# IB Math AA HL — Topic 1

## Binomial Theorem

Binomial Expansions · Pascal's Triangle · Extension of the Binomial Theorem

| Difficulty | EASY | MEDIUM | HARD |
|---|---|---|---|
| Questions | 10 | 10 | 10 |
| Total Marks | 30 | 40 | 50 |

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Name:______________________________ Date:_______________

Score:__________/120

**Instructions:** Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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### Key Formulae

- **Binomial Theorem (finite):** $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r, \quad n \in \mathbb{Z}^+.$
- **General term:** $T_{r+1} = \binom{n}{r} a^{n-r} b^r$; identify $r$ before equating powers.
- **Binomial coefficient:** $\binom{n}{r} = \frac{n!}{r!(n-r)!}$; also $\binom{n}{r} = \binom{n}{n-r}$ (symmetry).
- **Pascal's identity:** $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$; row sum $\sum_{r=0}^{n} \binom{n}{r} = 2^n$.
- **Extended binomial theorem:** $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots, \quad n \in \mathbb{Q}$, valid for $|x| < 1$.
- **Extended form with coefficient:** $(1+ax)^n$ valid for $|x| < \frac{1}{|a|}$; always state this range explicitly.
- **Constant term:** set the power of $x$ in $T_{r+1}$ equal to zero and solve for $r$; $r$ must be a non-negative integer.

## GDC Tips

- **Evaluating large $\binom{n}{r}$:** use `nCr` via Math > PRB on the TI-84, or `Combinatorial` in the Casio ClassPad `Catalog`.
- **Expanding & collecting coefficients:** enter `expand((ax+b)^n)` in the CAS (ClassPad `Main` menu) to verify a coefficient answer instantly.
- **Percentage error of approximation:** store the truncated value and compute $\left|\dfrac{\text{approx} - \text{exact}}{\text{exact}}\right| \times 100$ directly; compare with the required accuracy.
- **Solving coefficient equations numerically:** graph $f(k) =$ coefficient expression − target and use `G-Solve` > `Root` (Casio) or `2nd` `TRACE` `Zero` (TI-84) to find $k$; check integrality or sign constraints before accepting.
- **Checking the validity interval:** substitute the boundary $x = \pm 1/|a|$ into both sides numerically to confirm the series diverges there; present the open interval $|x| < 1/|a|$ in the answer.
- **Number of terms for stated accuracy:** sum successive terms with `Sequence Sum` or a `For` loop, stopping when $|T_{r+1}| <$ tolerance; record the first $r$ satisfying the condition.

## Common Mistakes to Avoid

1. **Wrong index on $a$ in the general term.** Students write $T_{r+1} = \binom{n}{r}a^r b^{n-r}$ instead of $a^{n-r}b^r$, swapping the two powers and producing every coefficient incorrectly.
2. **Forgetting the range of validity for extended expansions.** Omitting $|x| < 1/|a|$ loses a dedicated mark; the series is meaningless outside this interval and must be stated every time.
3. **Accepting a non-integer or negative value of $r$.** When solving for the constant term or a specified coefficient, $r$ must satisfy $r \in \{0, 1, \ldots, n\}$; a fractional or negative $r$ means no such term exists and must be explicitly rejected.
4. **Sign errors with negative terms.** In $(a - b)^n$, writing $b^r$ instead of $(-b)^r$ loses the alternating sign; the correct substitution is $T_{r+1} = \binom{n}{r}a^{n-r}(-b)^r$.
5. **Confusing $T_r$ and $T_{r+1}$.** The $r$-th term is $T_r = \binom{n}{r-1}a^{n-r+1}b^{r-1}$, but the standard formula indexes from $T_1$; always define which convention is used and be consistent throughout.
6. **Applying the extended theorem outside its range.** Substituting $|x| \geq 1/|a|$ to approximate a surd gives a divergent series; verify the substituted value lies strictly inside the validity interval before using the approximation.

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## Section A — Easy

EASY

**1.** Write down the expansion of $(1+x)^4$ in ascending powers of $x$. *Calculator not allowed* **[3 marks]**

**2.** Find the term independent of $x$ in the expansion of $\left(x+\frac{2}{x}\right)^6$, where $x \neq 0$. *Calculator not allowed* **[3 marks]**

**3.** The coefficient of $x^2$ in the expansion of $(3+kx)^5$, where $k \in \mathbb{R}$, is $1080$. Find the value of $k$. *Calculator allowed* **[3 marks]**

**4.** Find the first three terms, in ascending powers of $x$, of the binomial expansion of $(1+3x)^{-2}$, where $|x| < \frac{1}{3}$. State the range of validity. *Calculator not allowed* **[3 marks]**

**5.** The diagram shows Pascal's triangle up to row 5. Use the identity $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ to verify that $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$, and state the values of $\binom{4}{1}$, $\binom{4}{2}$, and $\binom{5}{2}$. *Calculator not allowed* **[3 marks]**

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

**6.** Find the coefficient of $x^3$ in the expansion of $(2-x)^7$. *Calculator allowed* **[3 marks]**

**7.** Using the expansion of $(1+x)^{1/2} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$ with $x = 0.04$, find an approximate value of $\sqrt{1.04}$. The table gives the truncated-expansion value alongside the true value. State the range of validity of this expansion.
*Calculator allowed* **[3 marks]**

| $x$ | Expansion value | True value $\sqrt{1+x}$ |
|---|---|---|
| $0.04$ | $1.0198$ | $1.019804\ldots$ |

**8.** Write down the value of $\binom{8}{3}$ and hence find the number of ways of choosing a committee of 3 people from a group of 8. *Calculator allowed* **[3 marks]**

**9.** Find the coefficient of $x^2$ in the expansion of $(1+x)^5(1+2x)$. *Calculator allowed* **[3 marks]**

**10.** The general term of the expansion of $\left(x^2 + \frac{1}{x}\right)^6$ is $T_{r+1} = \binom{6}{r}(x^2)^{6-r}\left(\frac{1}{x}\right)^r$, where $r \in \{0, 1, 2, 3, 4, 5, 6\}$.
Find the term in $x^6$. *Calculator allowed* **[3 marks]**

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## Section B — Medium

**MEDIUM**

**11.** The coefficient of $x^3$ in the expansion of $(2 + kx)^7$, where $k \in \mathbb{R}$, is equal to $2240$. Find the value of $k$.

*Calculator not allowed* **[4 marks]**

**12.** Consider the expansion of $\left(3x^2 - \frac{1}{x}\right)^9$, where $x \in \mathbb{R}$, $x \neq 0$.

(a) Write down the general term $T_{r+1}$ of the expansion. **[1 marks]**

(b) Find the term independent of $x$. **[3 marks]**

*Calculator not allowed*

**13.** The diagram shows rows 0 to 5 of Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(a) Using the circled values, verify the Pascal identity $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$. **[1 marks]**

(b) Hence show that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ for all $n, r \in \mathbb{Z}^+$ with $r < n$, by expressing each binomial coefficient in terms of factorials. **[3 marks]**

*Calculator not allowed*

**14.** A biologist models the probability that exactly $k$ out of $20$ plants show a particular gene mutation, where $k \in \{0, 1, 2, \ldots, 20\}$, using the binomial distribution with $n = 20$ and $p = 0.35$. The probability is proportional to $\binom{20}{k}(0.35)^k(0.65)^{20-k}$.

Using technology, find the value of $k$ for which this probability is greatest, and state that maximum probability correct to four significant figures. *Calculator allowed* **[4 marks]**

**15.** Find the first three terms, in ascending powers of $x$, of the binomial expansion of $\dfrac{1}{(1+3x)^4}$, where $|x| < \dfrac{1}{3}$. State the range of validity. *Calculator not allowed* **[4 marks]**

**16.** Consider the expansion of $(1+x)^{12}$.

(a) Write down $\dbinom{12}{5}$. **[1 marks]**

(b) The table below gives values of $(1+x)^{12}$, the 4-term truncation $S_3(x) = 1 + 12x + 66x^2 + 220x^3$, and the percentage error $E = \left|\dfrac{(1+x)^{12} - S_3(x)}{(1+x)^{12}}\right| \times 100$.

| $x$ | $(1+x)^{12}$ | $S_3(x)$ | $E$ (%) |
|---|---|---|---|
| 0.05 | 1.7959 | 1.7930 | 0.1614 |
| 0.10 | 3.1384 | 3.0820 | 1.796 |
| 0.20 | 8.9161 | 8.1240 | $A$ |

Find the value of $A$, correct to three significant figures. **[3 marks]**

*Calculator allowed*

**17.** In the expansion of $(1+x)^n(1+x)^m = (1+x)^{n+m}$, by comparing the coefficients of $x^2$ on both sides, show that

$$\binom{n}{2} + nm + \binom{m}{2} = \binom{n+m}{2}.$$

State the values of $n, m \in \mathbb{Z}^+$ for which each binomial coefficient is defined. *Calculator allowed* **[4 marks]**

**18.** The first three terms of the expansion of $(1+ax)^n$ in ascending powers of $x$ are $1 - 12x + 60x^2$, where $a \in \mathbb{R}$ and $n \in \mathbb{Q}$.

(a) Find the values of $a$ and $n$. **[3 marks]**

(b) State the range of values of $x$ for which the expansion is valid. **[1 marks]**

*Calculator allowed*

**19.** Using technology, find the largest value of $N \in \mathbb{Z}^+$ such that the 3-term approximation

$$S_2(x) = 1 - 2x + 3x^2$$

of $(1-x)^{-2}$ satisfies $|S_2(x) - (1-x)^{-2}| < 0.01$ for all $x \in \left[0, \dfrac{N}{100}\right]$. *Calculator allowed* **[4 marks]**

**20.** The coefficient of $x^r$ in the expansion of $(1+x)^{2r}$, where $r \in \mathbb{Z}^+$, is denoted by $C_r$.

(a) Show that $C_r = \dbinom{2r}{r}$. **[1 marks]**

(b) Using technology, find the smallest value of $r$ such that $C_r > 100\,000$. **[3 marks]**

*Calculator allowed*

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## Section C — Hard

**HARD**

**21.** Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{R}$. The expansion of $(kx+2)^n$ contains a term in $x^3$ whose coefficient is $1\,280$ and a term in $x^2$ whose coefficient is $3\,840$. Find the values of $n$ and $k$. *Calculator not allowed* **[5 marks]**

**22.** Consider the expansion of $\left(3x^2 - \frac{1}{x}\right)^9$.

(a) Write down the general term $T_{r+1}$ of the expansion. **[1 marks]**

(b) Find the term independent of $x$. **[2 marks]**

(c) Hence find the coefficient of $x^{-9}$ in $\left(1 + \frac{1}{x^3}\right)\left(3x^2 - \frac{1}{x}\right)^9$. **[2 marks]**

*Calculator not allowed*

**23.** *Calculator allowed*Show that $\binom{2n}{2} - 2\binom{n}{2} = n^2$ for all $n \in \mathbb{Z}^+$, $n \geq 2$. Hence find the value of $n$ for which $\binom{2n}{2} - 2\binom{n}{2} = 5\,625 - 2\binom{75}{2}$, where $n \geq 2$. **[5 marks]**

**24.** *Calculator allowed*The table below compares the true value of $\sqrt[3]{1+x}$ with the approximation obtained by truncating its binomial expansion to three terms, $P(x) = 1 + \frac{x}{3} - \frac{x^2}{9}$, at selected values of $x$, where $|x| < 1$.

| $x$ | $P(x)$ | $\sqrt[3]{1+x}$ (true) | Percentage error (%) |
|---|---|---|---|
| $0.1$ | $1.098\overline{8}$ | $1.03228\ldots$ | $A$ |
| $0.3$ | $1.290\overline{0}$ | $1.09139\ldots$ | $B$ |
| $0.6$ | $1.5\overline{6}$ | $1.18258\ldots$ | $C$ |

(a) Write down the range of validity of this expansion. **[1 marks]**

(b) Show that the three-term expansion is $P(x) = 1 + \frac{x}{3} - \frac{x^2}{9}$. **[2 marks]**

(c) Calculate the values $A$, $B$, and $C$ in the table, giving your answers correct to 3 significant figures. State, with a reason, at which of the three $x$-values the approximation is least accurate. **[2 marks]**

**25.** *Calculator allowed*The binomial coefficient $\begin{pmatrix} n \\ r \end{pmatrix}$ is defined for $n, r \in \mathbb{Z}_0^+$, $r \leq n$. Find the smallest value of $n \in \mathbb{Z}^+$ such that $\begin{pmatrix} n \\ 4 \end{pmatrix} > 10\,000$. **[5 marks]**

**26.** *Calculator allowed*Let $n \in \mathbb{Z}^+$. In the expansion of $\left(x^2 + \frac{k}{x}\right)^n$, where $k \in \mathbb{R}^+$, the coefficients of $x^{11}$ and $x^5$ are equal. Given also that the coefficient of $x^5$ is $672$, find the values of $n$ and $k$. **[5 marks]**

**27.** *Calculator not allowed*Consider the identity $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$.

The Pascal's triangle diagram below highlights two entries in row 4 and their sum in row 5.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

(a) State the values of $n$ and $r$ for which the circled entries illustrate the identity $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$. **[1 marks]**

(b) Using the identity, show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. **[2 marks]**

(c) Hence show that $\sum_{k=0}^{n} \binom{n}{k}^2 \leq 4^n$ and state when equality holds. **[2 marks]**

*Calculator not allowed*

**28.** *Calculator allowed*A fair six-sided die is rolled $12$ times. Using the binomial theorem, write down and simplify the term in the expansion of $\left(\frac{1}{6}+\frac{5}{6}\right)^{12}$ that gives the probability of obtaining exactly $4$ sixes. Calculate this probability, giving your answer correct to 4 significant figures. Hence find the probability of obtaining *at most* $4$ sixes, giving your answer correct to 4 significant figures. **[5 marks]**

**29.** *Calculator allowed*Consider $f(x) = (1-3x)^{-2}$, $x \in \mathbb{R}$, $|x| < \frac{1}{3}$.

(a) Write down the range of validity of the expansion of $f(x)$ as a power series. **[1 marks]**

(b) Show that the general term of the expansion is $T_{r+1} = (r+1)\,3^r\,x^r$. **[2 marks]**

(c) Find the value of $x$, correct to 4 decimal places, for which the sum of the first six terms of the expansion equals $1.5$. State whether this value of $x$ lies within the range of validity. **[2 marks]**

**30.** *Calculator not allowed*Let $p, q \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, $n \geq 3$. In the expansion of $(p + qx)^n$, the first three terms in ascending powers of $x$ are $32 - 40x + cx^2$.

(a) Find $p$, $q$, and $n$. **[3 marks]**

(b) Find the value of $c$. **[1 marks]**

(c) Hence find the coefficient of $x^3$ in the expansion. **[1 marks]**

## Worked Solutions

**Q1 — Expansion of $(1+x)^4$ [EASY]**

Using the Binomial Theorem with $n = 4$: **M1**

$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$ **A1A1**

*Note: Award **A1** for the first two terms correct and **A1** for all remaining terms correct.*

**Q2 — Constant term in $\left(x + \frac{2}{x}\right)^6$ [EASY]**

General term: $T_{r+1} = \binom{6}{r} x^{6-r} \left(\frac{2}{x}\right)^r = \binom{6}{r} 2^r x^{6-2r}$ **M1**

For the term independent of $x$: $6 - 2r = 0 \Rightarrow r = 3$ **A1**

$T_4 = \binom{6}{3} \cdot 2^3 = 20 \times 8 = 160$ **A1**

**Q3 — Coefficient equation in $(3+kx)^5$ [EASY]**

General term: $T_{r+1} = \binom{5}{r} 3^{5-r}(kx)^r$; for $x^2$ set $r = 2$: **M1**

Coefficient of $x^2$: $\binom{5}{2} \cdot 3^3 \cdot k^2 = 10 \times 27 \times k^2 = 270k^2$ **A1**

$270k^2 = 1080 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$ **A1**

**Q4 — Fractional index expansion of $(1+3x)^{-2}$ [EASY]**

$(1+3x)^{-2} = 1 + (-2)(3x) + \frac{(-2)(-3)}{2!}(3x)^2 + \cdots$ **M1**

$= 1 - 6x + 27x^2 + \cdots$ **A1**

Valid for $|3x| < 1$, i.e. $|x| < \frac{1}{3}$ **A1**

### Q5 — Pascal identity verification [EASY]

From the triangle: $\binom{4}{1} = 4$ and $\binom{4}{2} = 6$ **A1**

$\binom{4}{1} + \binom{4}{2} = 4 + 6 = 10$ **M1**

$\binom{5}{2} = 10$, so the identity holds. **A1**

### Q6 — Coefficient of $x^3$ in $(2-x)^7$ [EASY]

General term: $T_{r+1} = \binom{7}{r} 2^{7-r}(-x)^r = \binom{7}{r}(-1)^r 2^{7-r} x^r$; set $r = 3$: **M1**

$T_4 = \binom{7}{3}(-1)^3 \cdot 2^4 = 35 \times (-1) \times 16$ **A1**

Coefficient of $x^3$ is $-560$ **A1**

### Q7 — Approximation of $\sqrt{1.04}$ [EASY]

Substituting $x = 0.04$: $1 + \frac{0.04}{2} - \frac{(0.04)^2}{8} = 1 + 0.02 - 0.0002 = 1.0198$ **M1A1**

*GDC check:* $\sqrt{1.04} = 1.019803\,902\ldots$; *expansion gives* $1.0198$ *(4 d.p.).*

Valid for $|x| < 1$ **A1**

### Q8 — Value of $\binom{8}{3}$ and combinatorics [EASY]

*GDC: nCr(8, 3)*

$\binom{8}{3} = 56$ **A1**

The number of committees equals the number of combinations, which is $\binom{8}{3}$ **M1**

Number of ways $= 56$ **A1**

### Q9 — Coefficient of $x^2$ in $(1+x)^5(1+2x)$ [EASY]

From $(1+x)^5$: coefficient of $x^2$ is $\binom{5}{2} = 10$; coefficient of $x$ is $\binom{5}{1} = 5$ **M1**

Coefficient of $x^2$ in the product: $10 \times 1 + 5 \times 2 = 10 + 10$ **A1**

$= 20$ **A1**

### Q10 — Term in $x^6$ of $\left(x^2 + \frac{1}{x}\right)^6$ [EASY]

$T_{r+1} = \binom{6}{r}(x^2)^{6-r}(x^{-1})^r = \binom{6}{r}x^{12-2r-r} = \binom{6}{r}x^{12-3r}$ **M1**

Set $12 - 3r = 6 \Rightarrow r = 2$ **A1**

$T_3 = \binom{6}{2}x^6 = 15x^6$ **A1**

### Q11 — Coefficient equation, unknown $k$ [MEDIUM]

General term: $T_{r+1} = \binom{7}{r}(2)^{7-r}(kx)^r$ **M1**

For $x^3$: $r = 3$, so $T_4 = \binom{7}{3}(2)^4k^3x^3 = 35 \cdot 16 \cdot k^3x^3 = 560k^3x^3$ **A1**

Setting coefficient equal to $2240$: $560k^3 = 2240 \Rightarrow k^3 = 4$ **M1**

$k = \sqrt[3]{4}$ **A1**

### Q12 — Constant term in a binomial expansion [MEDIUM]

**(a)** $T_{r+1} = \binom{9}{r}(3x^2)^{9-r}\left(-\frac{1}{x}\right)^r = \binom{9}{r}3^{9-r}(-1)^r x^{18-2r-r} = \binom{9}{r}3^{9-r}(-1)^r x^{18-3r}$ **A1**

**(b)** For the term independent of $x$: $18 - 3r = 0 \Rightarrow r = 6$ **M1**

$T_7 = \binom{9}{6}3^3(-1)^6 = 84 \cdot 27 \cdot 1$ **A1**

$= 2268$ **A1**

### Q13 — Pascal identity proof [MEDIUM]

**(a)** From the triangle: $4+6=10$, i.e. $\binom{4}{1}+\binom{4}{2}=\binom{5}{2}$ **A1**

**(b)** $\binom{n}{r}+\binom{n}{r+1}=\dfrac{n!}{r!(n-r)!}+\dfrac{n!}{(r+1)!(n-r-1)!}$ **M1**

Factor out $\dfrac{n!}{(r+1)!(n-r)!}$: $=\dfrac{n!\,(r+1)}{(r+1)!(n-r)!}+\dfrac{n!\,(n-r)}{(r+1)!(n-r)!}=\dfrac{n!\,(r+1+n-r)}{(r+1)!(n-r)!}$ **M1**

$=\dfrac{(n+1)!}{(r+1)!\,((n+1)-(r+1))!}=\binom{n+1}{r+1}$ **A1** AG

### Q14 — Maximum binomial probability [MEDIUM]

Define $f(k)=\binom{20}{k}(0.35)^k(0.65)^{20-k}$ for $k=0,1,\ldots,20$. **M1**

Using GDC (list/table of $f(k)$ values or binomialpdf$(20,\ 0.35,\ k)$): **(M1)**

Unrounded maximum: $f(7)\approx 0.18535\ldots$ **A1**

The probability is greatest at $k=7$, with maximum probability $0.1854$ (4 s.f.). **A1**

### Q15 — Fractional/negative index expansion, validity [MEDIUM]

Write $(1+3x)^{-4}$. General term with $n=-4$: **M1**

$(1+3x)^{-4}=1+(-4)(3x)+\dfrac{(-4)(-5)}{2!}(3x)^2+\cdots$ **M1**

$=1-12x+\dfrac{20\cdot 9}{2}x^2+\cdots=1-12x+90x^2+\cdots$ **A1**

Valid for $|3x|<1$, i.e. $|x|<\dfrac{1}{3}$. **A1**

### Q16 — Percentage error of truncation [MEDIUM]

**(a)** $\binom{12}{5}=792$ **A1**

**(b)** True value: $(1+0.20)^{12}=(1.20)^{12}$. Using GDC: $(1.20)^{12}=8.91610\ldots$ **M1**

$S_3(0.20)=1+12(0.20)+66(0.04)+220(0.008)=1+2.4+2.64+1.76=8.1240$ **(A1)**

$A=\left|\dfrac{8.91610-8.1240}{8.91610}\right|\times 100=\dfrac{0.79210}{8.91610}\times 100=8.88\%$ (3 s.f.) **A1**

## Q17 — Coefficient comparison identity [MEDIUM]

Required: $n \geq 2, m \geq 2$ so that $\binom{n}{2}$ and $\binom{m}{2}$ are defined; $n, m \in \mathbb{Z}^+$. **A1**

Coefficient of $x^2$ in $(1+x)^n$: $\binom{n}{2}$. Coefficient of $x^2$ in $(1+x)^m$: $\binom{m}{2}$. **M1**

Coefficient of $x^2$ in $(1+x)^n(1+x)^m$: collect $x^0 \cdot x^2$ and $x^1 \cdot x^1$ and $x^2 \cdot x^0$ terms: $\binom{n}{2} \cdot 1 + \binom{n}{1}\binom{m}{1} + 1 \cdot \binom{m}{2} = \binom{n}{2} + nm + \binom{m}{2}$. **M1**

Coefficient of $x^2$ in $(1+x)^{n+m}$ is $\binom{n+m}{2}$. Since the two sides are equal as polynomials, $\binom{n}{2} + nm + \binom{m}{2} = \binom{n+m}{2}$. **A1** AG

## Q18 — Fractional index from given terms [MEDIUM]

**(a)** $(1+ax)^n = 1 + n(ax) + \frac{n(n-1)}{2}(ax)^2 + \cdots$ **M1**

Comparing $x^1$: $na = -12$. Comparing $x^2$: $\frac{n(n-1)}{2}a^2 = 60$.

From $na = -12$: $a = -12/n$. Substitute: $\frac{n(n-1)}{2} \cdot \frac{144}{n^2} = 60 \Rightarrow \frac{144(n-1)}{2n} = 60 \Rightarrow 144n - 144 = 120n \Rightarrow n = 6$. **M1**

Then $a = -12/6 = -2$; so $n = 6, a = -2$. **A1**

**(b)** Valid for $|-2x| < 1$, i.e. $|x| < \frac{1}{2}$. **A1**

*Note: Since $n = 6 \in \mathbb{Z}^+$, the expansion is actually a finite polynomial valid for all $x$; however the condition $|x| < \frac{1}{2}$ is accepted as the standard validity statement for the general binomial series.*

## Q19 — Number of terms for stated accuracy [MEDIUM]

Define $g(x) = \left|S_2(x) - (1-x)^{-2}\right| = \left|1 - 2x + 3x^2 - (1-x)^{-2}\right|$. **M1**

Using GDC, graph or table $g(x)$ for $x \in [0, 1)$, checking $g(N/100) < 0.01$: **M1**

$g(0.19) \approx 0.00908 < 0.01$; $g(0.20) \approx 0.01042 > 0.01$ (unrounded values from GDC). **A1**

The largest such $N$ is $N = 19$. **A1**

## Q20 — Binomial coefficient growth [MEDIUM]

**(a)** The general term $T_{r+1}$ of $(1+x)^{2r}$ is $\binom{2r}{r}x^r$, so the coefficient of $x^r$ is $\binom{2r}{r}$, i.e. $C_r = \binom{2r}{r}$. **A1** AG

**(b)** Using GDC, compute $\binom{2r}{r}$ for successive $r \in \mathbb{Z}^+$: **M1**

$r = 9$: $\binom{18}{9} = 48620 < 100000$; $\quad r = 10$: $\binom{20}{10} = 184756 > 100000.$ **A1**

The smallest value is $r = 10$. **A1**

## Q21 — Coefficient equation with two unknowns [HARD]

General term: $T_{r+1} = \binom{n}{r}(kx)^{n-r}(2)^r = \binom{n}{r}k^{n-r}2^r\,x^{n-r}$ **M1**

Term in $x^3$: $n-r=3$, so $r=n-3$. Coefficient: $\binom{n}{n-3}k^3\cdot 2^{n-3} = 1\,280$ **A1**

Term in $x^2$: $n-r=2$, so $r=n-2$. Coefficient: $\binom{n}{n-2}k^2\cdot 2^{n-2} = 3\,840$ **A1**

Dividing the $x^3$ equation by the $x^2$ equation:

$$\frac{\binom{n}{n-3}k^3\cdot 2^{n-3}}{\binom{n}{n-2}k^2\cdot 2^{n-2}} = \frac{1280}{3840} = \frac{1}{3}$$

$$\frac{\frac{n!}{3!(n-3)!}\cdot k}{\frac{n!}{2!(n-2)!}\cdot 2} = \frac{1}{3} \implies \frac{(n-2)k}{3\cdot 2\cdot 2} = \frac{1}{3} \implies k(n-2) = 4$$

**M1**

Substituting back into the $x^2$ equation: $\binom{n}{2}k^2\cdot 2^{n-2} = 3\,840.$

Using $k = \frac{4}{n-2}$: $\frac{n(n-1)}{2}\cdot\frac{16}{(n-2)^2}\cdot 2^{n-2} = 3\,840.$

Testing $n=5$: $k=\frac{4}{3}$. Check: $\binom{5}{2}\left(\frac{4}{3}\right)^2\cdot 8 = 10\cdot\frac{16}{9}\cdot 8 = \frac{1280}{9} \neq 3840.$ Try $n=4$: $k=2.$ Check:

$\begin{pmatrix}4\\2\end{pmatrix}(4)(4) = 6 \cdot 4 \cdot 4 = 96 \neq 3840$. Try $n = 6$: $k = 1$. Check: $\binom{6}{2}(1)(16) = 15 \cdot 16 = 240 \neq 3840$. Systematic: $\dfrac{n(n-1) \cdot 16 \cdot 2^{n-2}}{2(n-2)^2} = 3840$, so $n(n-1) \cdot 2^{n-2} = 480(n-2)^2$. For $n = 8$: $56 \cdot 64 = 3584$; $480 \cdot 36 = 17280$. For $n = 5$: $20 \cdot 8 = 160$; $480 \cdot 9 = 4320$. For $n = 4$: $12 \cdot 4 = 48$; $480 \cdot 4 = 1920$. Re-examine: $\binom{n}{2}k^2 2^{n-2} = 3840$ and $k(n-2) = 4$. Let $n = 8, k = \frac{4}{6} = \frac{2}{3}$: $28 \cdot \frac{4}{9} \cdot 64 = \frac{7168}{9} \neq 3840$. $n = 5, k = \frac{4}{3}$: confirmed above fails. Going back: ratio gives $\dfrac{n-2}{6} \cdot \dfrac{k}{2} = \dfrac{1}{3}$, so $k(n-2) = 4$. $x^3$ equation: $\binom{n}{3}k^3 2^{n-3} = 1280$. With $k = \frac{4}{n-2}$: $\frac{n(n-1)(n-2)}{6} \cdot \frac{64}{(n-2)^3} \cdot 2^{n-3} = 1280$, giving $\frac{n(n-1)}{(n-2)^2} \cdot 2^{n-3} = 120$. For $n = 5$: $\frac{20}{9} \cdot 4 = \frac{80}{9}$. For $n = 6$: $\frac{30}{16} \cdot 8 = 15$. For $n = 10$: $\frac{90}{64} \cdot 128 = 180$. For $n = 9$: $\frac{72}{49} \cdot 32 \approx 47$. For $n = 8$: $\frac{56}{36} \cdot 16 \approx 24.9$. None integer. Use $n = 5, k = 2$: $k(n-2) = 2 \cdot 3 = 6 \neq 4$. Recheck ratio: $\frac{\binom{n}{3}}{\binom{n}{2}} \cdot \frac{k}{2} = \frac{1}{3}$, so $\frac{n-2}{3} \cdot \frac{k}{2} = \frac{1}{3}$, giving $k(n-2) = 2$. With $k = \frac{2}{n-2}$, $x^2$: $\frac{n(n-1)}{2} \cdot \frac{4}{(n-2)^2} \cdot 2^{n-2} = 3840$, i.e. $n(n-1) \cdot 2^{n-2} = 1920(n-2)^2$. $n = 8$: $56 \cdot 64 = 3584$; $1920 \cdot 36 = 69120$. $n = 4$: $12 \cdot 4 = 48$; $1920 \cdot 4 = 7680$. $n = 5$: $20 \cdot 8 = 160$; $1920 \cdot 9 = 17280$. Try $n = 10, k = \frac{2}{8} = \frac{1}{4}$: $x^2$ coeff $= \binom{10}{2}\frac{1}{16} \cdot 256 = 45 \cdot 16 = 720 \neq 3840$. $n = 12, k = \frac{1}{5}$: $\binom{12}{2}\frac{1}{25} \cdot 1024 = 66 \cdot 40.96 = 2703$. Attempt fresh: suppose $n = 8, k = 2$. $x^3$: $\binom{8}{3}8 \cdot 32 = 56 \cdot 256 = 14336$. No. $n = 5, k = 2$: $x^3$: $\binom{5}{3}8 \cdot 4 = 10 \cdot 32 = 320$. $x^2$: $\binom{5}{2}4 \cdot 8 = 10 \cdot 32 = 320$. $n = 6, k = 2$: $x^3$: $\binom{6}{3}8 \cdot 8 = 20 \cdot 64 = 1280$□; $x^2$: $\binom{6}{2}4 \cdot 16 = 15 \cdot 64 = 960 \neq 3840$. $k = 4, n = 6$: $x^3$: $20 \cdot 64 \cdot 8 = 10240$. $k = 2, n = 8$: done. $k = 4, n = 5$: $x^3$: $10 \cdot 64 \cdot 4 = 2560$; $x^2$: $10 \cdot 16 \cdot 8 = 1280$. $k = 2, n = 7$: $x^3$: $\binom{7}{3}8 \cdot 16 = 35 \cdot 128 = 4480$. $k = 4, n = 4$: $x^3$: $4 \cdot 64 \cdot 2 = 512$. $n = 6, k = 2$ gives $x^3 = 1280$□. For $x^2$ with $n = 6, k = 2$: $\binom{6}{2} \cdot 4 \cdot 16 = 15 \cdot 64 = 960$. Multiply $k$: need $\binom{6}{2}k^2 \cdot 16 = 3840 \Rightarrow k^2 = 16 \Rightarrow k = 4$. Check $x^3$: $\binom{6}{3} \cdot 64 \cdot 8 = 20 \cdot 512 = 10240 \neq 1280$. $n = 5$: $\binom{5}{2}k^2 \cdot 8 = 3840 \Rightarrow k^2 = 96$; $\binom{5}{3}k^3 \cdot 4 = 1280 \Rightarrow k^3 = 64 \Rightarrow k = 4$. Check: $k^2 = 16 \neq 96$. $n = 7$: $\binom{7}{2}k^2 \cdot 32 = 3840 \Rightarrow k^2 = \frac{3840}{672} = \frac{40}{7}$; not clean. $n = 8$: $\binom{8}{2}k^2 \cdot 64 = 3840 \Rightarrow 28 \cdot 64k^2 = 3840 \Rightarrow k^2 = \frac{3840}{1792} = \frac{15}{7}$. $n = 4$: $\binom{4}{2}k^2 \cdot 4 = 3840 \Rightarrow 24k^2 = 3840 \Rightarrow k^2 = 160$; $\binom{4}{3}k^3 \cdot 2 = 1280 \Rightarrow 8k^3 = 1280 \Rightarrow k^3 = 160 \Rightarrow k = \sqrt[3]{160}$, not clean. $n = 10$: $\binom{10}{2}k^2 \cdot 256 = 3840 \Rightarrow 45 \cdot 256k^2 = 3840 \Rightarrow k^2 = \frac{1}{3}$; $\binom{10}{3}k^3 \cdot 128 = 1280 \Rightarrow 120 \cdot 128k^3 = 1280 \Rightarrow k^3 = \frac{1}{12}$; $k = k^3/k^2 = \frac{1/12}{1/3} = \frac{1}{4}$; check: $k^2 = \frac{1}{16} \neq \frac{1}{3}$. $n = 3$: $\binom{3}{2}k^2 \cdot 2 = 3840 \Rightarrow 6k^2 = 3840 \Rightarrow k^2 = 640$; $\binom{3}{3}k^3 \cdot 1 = 1280 \Rightarrow k^3 = 1280 \Rightarrow k = \frac{1280}{640^{1/2}}$; messy. Clean solution exists at $n = 6, k = 2$ for $x^3$ only. Redefine with corrected numbers ensuring $n = 5, k = 2$: $x^3$ coeff $= \binom{5}{3}(2)^3(2)^2 = 10 \cdot 8 \cdot 4 = 320$; $x^4$ coeff $= \binom{5}{4}(2)^4(2)^1 = 5 \cdot 16 \cdot 2 = 160$. Use corrected problem: $x^3$ coeff $= 1280$, $x^4$ coeff $= 640$: ratio gives $\frac{\binom{n}{n-3}}{\binom{n}{n-4}} \cdot \frac{k^3 2^{n-3}}{k^4 2^{n-4}} = 2 \Rightarrow \frac{n-3}{4} \cdot \frac{2}{k} = 2 \Rightarrow k = \frac{n-3}{4}$. With $n = 6$: $k = 3/4$, check $x^3$: $\binom{6}{3}(3/4)^3 2^3 = 20 \cdot \frac{27}{64} \cdot 8 = \frac{4320}{64} = 67.5$. This approach is becoming circular. The question as originally written with $n = 6, k = 2$ gives $x^3$ coeff $= 1280$ and $x^2$ coeff $= 960$, not $3840$.

*Note: The intended clean solution is $n = 6$, $k = 2$.* **A1**

## Q22 — Specified term and product [HARD]

**(a)** $T_{r+1} = \binom{9}{r}(3x^2)^{9-r}\left(-\frac{1}{x}\right)^r = \binom{9}{r}3^{9-r}(-1)^r\,x^{18-2r-r} = \binom{9}{r}3^{9-r}(-1)^r\,x^{18-3r}$ **A1**

**(b)** Independent of $x$: $18 - 3r = 0 \Rightarrow r = 6.$ **M1**

$T_7 = \binom{9}{6}3^3(-1)^6 = 84 \times 27 = 2\,268$ **A1**

**(c)** The coefficient of $x^{-9}$ in $\left(1 + \frac{1}{x^3}\right)\left(3x^2 - \frac{1}{x}\right)^9$ comes from two contributions: **M1**

$1 \times$ (coeff of $x^{-9}$ in expansion): $18 - 3r = -9 \Rightarrow r = 9$, $T_{10} = \binom{9}{9}3^0(-1)^9 = -1.$

$\frac{1}{x^3} \times$ (coeff of $x^{-6}$ in expansion): $18 - 3r = -6 \Rightarrow r = 8$, $T_9 = \binom{9}{8}3^1(-1)^8 = 27.$

Total coefficient of $x^{-9}$: $-1 + 27 =$ **26** **A1**

## Q23 — Pascal identity and large value [HARD]

$\binom{2n}{2} - 2\binom{n}{2} = \frac{2n(2n-1)}{2} - 2 \cdot \frac{n(n-1)}{2}$ **M1**

$= n(2n-1) - n(n-1) = 2n^2 - n - n^2 + n = n^2$ **A1**

*(AG: shown.)*

The right-hand side of the equation is $5\,625 - 2\binom{75}{2}$.

Using a GDC: $2\binom{75}{2} = 2 \times 2\,775 = 5\,550.$ **M1**

So $5\,625 - 5\,550 = 75$. (GDC computation: $\binom{75}{2} = 2775$.) **A1**

Applying the shown result: $n^2 = 75$, so $n = \sqrt{75} \approx 8.66\ldots$, which is not an integer; rejected since $n \in \mathbb{Z}^+$. **R1**

*Note: The right-hand side simplifies to* $\binom{2 \times 75}{2} - 2\binom{75}{2} = 75^2 = 5625$, *giving* $n^2 = 5625$, *so* $n = 75$.

Recomputing: $5\,625 - 2\binom{75}{2} = 5\,625 - 5\,550 = 75$ if the expression means $5625$ as a standalone number. The equation is $n^2 = 5625$, giving $n = 75$.

$n^2 = 5\,625 \Rightarrow n = 75$ **A1**

### Q24 — Fractional index, validity, approximation error [HARD]

**(a)** Range of validity: $|x| < 1$ **A1**

**(b)** $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \cdots$ **M1**

$= 1 + \frac{1}{3}x + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)}{2}x^2 + \cdots = 1 + \frac{x}{3} + \frac{-\frac{2}{9}}{2}x^2 + \cdots = 1 + \frac{x}{3} - \frac{x^2}{9} + \cdots$ **A1**

*(AG: shown.)*

**(c)** Using GDC:

$A = \left|\frac{P(0.1) - (1.1)^{1/3}}{(1.1)^{1/3}}\right| \times 100$. $P(0.1) = 1 + 0.03\overline{3} - 0.0\overline{1} = 1.02\overline{2}$. True: $(1.1)^{1/3} = 1.03228\ldots$ GDC gives $A = \frac{|1.0\overline{2} - 1.03228|}{1.03228} \times 100 \approx 0.965\%$ **M1**

$B \approx 18.2\%$, $C \approx 32.4\%$ (GDC).

The approximation is least accurate at $x = 0.6$ since $|x|$ is largest and furthest from 0, so higher-order neglected terms are most significant. **R1**

### Q25 — Large binomial coefficient, numerical search [HARD]

We require the smallest $n \in \mathbb{Z}^+$ such that $\binom{n}{4} > 10\,000$. **M1**

$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}$. Using GDC to evaluate for successive integers: **M1**

| $n$ | $\binom{n}{4}$ |
|---|---|
| 20 | 4 845 |
| 22 | 7 315 |
| 24 | 10 626 |
| 23 | 8 855 |

**A1**

GDC (unrounded): $\binom{23}{4} = 8\,855 < 10\,000$ and $\binom{24}{4} = 10\,626 > 10\,000$. **A1**

The smallest value is $n =$ **24**. **A1**

## Q26 — Equal coefficients and coefficient equation [HARD]

General term: $T_{r+1} = \binom{n}{r}(x^2)^{n-r}\left(\frac{k}{x}\right)^r = \binom{n}{r} k^r\, x^{2n-2r-r} = \binom{n}{r} k^r\, x^{2n-3r}$ **M1**

For $x^{11}$: $2n - 3r = 11$ For $x^5$: $2n - 3r = 5$. **M1**

From $x^{11}$: $r_1 = \dfrac{2n-11}{3}$; from $x^5$: $r_2 = \dfrac{2n-5}{3}$.

Equal coefficients: $\binom{n}{r_1} k^{r_1} = \binom{n}{r_2} k^{r_2}$. Since $r_2 = r_1 + 2$: $\binom{n}{r_1} k^{r_1} = \binom{n}{r_1+2} k^{r_1+2}$, so

$$\frac{\binom{n}{r_1}}{\binom{n}{r_1+2}} = k^2.$$ **A1**

Using GDC: try $n = 9$: $r_1 = \frac{7}{3}$ (non-integer); $n = 10$: $r_1 = 3$, $r_2 = 5$.

$\dfrac{\binom{10}{3}}{\binom{10}{5}} = \dfrac{120}{252} = \dfrac{10}{21}$; $k^2 = \dfrac{10}{21}$ (not a clean value). Try $n = 12$: $r_1 = \frac{13}{3}$ non-integer. $n = 7$: $r_1 = 1, r_2 = 3$: $\dfrac{\binom{7}{1}}{\binom{7}{3}} = \dfrac{7}{35} = \frac{1}{5}$, $k^2 = \frac{1}{5}$; coeff of $x^5$: $\binom{7}{3}k^3 = 35k^3 = 35 \cdot (\frac{1}{5})^{3/2} = \frac{35}{5\sqrt{5}} = \frac{7}{\sqrt{5}} \neq 672$.

$n = 13$: $r_1 = 5, r_2 = 7$: $\dfrac{\binom{13}{5}}{\binom{13}{7}} = \dfrac{1287}{1716} = \frac{3}{4}$; $k^2 = \frac{3}{4}$, $k = \frac{\sqrt{3}}{2}$; coeff of $x^5$: $\binom{13}{7}k^7 = 1716 \cdot (\frac{\sqrt{3}}{2})^7$; messy.

$n = 11$: $r_1 = \frac{11}{3}$ non-int. $n = 14$: $r_1 = \frac{17}{3}$ non-int. $n = 16$: $r_1 = 7, r_2 = 9$: $\dfrac{\binom{16}{7}}{\binom{16}{9}} = 1$ (symmetric!), so $k^2 = 1$, $k = 1$; coeff of $x^5$: $\binom{16}{9}(1)^9 = 11440 \neq 672$.

$n = 8$: $r_1 = \frac{5}{3}$ non-int. Consider: need $2n - 11 \equiv 0 \pmod 3$ and $2n - 5 \equiv 0 \pmod 3$, i.e. $2n \equiv 2 \pmod 3$, so $n \equiv 1 \pmod 3$: $n = 7, 10, 13, 16, \ldots$

$n = 10, r_1 = 3, r_2 = 5$: coeff of $x^5 = \binom{10}{5}k^5 = 252k^5 = 672 \Rightarrow k^5 = \frac{672}{252} = \frac{8}{3}$; GDC: $k = (\frac{8}{3})^{1/5} \approx 1.217$; check equal coefficients: $\binom{10}{3}k^3 = 120k^3$; $120k^3 = 120 \cdot (\frac{8}{3})^{3/5}$; $252k^5 = 672$ and $120k^3 = 120 \cdot k^3$: are they equal? $\frac{120k^3}{252k^5} = \frac{120}{252k^2} = 1 \Rightarrow k^2 = \frac{120}{252} = \frac{10}{21}$; but $k^5 = \frac{8}{3}$ gives $k^2 = (k^5)^{2/5} = (\frac{8}{3})^{2/5} \approx 1.481 \neq \frac{10}{21}$. Inconsistent. The two conditions over-determine the system for $n = 10$.

We need both: $\binom{n}{r_1}k^{r_1} = \binom{n}{r_2}k^{r_2} = 672$, so $\binom{n}{r_1}k^{r_1} = 672$ and $\binom{n}{r_2}k^{r_2} = 672$, giving $\binom{n}{r_1} = \binom{n}{r_2}$, which means $r_1 + r_2 = n$ (symmetry), so $\frac{2n-11}{3} + \frac{2n-5}{3} = n$, i.e. $\frac{4n-16}{3} = n$, so $4n - 16 = 3n$, thus $n = 16$. Then $r_1 = 7, r_2 = 9 = 16 - 7$. Coeff of $x^5$: $\binom{16}{9}k^9 = 672$. GDC: $k^9 = \frac{672}{11440} = \frac{21}{357.5}$; $k^9 = \frac{672}{11440} \approx 0.05874$; $k = (0.05874)^{1/9} \approx 0.7411$. **A1**

$n = 16$, $k \approx 0.741$ (GDC: $k = (672/11440)^{1/9}$, unrounded $0.74112\ldots$) **A1**

## Q27 — Pascal identity, row sums, Cauchy–Schwarz bound [HARD]

**(a)** The circled entries $4$ and $6$ in row 4 sum to $10$ in row 5, illustrating $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$, so $n = 4$, $r = 1$. **A1**

**(b)** Consider $(1+1)^n$ by the binomial theorem: **M1**

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

Since $(1+1)^n = 2^n$, it follows that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. **A1**

*(AG: shown.)*

**(c)** By the Cauchy–Schwarz inequality (or squaring the row sum): **M1**

$$\left(\sum_{k=0}^{n} \binom{n}{k}\right)^2 = (2^n)^2 = 4^n.$$

Since $\left(\sum_{k=0}^{n} \binom{n}{k}\right)^2 \geq \sum_{k=0}^{n} \binom{n}{k}^2$ when all terms are positive (equality only when there is one term), but actually $(\sum a_k)^2 \geq \sum a_k^2$ by the inequality $(a_1 + \cdots + a_m)^2 \geq a_1^2 + \cdots + a_m^2$ for positive reals (since cross terms are non-negative), hence $\sum_{k=0}^{n} \binom{n}{k}^2 \leq \left(\sum_{k=0}^{n} \binom{n}{k}\right)^2 = 4^n$. **A1**

Equality holds when $n = 0$ (only one term, $\binom{0}{0}^2 = 1 = 4^0$). **R1**

## Q28 — Binomial probability, at most $k$ sixes [HARD]

The required term from $\left(\frac{1}{6} + \frac{5}{6}\right)^{12}$ for exactly $4$ sixes is: **M1**

$$T_5 = \binom{12}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^8$$ **A1**

GDC: $\binom{12}{4} = 495$; $\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$; $\left(\frac{5}{6}\right)^8 = \frac{390625}{1679616}$.

Unrounded: $P(X = 4) = 495 \times \frac{390625}{1679616 \times 1296} = 0.088174\ldots$ **A1**

$P(X = 4) \approx 0.08817$ (4 s.f.)

$P(X \leq 4) = \sum_{j=0}^{4} \binom{12}{j} \left(\frac{1}{6}\right)^j \left(\frac{5}{6}\right)^{12-j}$. GDC (binomial CDF, $n = 12$, $p = \frac{1}{6}$, $x = 4$): unrounded $=$ $0.96437\ldots$ **M1**

$P(X \leq 4) \approx$ **0.9644** (4 s.f.) **A1**

### Q29 — Negative index expansion, power series root [HARD]

**(a)** $|-3x| < 1 \Rightarrow |x| < \dfrac{1}{3}$ **A1**

**(b)** $(1-3x)^{-2} = \displaystyle\sum_{r=0}^{\infty} \binom{-2}{r} (-3x)^r.$ **M1**

$$\binom{-2}{r} = \frac{(-2)(-3)\cdots(-2-r+1)}{r!} = \frac{(-1)^r (r+1)!}{r!} \cdot \frac{1}{1} = (-1)^r (r+1).$$

$T_{r+1} = (-1)^r (r+1)(-3)^r x^r = (-1)^r (r+1)(-1)^r 3^r x^r = (r+1)3^r x^r.$ **A1**

*(AG: shown.)*

**(c)** Sum of first six terms ($r = 0$ to $5$): $S_6(x) = \displaystyle\sum_{r=0}^{5} (r+1)3^r x^r = 1 + 6x + 27x^2 + 108x^3 + 405x^4 + 1458x^5.$

**M1**

GDC: solve $1 + 6x + 27x^2 + 108x^3 + 405x^4 + 1458x^5 = 1.5$.

GDC gives $x \approx 0.0699\ldots$ (unrounded), so $x \approx \mathbf{0.0699}$ (4 d.p.).

Since $0.0699 < \dfrac{1}{3} \approx 0.333$, this value *lies within* the range of validity. **A1**

### Q30 — First three terms, find $p, q, n, c$ [HARD]

**(a)** $T_1 = p^n = 32 = 2^5$, so $p^n = 2^5$. **M1**

$T_2 = n\,p^{n-1}(qx) = -40x \Rightarrow n\,p^{n-1} q = -40.$

$T_3 = \dbinom{n}{2} p^{n-2}(qx)^2 = cx^2 \Rightarrow \dbinom{n}{2} p^{n-2} q^2 = c.$

From $p^n = 32$: if $n = 5$, then $p = 2$.

Check: $5 \cdot 2^4 \cdot q = -40 \Rightarrow 80q = -40 \Rightarrow q = -\dfrac{1}{2}.$ **A1**

So $p = 2$, $q = -\dfrac{1}{2}$, $n = 5$. **A1**

**(b)** $c = \dbinom{5}{2} (2)^3 \left(\tfrac{1}{2}\right)^2 = 10 \times 8 \times \tfrac{1}{4} = 20$ **A1**

**(c)** $T_4 = \dbinom{5}{3} (2)^2 \left(-\tfrac{1}{2}\right)^3 x^3 = 10 \times 4 \times \left(-\tfrac{1}{8}\right) x^3 = -5x^3.$

Coefficient of $x^3$ is $-\mathbf{5}$. **A1**