1**

Q21 — Coefficient Equation with Two Unknowns HARD

General term: $T_{r+1} = \binom{n}{r}(kx)^{n-r}(2)^r = \binom{n}{r}k^{n-r}2^r x^{n-r}$ **M1**

Term in x^3: $n-r=3$, so $r=n-3$. Coefficient: $\binom{n}{n-3}k^3\cdot 2^{n-3}=\binom{n}{3}k^3\cdot 2^{n-3}=1280$ **A1**

Term in x^2: $n-r=2$, so $r=n-2$. Coefficient: $\binom{n}{n-2}k^2\cdot 2^{n-2}=\binom{n}{2}k^2\cdot 2^{n-2}=960$ **A1**

Dividing the x^3 equation by the x^2 equation, and using $\binom{n}{3}/\binom{n}{2}=\frac{n-2}{3}$:

$$\frac{n-2}{3}\cdot k\cdot\frac{1}{2}=\frac{1280}{960}=\frac{4}{3}\implies k(n-2)=8.$$

M1

Testing $n=6$: $k=8/4=2$. Check x^2 equation: $\binom{6}{2}(2)^2(2)^4=15\cdot 4\cdot 16=960$ ✓; check x^3 equation: $\binom{6}{3}(2)^3(2)^3=20\cdot 8\cdot 8=1280$ ✓. **A1**

So $n=6$, $k=2$.

Q22 — Specified Term in a Product Expansion HARD

(a) $T_{r+1}=\binom{9}{r}(3x^2)^{9-r}\left(-\frac{1}{x}\right)^r=\binom{9}{r}3^{9-r}(-1)^r x^{18-2r-r}=\binom{9}{r}3^{9-r}(-1)^r x^{18-3r}$ **A1**

(b) Independent of x: $18-3r=0\Rightarrow r=6$. **M1**

$T_7=\binom{9}{6}3^3(-1)^6=84\times 27=2268$ **A1**

(c) The coefficient of x^{-9} in $\left(1+\frac{1}{x^3}\right)\left(3x^2-\frac{1}{x}\right)^9$ comes from two contributions: **M1**

$1\times$(coeff of x^{-9} in expansion): $18-3r=-9\Rightarrow r=9$, $T_{10}=\binom{9}{9}3^0(-1)^9=-1$.

$\frac{1}{x^3}\times$(coeff of x^{-6} in expansion): $18-3r=-6\Rightarrow r=8$, $T_9=\binom{9}{8}3^1(-1)^8=27$.

Total coefficient of x^{-9}: $-1+27=26$ **A1**

Q23 — Pascal Identity and Large Value HARD

$\binom{2n}{2}-2\binom{n}{2}=\frac{2n(2n-1)}{2}-2\cdot\frac{n(n-1)}{2}$ **M1**

$=n(2n-1)-n(n-1)=2n^2-n-n^2+n=n^2$ **A1**
(AG: shown.)

Hence solve $n^2=5\,625$: **M1**

$n = \sqrt{5625} = 75$ (rejecting $n = -75$ since $n \in \mathbb{Z}^+$) **A1**

Check: $\binom{150}{2} - 2\binom{75}{2} = 11175 - 2(2775) = 11175 - 5550 = 5625$ ✓ **A1**

Q24 — Fractional Index Validity and Error HARD

(a) Range of validity: $|x| < 1$ **A1**

(b) $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}x^2 + \cdots$ **M1**

$= 1 + \frac{x}{3} + \frac{\frac{1}{3}\cdot(-\frac{2}{3})}{2}x^2 + \cdots = 1 + \frac{x}{3} - \frac{x^2}{9} + \cdots$ **A1**

(AG: shown.)

(c) Using GDC:

$A = \left|\frac{1.0322 - 1.03228}{1.03228}\right| \times 100 \approx 0.00561\%$ **M1**

$B = \left|\frac{1.0900 - 1.09139}{1.09139}\right| \times 100 \approx 0.128\%$

$C = \left|\frac{1.1600 - 1.16961}{1.16961}\right| \times 100 \approx 0.821\%$ (GDC).

The approximation is least accurate at $x = 0.6$ since $|x|$ is largest and furthest from 0, so higher-order neglected terms are most significant. **A1**

Q25 — Large Binomial Coefficient Search HARD

We require the smallest $n \in \mathbb{Z}^+$ such that $\binom{n}{4} > 10\,000$. **M1**

$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$. Using GDC to evaluate for successive integers: **M1**

n	$\binom{n}{4}$
20	4 845
22	7 315
23	8 855
24	10 626

A1

GDC (unrounded): $\binom{23}{4} = 8\,855 < 10\,000$ and $\binom{24}{4} = 10\,626 > 10\,000$. **A1**

The smallest value is $n = 24$. **A1**

Q26 — Equal Coefficients Equation HARD

General term: $T_{r+1} = \binom{n}{r}(x^2)^{n-r}\left(\frac{k}{x}\right)^r = \binom{n}{r}k^r x^{2n-3r}$ **M1**

Coefficient of x^{11}: $2n - 3r_1 = 11 \Rightarrow \binom{n}{r_1}k^{r_1} = 960.$

Coefficient of x^5: $2n - 3r_2 = 5 \Rightarrow \binom{n}{r_2}k^{r_2} = 8\,064$. Subtracting exponents gives $r_2 = r_1 + 2$. **M1**

Dividing: $\dfrac{\binom{n}{r_2}}{\binom{n}{r_1}}k^2 = \dfrac{8064}{960} = 8.4.$ **A1**

Using GDC: try $n = 10$: $r_1 = 3$, $r_2 = 5$. $\dfrac{\binom{10}{5}}{\binom{10}{3}} = \dfrac{252}{120} = 2.1$, so $2.1k^2 = 8.4 \Rightarrow k^2 = 4 \Rightarrow k = 2$ ($k \in \mathbb{R}^+$). **A1**

Check: $\binom{10}{3}(2)^3 = 120 \times 8 = 960$ ✓; exponent $2(10) - 3(3) = 11$ ✓. So $n = 10$, $k = 2$. **A1**

Q27 — Pascal Identity Row Sums and Bound HARD

(a) The circled entries 4 and 6 in row 4 sum to 10 in row 5, illustrating $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$, so $n = 4$, $r = 1$. **A1**

(b) Consider $(1+1)^n$ by the binomial theorem: **M1**

$$(1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k = \sum_{k=0}^{n}\binom{n}{k}$$

Since $(1+1)^n = 2^n$, it follows that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. **A1**

(AG: shown.)

(c) By squaring the row sum: **M1**

$$\left(\sum_{k=0}^{n}\binom{n}{k}\right)^2 = (2^n)^2 = 4^n.$$

Since $(a_1 + \cdots + a_m)^2 \geq a_1^2 + \cdots + a_m^2$ for positive reals (as cross terms are non-negative), $\sum_{k=0}^{n}\binom{n}{k}^2 \leq \left(\sum_{k=0}^{n}\binom{n}{k}\right)^2 = 4^n.$ **A1**

Equality holds when $n = 0$ (only one term, $\binom{0}{0}^2 = 1 = 4^0$). **R1**

Q28 — Binomial Probability of Sixes HARD

The required term from $\left(\frac{1}{6} + \frac{5}{6}\right)^{12}$ for exactly 4 sixes is: **M1**

$T_5 = \binom{12}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^8$ **A1**

GDC: $\binom{12}{4} = 495$; $\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$; $\left(\frac{5}{6}\right)^8 = \frac{390625}{1679616}$.

Unrounded: $P(X = 4) = 495 \times \dfrac{390625}{1679616 \times 1296} = 0.088828\ldots$ **A1**

$P(X = 4) \approx 0.08883$ (4 s.f.)

$P(X \le 4) = \sum_{j=0}^{4}\binom{12}{j}\left(\frac{1}{6}\right)^j\left(\frac{5}{6}\right)^{12-j}$. GDC (binomial CDF, $n = 12$, $p = \frac{1}{6}$, $x = 4$): unrounded $= 0.96365\ldots$ **M1**

$P(X \le 4) \approx 0.9636$ (4 s.f.) **A1**

Q29 — Negative Index Series Root HARD

(a) $|-3x| < 1 \Rightarrow |x| < \dfrac{1}{3}$ **A1**

(b) $(1-3x)^{-2} = \sum_{r=0}^{\infty}\binom{-2}{r}(-3x)^r$. **M1**

$\binom{-2}{r} = \dfrac{(-2)(-3)\cdots(-2-r+1)}{r!} = (-1)^r(r+1)$.
$T_{r+1} = (-1)^r(r+1)(-3)^r x^r = (r+1)3^r x^r$. **A1**
(AG: shown.)

(c) Sum of first six terms ($r = 0$ to 5): $S_6(x) = \sum_{r=0}^{5}(r+1)3^r x^r = 1+6x+27x^2+108x^3+405x^4+1458x^5$.
M1

GDC: solve $1 + 6x + 27x^2 + 108x^3 + 405x^4 + 1458x^5 = 1.5$.
GDC gives $x \approx 0.0612\ldots$ (unrounded $0.061199\ldots$), so $x \approx 0.0612$ (4 d.p.).

Since $0.0612 < \dfrac{1}{3} \approx 0.333$, this value lies within the range of validity. **A1**

Q30 — First Three Terms of an Expansion HARD

(a) $T_1 = p^n = 32 = 2^5$, so $p^n = 2^5$. **M1**

$T_2 = np^{n-1}(qx) = -40x \Rightarrow np^{n-1}q = -40$.

$T_3 = \binom{n}{2}p^{n-2}(qx)^2 = cx^2 \Rightarrow \binom{n}{2}p^{n-2}q^2 = c$.

From $p^n = 32$: if $n = 5$, then $p = 2$.

Check: $5 \cdot 2^4 \cdot q = -40 \Rightarrow 80q = -40 \Rightarrow q = -1/2$. **A1**

So $p = 2$, $q = -1/2$, $n = 5$. **A1**

(b) $\binom{5}{2}(2)^3\left(\frac{1}{2}\right)^2 = 10 \times 8 \times \frac{1}{4} = 20$, so $c = 20$ **A1**

(c) $T_4 = \binom{5}{3}(2)^2\left(-\frac{1}{2}\right)^3 x^3 = 10 \times 4 \times \left(-\frac{1}{8}\right)x^3 = -5x^3$.

Coefficient of x^3 is -5. **A1**