IB Math AA HL — Topic 1

Further Complex Numbers

Polar & Euler Form · De Moivre · Roots of Complex Numbers · Complex Roots of Polynomials

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Scan me for help

Name:____________________ Date:____________

Score:________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

- Polar/Euler form: $z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta = re^{i\theta}$, where $r = |z| \geq 0$ and $\theta = \arg(z) \in (-\pi, \pi]$.
- De Moivre's Theorem: $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$ for all $n \in \mathbb{Z}$ (and by extension $n \in \mathbb{Q}$).
- Multiplication & division: $z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$; $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$.
- n-th roots of $z = re^{i\theta}$: $z_k = r^{1/n}\,\text{cis}\left(\dfrac{\theta + 2\pi k}{n}\right)$, $k = 0, 1, \ldots, n-1$; arguments spaced $\dfrac{2\pi}{n}$ apart.
- Real-coefficient polynomials: complex roots occur in conjugate pairs; if $a + bi$ is a root then $a - bi$ is also a root.
- Multiple-angle identities: expand $(\cos\theta + i\sin\theta)^n$ by the binomial theorem, then equate real and imaginary parts to obtain $\cos(n\theta)$ and $\sin(n\theta)$.
- Loci: $|z - a| = k$ is a circle, centre a, radius k; $\arg(z - a) = \alpha$ is a ray from a; $|z - a| = |z - b|$ is the perpendicular bisector of ab.

GDC Tips

- Rectangular $\leftrightarrow$ polar conversion: on the TI-Nspire use Menu $>$ Number $>$ Convert to Polar; on the Casio ClassPad tap Action $>$ Complex $>$ arg and abs to extract θ and r separately.
- High powers as decimals: enter z^n directly in the CAS; confirm the imaginary part is zero (or real part is zero) to verify the answer before rounding to 3 s.f.
- Polynomial roots: use polyRoots(on the TI-Nspire or solve(in complex mode on the ClassPad to find all roots; identify conjugate pairs and verify their product equals the constant term divided by the leading coefficient.
- Smallest n by numerical search: store z and evaluate Im(z^n) or Re(z^n) in a spreadsheet column for $n = 1, 2, \dots$ until the required condition is met to sufficient accuracy.
- Roots-of-unity polygon area / side length: plot the n roots as coordinates $(r\cos\theta_k, r\sin\theta_k)$; use the distance formula between adjacent vertices or the regular-polygon area formula $\frac{1}{2}nr^2\sin(2\pi/n)$, evaluated numerically.
- Principal argument check: after computing $\theta = \arctan(y/x)$ on the GDC, always verify the quadrant using the signs of Re(z) and Im(z); adjust by $\pm\pi$ if needed to place $\theta \in (-\pi, \pi]$.

Common Mistakes to Avoid

1. **Wrong principal argument quadrant.** Students apply $\theta = \arctan(y/x)$ without checking the signs of x and y, placing the argument in the wrong quadrant; always draw a quick sketch and adjust by $\pm\pi$.
2. **Missing roots when solving $z^n = w$.** Only one or two roots are found instead of all n; every n-th root question requires all n values with arguments $\theta_0 + \dfrac{2\pi k}{n}$ for $k = 0, 1, \dots, n-1$.
3. **Incomplete De Moivre induction.** Students omit one of the four required stages (base case, inductive hypothesis stated, inductive step shown, conclusion named); all four must appear for full marks.
4. **Forgetting the conjugate root.** When a polynomial with real coefficients has complex root $a + bi$, the conjugate $a - bi$ must also be stated as a root and used to build the quadratic factor $(z-a)^2 + b^2$.
5. **Applying De Moivre to sums instead of products.** Writing $(z_1 + z_2)^n = z_1^n + z_2^n$; De Moivre applies only to a single complex number raised to a power, not to a sum.
6. **Confusing modulus and argument after multiplication.** Students add moduli or multiply arguments instead of multiplying moduli and adding arguments; remember $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

Scan me for help

1. EASY No Calculator **[3 marks]**

Let $z = -3 + 3i$, where $i^2 = -1$. Write down the modulus and principal argument of z, giving the argument in radians.

2. EASY No Calculator **[3 marks]**

Let $z = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. Write z in the form $a + bi$, where $a, b \in \mathbb{R}$, giving exact values.

3. EASY Calculator **[3 marks]**

Let $z = 4e^{i\pi/3}$. Write z in the form $re^{i\theta}$ and hence state its Cartesian form $a + bi$, giving exact values.

4. EASY Calculator **[3 marks]**

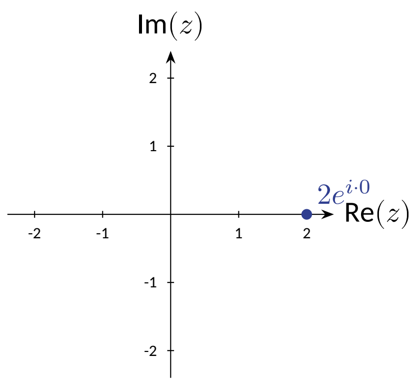
The complex number z has modulus 2 and argument $\frac{\pi}{4}$. Find z^6, giving your answer in the form $a + bi$ where $a, b \in \mathbb{R}$.

5. EASY Calculator **[3 marks]**

The polynomial $p(x) = x^3 - 4x^2 + 6x - 4$, where $x \in \mathbb{C}$, has a root at $z = 1 + i$. Write down all three roots of $p(x)$.

6. EASY No Calculator **[3 marks]**

Find all cube roots of 8, giving your answers in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$.
The diagram shows the three cube roots of 8 plotted on an Argand diagram.



7. EASY Calculator **[3 marks]**

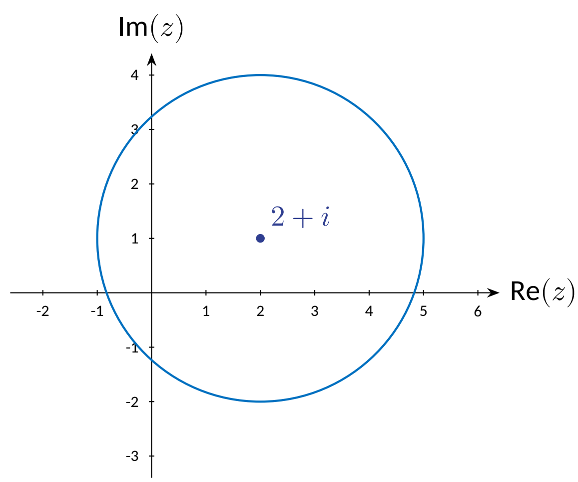
Let $z_1 = 3 + 4i$ and $z_2 = 1 + 2i$, where $i^2 = -1$. Find $|z_1 z_2|$ and $\arg(z_1 z_2)$, giving the argument in radians to three significant figures.

8. EASY Calculator **[3 marks]**

The complex number $w = \sqrt{3} - i$, where $i^2 = -1$. Find the modulus and principal argument of w^4, giving the argument in radians to three significant figures.

9. EASY Calculator **[3 marks]**

The locus of points $z \in \mathbb{C}$ satisfying $|z - (2 + i)| = 3$ is shown on the Argand diagram below. Write down the centre and radius of this locus.



10. EASY Calculator **[3 marks]**

A complex number z satisfies $z^2 = -9$. Find both values of z, where $z \in \mathbb{C}$.

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

Scan me for help

11. MEDIUM No Calculator **[4 marks]**

Let $z = -3 + 3i$, where $i^2 = -1$.

(a) Write z in the form $r \operatorname{cis} \theta$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

(b) Hence write z in the form $re^{i\theta}$. [1]

(c) State the modulus of z^2. [1]

12. MEDIUM Calculator **[4 marks]**

Let $w = 1 - \sqrt{3}i$, where $i^2 = -1$. Find the smallest positive integer n such that w^n is a positive real number, showing your reasoning.

13. MEDIUM Calculator **[4 marks]**

The complex number $z \in \mathbb{C}$ satisfies $z^2 = 5 + 12i$, where $i^2 = -1$. Find the two values of z, giving your answers in the form $a + bi$ where $a, b \in \mathbb{R}$.

14. MEDIUM Calculator **[4 marks]**

A polynomial $p(x) = x^3 - 6x^2 + 25x - 68$, where $x \in \mathbb{C}$. Given that $z_1 = 1 + 4i$ is a root of $p(x)$, find all three roots, justifying any complex roots obtained.

15. MEDIUM No Calculator **[4 marks]**

Use De Moivre's theorem to show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

16. MEDIUM Calculator **[4 marks]**

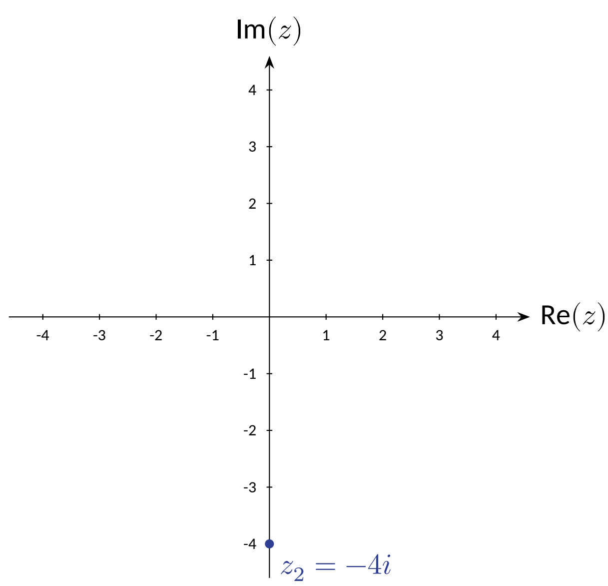
Let $z = 2e^{i\pi/6}$ and $w = \sqrt{2}\, e^{i\pi/4}$, where $i^2 = -1$.

(a) Find $\left|\dfrac{z^3}{w^2}\right|$. [2]

(b) Find $\arg\left(\dfrac{z^3}{w^2}\right)$, giving your answer in the form $k\pi$ where $k \in \mathbb{Q}$ and $-1 < k \leq 1$. [2]

17. MEDIUM Calculator **[4 marks]**

The diagram shows the three cube roots of $w = 64i$ plotted on an Argand diagram.



Find the other two cube roots of $64i$, giving exact values in the form $a + bi$ where $a, b \in \mathbb{R}$.

18. MEDIUM Calculator **[4 marks]**

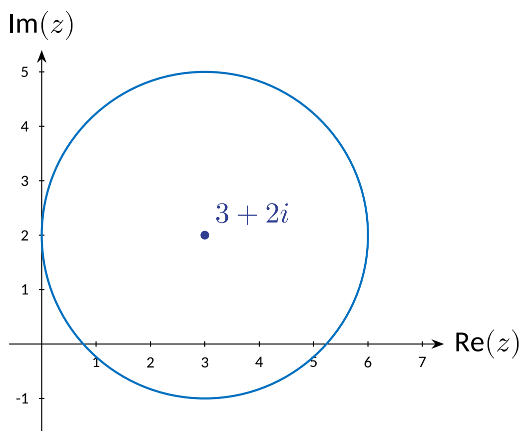
The complex number $z \in \mathbb{C}$ satisfies $|z-8| = |z-4i|$.

(a) Show that the locus of z can be written as $y = 2x - 6$, where $z = x + iy$ and $x, y \in \mathbb{R}$. [3]

(b) State the y-intercept of this locus. [1]

19. MEDIUM Calculator **[4 marks]**

The complex number $z \in \mathbb{C}$ satisfies $|z-(3+2i)| = 3$. The diagram shows the locus of z on an Argand diagram.



(a) Write down the centre and radius of this circle. [1]

(b) Find the two values of z with the greatest and least modulus, giving answers in the form $a + bi$. [3]

20. MEDIUM No Calculator **[4 marks]**

Use mathematical induction to prove De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all positive integers n, where $\theta \in \mathbb{R}$ and $i^2 = -1$.

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

Scan me for help

21. HARD No Calculator **[5 marks]**

Let $z = 1 + i\sqrt{3}$, where $i^2 = -1$.

(a) Express z in the form $re^{i\theta}$, where $r \in \mathbb{R}^+$ and $\theta \in (-\pi, \pi]$. [2]

(b) Using De Moivre's theorem, show that $z^6 = 64$. [2]

(c) Hence state the value of $(1 + i\sqrt{3})^6 + \overline{(1 + i\sqrt{3})^6}$. [1]

22. HARD No Calculator **[5 marks]**

Use mathematical induction to prove that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all $n \in \mathbb{Z}^+$.

23. HARD Calculator **[5 marks]**

The complex number $w = 3 + 4i$, where $i^2 = -1$.

(a) Write down $|w|$ and arg(w), giving arg(w) in radians to three significant figures. [2]

(b) The complex number u satisfies $|u| = 5$ and arg$(u) =$ arg$(w) + \frac{\pi}{4}$. Express u in the form $a + bi$, where $a, b \in \mathbb{R}$, giving your values to three significant figures. [3]

24. HARD Calculator **[5 marks]**

Let $p(z) = z^4 - 8z^3 + 27z^2 - 50z + 50$, where $z \in \mathbb{C}$ and all coefficients are real. It is known that $z_1 = 1 + 2i$ is a root of $p(z)$.

(a) Write down another root z_2 of $p(z)$, justifying your answer. [1]

(b) Show that $p(z) = (z^2 - 2z + 5)(z^2 - az + b)$ for integers a and b to be determined. [2]

(c) Hence find all four roots of $p(z)$. [2]

25. HARD No Calculator **[5 marks]**

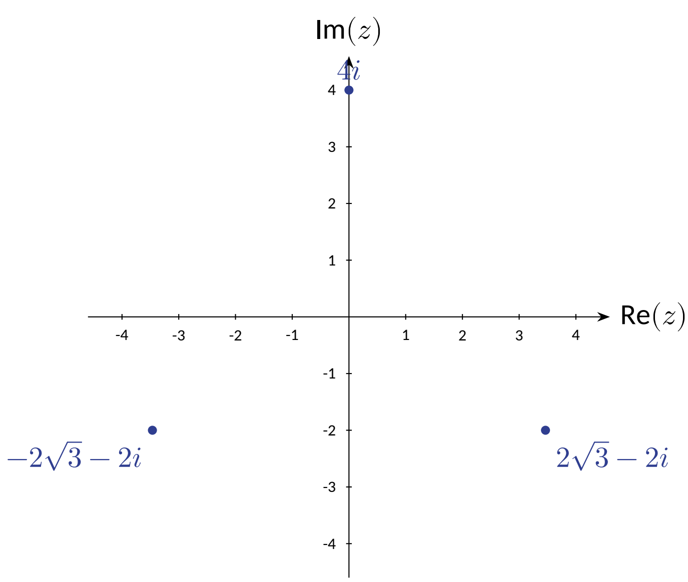
Use De Moivre's theorem to show that

$$\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Hence, given that $\cos\left(\frac{\pi}{8}\right) > 0$ and that $8x^4 - 8x^2 + 1 = 0$ has a solution $x = \cos\left(\frac{\pi}{8}\right)$, find the exact value of $\cos\left(\frac{\pi}{8}\right)$.

26. HARD Calculator **[5 marks]**

Find all cube roots of $-64i$, where $i^2 = -1$. Express each root in the form $a + bi$, where $a, b \in \mathbb{R}$, giving exact values. The diagram shows the three cube roots plotted on an Argand diagram.



27. HARD Calculator **[5 marks]**

The locus of the complex number $z \in \mathbb{C}$ satisfies both

$$|z-(3+2i)| = |z-(7+2i)|$$

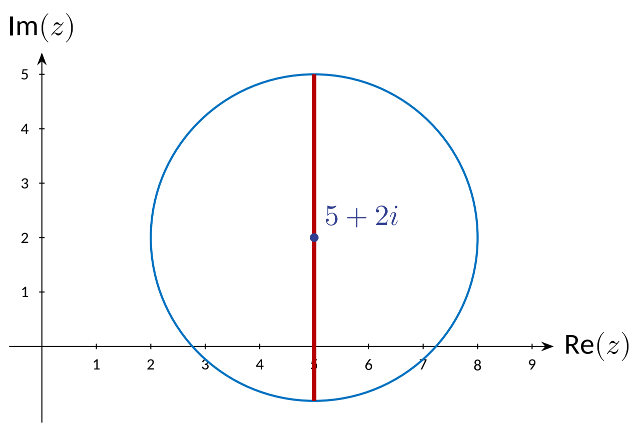
and

$$|z-(5+2i)| \leq 3.$$

(a) Describe geometrically the locus defined by $|z-(3+2i)| = |z-(7+2i)|$. [1]

(b) On the Argand diagram below, draw both loci and shade the region satisfying both conditions simultaneously. [2]

(c) Find the exact length of the chord along which the two loci intersect, giving your answer in the form $k\sqrt{m}$, $k, m \in \mathbb{Z}^+$. [2]



28. HARD Calculator **[5 marks]**

Let $\omega = e^{2\pi i/7}$, where $i^2 = -1$.

(a) State the value of ω^7. [1]

(b) Show that $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = 0$. [2]

(c) Hence find the numerical value of $\sum_{k=1}^{6} \cos\left(\frac{2\pi k}{7}\right)$, giving a clear justification for your answer. [2]

29. HARD Calculator **[5 marks]**

The complex number z satisfies $z^2 - (6 + 2i)z + (13 + 18i) = 0$, where $i^2 = -1$ and $z \in \mathbb{C}$.

(a) Using the quadratic formula, show that the discriminant of the equation is $\Delta = -20 - 48i$. [1]

(b) Find $\sqrt{\Delta}$ in the form $a + bi$, $a, b \in \mathbb{R}$, showing your method clearly. [2]

(c) Hence find both roots of the equation. [2]

30. HARD Calculator **[5 marks]**

A transformation T maps the complex number z to $w = (1 + i)z^3$, where $i^2 = -1$. Let $z = re^{i\theta}$ where $r \in \mathbb{R}^+$ and $\theta \in (-\pi, \pi]$.

(a) Express w in the form $Re^{i\phi}$, giving R and ϕ in terms of r and θ. [2]

(b) Given that $|z| = 2$ and $\arg(z) = \dfrac{\pi}{9}$, find $|w|$ and $\arg(w)$, giving $\arg(w)$ in radians to three significant figures. [2]

(c) State the geometric effect of T on z in terms of a scaling factor and a rotation angle. [1]

Mark Schemes

Q1 — Modulus and Argument of a Complex Number EASY

$|z| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$ **A1**

Reference angle: arctan $\left(\frac{3}{3}\right) = \frac{\pi}{4}$; z is in the second quadrant, **M1**

so arg$(z) = \pi - \frac{\pi}{4} = 3\pi/4$ **A1**

Q2 — Polar to Cartesian (Exact) EASY

$a = 5\cos\frac{\pi}{6} = 5 \cdot \frac{\sqrt{3}}{2} = 5\sqrt{3}/2$ **M1**

$b = 5\sin\frac{\pi}{6} = 5 \cdot \frac{1}{2} = 5/2$ **A1**

$z = 5\sqrt{3}/2 + 5/2\,i$ **A1**

Q3 — Euler Form to Cartesian EASY

$z = 4e^{i\pi/3}$ so $r = 4$, $\theta = \frac{\pi}{3}$ **A1**

$a = 4\cos\frac{\pi}{3} = 4 \times 0.5 = 2, \quad b = 4\sin\frac{\pi}{3} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$ **M1**

$z = 2 + 2\sqrt{3}i$ **A1**

Q4 — Power by De Moivre EASY

By De Moivre's theorem: $z^6 = 2^6\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = 64\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$ **M1**

$\cos\frac{3\pi}{2} = 0$, $\sin\frac{3\pi}{2} = -1$ **A1**

$z^6 = 0 - 64i = -64i$ **A1**

Q5 — Complex Roots of a Polynomial EASY

Since $p(x)$ has real coefficients and $z = 1 + i$ is a root, $z = 1 - i$ is also a root. **M1A1**

Note: Award M1 for stating the conjugate root theorem; A1 for writing $1 - i$.

The three roots are $z = 1 + i, 1 - i, 2$. **A1**

Note: Award final A1 for correctly identifying the real root $z = 2$.

Q6 — Cube Roots of 8 in Euler Form EASY

Write $8 = 8e^{i \cdot 0}$. The three cube roots have modulus $8^{1/3} = 2$ and arguments $\dfrac{0 + 2k\pi}{3}$ for $k = 0, 1, 2$. **M1**

$k = 0$: $2e^{i0}$; $k = 1$: $2e^{i2\pi/3}$; $k = 2$ gives argument $\dfrac{4\pi}{3}$, adjusted to $-\dfrac{2\pi}{3}$: $2e^{-i2\pi/3}$ **A1**

All three roots: $2e^{i0}$, $2e^{i2\pi/3}$, $2e^{-i2\pi/3}$ **A1**

Q7 — Modulus and Argument of a Product EASY

$|z_1| = \sqrt{3^2 + 4^2} = 5$, $|z_2| = \sqrt{1^2 + 2^2} = \sqrt{5}$, so $|z_1 z_2| = 5\sqrt{5} \approx 11.180\ldots$ **M1**

$|z_1 z_2| = 11.2$ (3 s.f.) **A1**

$\arg(z_1 z_2) = \arctan(4/3) + \arctan(1/2) = 0.9273\ldots + 1.1071\ldots = 2.0344\ldots \approx 2.03$ (3 s.f.) **A1**

Q8 — Power via De Moivre EASY

$|w| = \sqrt{3 + 1} = 2$, $\arg(w) = -\dfrac{\pi}{6}$ (fourth quadrant). By De Moivre: $|w^4| = 2^4 = 16$ **M1**

$\arg(w^4) = 4 \times \left(-\dfrac{\pi}{6}\right) = -\dfrac{2\pi}{3} \approx -2.094\ldots \approx -2.09$ (3 s.f.) **A1**

Modulus $= 16$, argument $= -2.09$ radians (3 s.f.) **A1**

Q9 — Locus Circle – Centre and Radius EASY

The equation $|z - (2 + i)| = 3$ represents a circle. **M1**

Centre $= 2 + i$ (i.e. the point $(2, 1)$ in the Argand diagram) **A1**

Radius $= 3$ **A1**

Q10 — Square Roots of -9 EASY

$z^2 = -9 \implies z^2 + 9 = 0$ **M1**

$(z - 3i)(z + 3i) = 0$ **A1**

$z = 3i$ or $z = -3i$ **A1**

Q11 — Polar and Euler Forms of a Complex Number MEDIUM

(a) $r = |z| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$ **A1**

θ: since z lies in the second quadrant, $\theta = \pi - \arctan\left(\frac{3}{3}\right) = \pi - \frac{\pi}{4} = 3\pi/4$ **A1**

$z = 3\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$

(b) Using the result from (a): $z = 3\sqrt{2}\,e^{i3\pi/4}$ **A1**

(c) $|z^2| = r^2 = (3\sqrt{2})^2 = 18$ **A1**

Q12 — Smallest n for a Real Power MEDIUM

Write w in polar form: $|w| = \sqrt{1+3} = 2$, arg$(w) = -\frac{\pi}{3}$ (fourth quadrant). **M1**

So $w = 2e^{-i\pi/3}$ and $w^n = 2^n e^{-in\pi/3}$.

For w^n to be a positive real number, arg$(w^n) = -\frac{n\pi}{3} \equiv 0 \pmod{2\pi}$, so n must be a multiple of 6. **M1**

Checking $n = 6$: arg$(w^6) = -2\pi \equiv 0$, and $2^6 = 64 > 0$. **A1**

Smallest positive integer is $n = 6$. **A1**

Q13 — Square Roots of a Complex Number MEDIUM

Let $z = a + bi$. Then $(a+bi)^2 = a^2 - b^2 + 2abi = 5 + 12i$. **M1**

Equating real and imaginary parts: $a^2 - b^2 = 5$ and $2ab = 12$, so $b = 6/a$.

Substituting: $a^2 - 36/a^2 = 5 \implies a^4 - 5a^2 - 36 = 0$.

Using GDC or factorisation: $(a^2 - 9)(a^2 + 4) = 0 \implies a^2 = 9$, so $a = \pm 3$. **M1**

Note: Reject $a^2 = -4$ since $a \in \mathbb{R}$.

When $a = 3$: $b = 2$; when $a = -3$: $b = -2$. **A1**

$z = 3 + 2i$ or $z = -3 - 2i$ **A1**

Q14 — Complex Roots of a Cubic Polynomial MEDIUM

Since $p(x)$ has real coefficients and $z_1 = 1 + 4i$ is a root, the complex conjugate $z_2 = 1 - 4i$ is also a root. **R1**

The corresponding quadratic factor is $(x - (1+4i))(x - (1-4i)) = (x-1)^2 + 16 = x^2 - 2x + 17$. **M1**

Dividing $p(x)$ by $x^2 - 2x + 17$ using GDC or long division: $p(x) = (x^2 - 2x + 17)(x - 4)$. **A1**

The three roots are $z = 4$, $z = 1 + 4i$, and $z = 1 - 4i$. **A1**

Q15 — Triple Angle Identity via De Moivre MEDIUM

By De Moivre's theorem, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$. **M1**

Expanding the left side by the binomial theorem:

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

M1

Equating real parts: $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ **M1**

Substituting $\sin^2\theta = 1 - \cos^2\theta$:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos\theta \qquad \text{AG}$$

A1

Q16 — Modulus and Argument of a Quotient MEDIUM

(a) $|z^3| = |z|^3 = 2^3 = 8$; $|w^2| = |w|^2 = (\sqrt{2})^2 = 2$. **M1**

$\left|\dfrac{z^3}{w^2}\right| = \dfrac{8}{2} = 4$ **A1**

(b) $\arg(z^3) = 3 \times \dfrac{\pi}{6} = \dfrac{\pi}{2}$; $\quad \arg(w^2) = 2 \times \dfrac{\pi}{4} = \dfrac{\pi}{2}$. **M1**

$\arg\left(\dfrac{z^3}{w^2}\right) = \dfrac{\pi}{2} - \dfrac{\pi}{2} = 0$ **A1**

Q17 — Cube Roots of 64i MEDIUM

Write $64i = 64e^{i\pi/2}$. The general cube root is $z_k = 4e^{i(\pi/6 + 2k\pi/3)}$, $k = 0, 1, 2$. **M1**

$k = 0$: $z_0 = 4e^{i\pi/6} = 2\sqrt{3} + 2i$; $\quad k = 1$: $z_1 = 4e^{i5\pi/6} = -2\sqrt{3} + 2i$; $\quad k = 2$: $z_2 = 4e^{-i\pi/2} = -4i$. **A1**

Note: $z_2 = -4i$ matches the diagram, confirming $(-4i)^3 = 64i$.

The other two roots are $z = 2\sqrt{3} + 2i$ and $z = -2\sqrt{3} + 2i$. **A1**

Q18 — Locus as Perpendicular Bisector MEDIUM

(a) Let $z = x + iy$. Then $|z - 8|^2 = (x - 8)^2 + y^2$ and $|z - 4i|^2 = x^2 + (y - 4)^2$. **M1**

Setting equal: $(x - 8)^2 + y^2 = x^2 + (y - 4)^2$ **M1**

Expanding: $x^2 - 16x + 64 + y^2 = x^2 + y^2 - 8y + 16$

$-16x + 64 = -8y + 16$
$8y = 16x - 48$
$y = 2x - 6$ **A1 AG**

(b) y-intercept: set $x = 0$: $y = -6$, so y-intercept is $(0, -6)$. **A1**

Q19 — Circle Locus – Greatest and Least Modulus MEDIUM

(a) Centre $3 + 2i$, radius 3. **A1**

(b) The modulus $|z|$ is the distance from the origin to z. **M1**

Distance from origin to centre: $|3 + 2i| = \sqrt{9 + 4} = \sqrt{13}$. **A1**

GDC gives $\sqrt{13} \approx 3.606$.

Greatest modulus occurs along the ray from origin through centre, at distance $\sqrt{13} + 3$; least modulus at $\sqrt{13} - 3$.

Unit vector in direction of centre: $\dfrac{3 + 2i}{\sqrt{13}}$.

$z_{\text{max}} = (3 + 2i)\left(1 + \dfrac{3}{\sqrt{13}}\right) = \left(3 + \dfrac{9}{\sqrt{13}}\right) + i\left(2 + \dfrac{6}{\sqrt{13}}\right)$

$z_{\text{min}} = \left(3 - \dfrac{9}{\sqrt{13}}\right) + i\left(2 - \dfrac{6}{\sqrt{13}}\right)$ **A1**

Note: Accept decimal equivalents to 3 s.f.: $z_{\text{max}} \approx 5.50 + 3.66i$, $z_{\text{min}} \approx 0.504 + 0.336i$.

Q20 — Proof of De Moivre's Theorem by Induction MEDIUM

Basis step ($n = 1$): LHS $= \cos\theta + i\sin\theta$; RHS $= \cos\theta + i\sin\theta$. True for $n = 1$. **A1**

Inductive hypothesis: Assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ for some $k \in \mathbb{Z}^+$. **M1**

Inductive step: Consider $n = k + 1$:

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

using the compound angle formulae. **M1**

Conclusion: Since the statement holds for $n = 1$, and whenever it holds for $n = k$ it holds for $n = k + 1$, by the principle of mathematical induction it holds for all positive integers n. **R1**

Q21 — Polar/Euler Form and De Moivre HARD

(a) $r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$ **A1**

$\theta = \arg(z) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$ (first quadrant, both parts positive) **A1**

$z = 2e^{i\pi/3}$

(b) By De Moivre's theorem, $z^6 = 2^6 e^{i \cdot 6 \cdot \pi/3} = 64e^{i \cdot 2\pi}$ **M1**

$= 64(\cos 2\pi + i \sin 2\pi) = 64(1 + 0) = 64$ **AG**

(c) $(1 + i\sqrt{3})^6 = 64 \in \mathbb{R}$, so $\overline{(1 + i\sqrt{3})^6} = 64.$ **A1**

$(1 + i\sqrt{3})^6 + \overline{(1 + i\sqrt{3})^6} = 64 + 64 = 128$

Q22 — De Moivre Induction Proof HARD

Base case: $n = 1$: LHS $= \cos\theta + i\sin\theta$, RHS $= \cos(1\cdot\theta) + i\sin(1\cdot\theta) = \cos\theta + i\sin\theta$. True. **M1**

Inductive hypothesis: Assume the statement holds for $n = k$, i.e. $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$ for some $k \in \mathbb{Z}^+$. **M1**

Inductive step:

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$$
$$= [\cos(k\theta) + i\sin(k\theta)](\cos\theta + i\sin\theta) \quad \text{(by hypothesis)}$$

M1

$$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + i[\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta]$$
$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos((k+1)\theta) + i\sin((k+1)\theta)$$

A1

Conclusion: Since the statement holds for $n = 1$, and whenever it holds for $n = k$ it holds for $n = k+1$, by the principle of mathematical induction it holds for all $n \in \mathbb{Z}^+$. **R1**

Q23 — Modulus and Argument – Complex Rotation HARD

(a) $|w| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ **A1**

$\arg(w) = \arctan\left(\frac{4}{3}\right) = 0.927$ rad (to 3 s.f.) **A1**

Note: GDC gives $\arctan(4/3) = 0.92729\ldots$ *rad.*

(b) $\arg(u) = 0.92729\ldots + \frac{\pi}{4} = 0.92729\ldots + 0.78540\ldots = 1.71269\ldots$ rad **M1**

$u = 5\cos(1.71269\ldots) + 5i\sin(1.71269\ldots)$ **M1**

$= 5(-0.14142\ldots) + 5i(0.98995\ldots)$

$u = -0.707 + 4.95i$ (to 3 s.f.) **A1**

Q24 — Complex Roots of a Real Polynomial HARD

(a) Since all coefficients are real and $z_1 = 1 + 2i$ is a root, the conjugate $z_2 = 1 - 2i$ is also a root, by the conjugate root theorem. **R1**

(b) $(z - (1+2i))(z - (1-2i)) = (z-1)^2 + 4 = z^2 - 2z + 5$ **M1**

Dividing: $z^4 - 8z^3 + 27z^2 - 50z + 50 \div (z^2 - 2z + 5)$ gives quotient $z^2 - 6z + 10$ with zero remainder. **A1**

So $p(z) = (z^2 - 2z + 5)(z^2 - 6z + 10)$, i.e. $a = 6$, $b = 10$.

(c) From $z^2 - 6z + 10 = 0$: $z = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$. **M1**

All four roots: $z = 1 + 2i, 1 - 2i, 3 + i, 3 - i$. **A1**

Q25 — Multiple-Angle Identity and Exact Value HARD

Let $c = \cos\theta$, $s = \sin\theta$. By De Moivre's theorem:

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4$$

M1

Expanding by the binomial theorem:

$$= c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$$

$$= c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$$

M1

Taking real parts and using $s^2 = 1 - c^2$:

$$\cos(4\theta) = c^4 - 6c^2s^2 + s^4 = c^4 - 6c^2(1-c^2) + (1-c^2)^2$$

$$= c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4 = 8c^4 - 8c^2 + 1 \qquad \text{AG}$$

A1

For $\theta = \frac{\pi}{8}$: $\cos\left(\frac{\pi}{2}\right) = 0$, so $8\cos^4\left(\frac{\pi}{8}\right) - 8\cos^2\left(\frac{\pi}{8}\right) + 1 = 0$, i.e. $x = \cos\left(\frac{\pi}{8}\right)$ satisfies $8x^4 - 8x^2 + 1 = 0$.

Using the quadratic formula in x^2: $x^2 = \frac{8 \pm \sqrt{64 - 32}}{16} = \frac{8 \pm \sqrt{32}}{16} = \frac{2 \pm \sqrt{2}}{4}$ **M1**

Since $\cos(\pi/8) > 0$ and $\cos(\pi/8) = \cos 22.5^\circ \approx 0.924$ (close to 1), take the larger root:

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{4}, \quad \text{so} \quad \cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

A1

Q26 — Cube Roots of a Complex Number HARD

Write $-64i = 64e^{i(-\pi/2)}$ (since $|-64i| = 64$, $\arg(-64i) = -\pi/2$). **M1**

The three cube roots have modulus $64^{1/3} = 4$ and arguments $\dfrac{-\pi/2 + 2\pi k}{3}$, $k = 0, 1, 2$. **M1**

$k = 0$: $\theta = -\pi/6$: $z_0 = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 4\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = 2\sqrt{3} - 2i$

$k = 1$: $\theta = \pi/2$: $z_1 = 4(\cos 90^\circ + i\sin 90^\circ) = 4i$

$k = 2$: $\theta = 7\pi/6 \equiv -5\pi/6$: $z_2 = 4\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -2\sqrt{3} - 2i$ **A1**

The three cube roots of $-64i$ are $2\sqrt{3} - 2i$, $4i$, $-2\sqrt{3} - 2i$. **A1**

Note: verified by cubing each root: $(2\sqrt{3} - 2i)^3 = (4i)^3 = (-2\sqrt{3} - 2i)^3 = -64i$, *consistent with the diagram.* **A1**

Q27 — Loci Intersection and Chord Length HARD

(a) $|z - (3+2i)| = |z - (7+2i)|$ is the perpendicular bisector of the segment joining $3 + 2i$ and $7 + 2i$, which is the vertical line $\text{Re}(z) = 5$ (i.e. $x = 5$). **R1**

(b) The circle has centre $5 + 2i$ and radius 3; the perpendicular bisector $x = 5$ passes through the centre. Both are drawn on the diagram; the shaded segment is the diameter of the circle along $x = 5$. **M1A1**

(c) Since $x = 5$ passes through the centre $(5, 2)$ of the circle of radius 3, the chord is in fact a diameter. **M1**

Length of chord $= 2 \times 3 = 6$. In the form $k\sqrt{m}$: $6 = 6\sqrt{1}$, so $k = 6$, $m = 1$.

Chord length $= 6\sqrt{1} = 6$ (units). **A1**

Q28 — Roots of Unity and Sum of Cosines HARD

(a) $\omega^7 = e^{2\pi i \cdot 7/7} = e^{2\pi i} = 1$ **A1**

(b) ω is a primitive 7th root of unity, so $\omega^7 - 1 = 0$. **M1**

Since $\omega \neq 1$, dividing $z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$ by $(z-1)$:

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0.$$

A1 AG

(c) Writing $\omega^k = e^{2\pi i k/7} = \cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right)$, taking real parts of the sum in (b):

$$\sum_{k=0}^{6} \cos\left(\frac{2\pi k}{7}\right) = 0; \quad \text{the } k = 0 \text{ term is } \cos(0) = 1, \text{ so}$$

M1

$$1 + \sum_{k=1}^{6} \cos\left(\frac{2\pi k}{7}\right) = 0 \implies \sum_{k=1}^{6} \cos\left(\frac{2\pi k}{7}\right) = -1.$$

The sum equals -1; this follows because the real parts of all seven 7th roots of unity sum to zero, and the $k = 0$ root contributes 1. **A1**

Q29 — Quadratic over C via Discriminant HARD

(a) $\Delta = (6+2i)^2 - 4(1)(13+18i) = (36+24i-4) - (52+72i) = (32+24i) - (52+72i) = -20-48i$ **A1 AG**

(b) Let $\Delta = a + bi$; then $(a+bi)^2 = -20 - 48i$: **M1**

$a^2 - b^2 = -20$ and $2ab = -48 \implies b = -24/a$.

Substituting: $a^2 - 576/a^2 = -20 \implies a^4 + 20a^2 - 576 = 0 \implies (a^2 - 16)(a^2 + 36) = 0$.

Since $a \in \mathbb{R}$, $a^2 = 16 \implies a = \pm 4$; $b = \mp 6$.

$\sqrt{\Delta} = -4 + 6i$ (taking this branch). **A1**

(c) $z = \dfrac{(6+2i) \pm (-4+6i)}{2}$ **M1**

$z_1 = \dfrac{2+8i}{2} = 1 + 4i; \quad z_2 = \dfrac{10-4i}{2} = 5 - 2i.$ **A1**

Q30 — Transformation as Rotation and Scaling HARD

(a) $1 + i = \sqrt{2}\, e^{i\pi/4}$, so **M1**

$w = \sqrt{2}e^{i\pi/4} \cdot r^3 e^{i3\theta} = \sqrt{2}\, r^3\, e^{i(3\theta + \pi/4)}$.

$R = \sqrt{2} r^3$, $\phi = 3\theta + \pi/4$. **A1**

(b) $r = 2$, $\theta = \pi/9$:

$|w| = R = \sqrt{2} \cdot 2^3 = 8\sqrt{2} = 11.313\ldots \approx 11.3$ (to 3 s.f.) **A1**

$\arg(w) = \phi = 3 \cdot \dfrac{\pi}{9} + \dfrac{\pi}{4} = \dfrac{\pi}{3} + \dfrac{\pi}{4} = \dfrac{7\pi}{12} = 1.8326\ldots \approx 1.83$ rad (to 3 s.f.) **A1**

Note: GDC gives $7\pi/12 = 1.83259\ldots$ rad.

(c) T scales z by a factor of $\sqrt{2}r^2$ and rotates it by $\dfrac{\pi}{4}$ radians (anticlockwise), in addition to the cubing operation which scales by r^2 and rotates by 2θ. Overall, T multiplies the modulus of z by $\sqrt{2}|z|^2$ and increases the argument by $\dfrac{\pi}{4}$. **A1**