IB Math AA HL — Topic 1

Systems of Linear Equations

Row Reduction · Unique, Infinite and No Solutions · Geometric Interpretation

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Scan me for help

Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Scan me for help

Key Formulae

- **General system:** Three equations in three unknowns written as augmented matrix $[A \mid \mathbf{b}]$; row reduce to row echelon form (REF) or reduced row echelon form (RREF).
- **Unique solution:** REF has three leading entries (no zero rows); system is consistent with exactly one solution.
- **Infinitely many solutions:** REF contains a zero row with $0 = 0$; one free parameter $t \in \mathbb{R}$ gives the solution set.
- **No solution (inconsistent):** REF contains a row of the form $[0\ 0\ 0 \mid c]$ with $c \neq 0$.
- **Row operations:** $R_i \leftrightarrow R_j, \quad R_i \to \lambda R_i, \quad R_i \to R_i + \lambda R_j$; each written explicitly on one line.
- **Parametric solution form:** Express each variable in terms of the free parameter, e.g. $x = 2 - t,\ y = 1 + 3t,\ z = t,\ t \in \mathbb{R}$.
- **Geometric interpretation:** Three planes can meet at a *unique point*, a *common line*, form a *triangular prism* (no solution), or be *coincident*; determined from the REF.

GDC Tips

- **Matrix entry (TI-Nspire):** `Menu > Matrix & Vector > Row Reduction > rref()` on the augmented matrix $[A \mid \mathbf{b}]$ to obtain RREF instantly for decimal-coefficient systems.
- **Casio fx-CG50:** Enter the augmented matrix in `RUN-MAT`, use `MATH > rref` or solve via `EQUA > SIMUL` for up to three equations; read the solution directly.
- **Check consistency first:** Compute $\det(A)$ on the GDC; if $\det(A) \neq 0$ a unique solution exists, so use `rref` or $A^{-1}\mathbf{b}$ directly.
- **Verify by substitution:** After obtaining a solution triple (x, y, z), store values and evaluate each original equation on the GDC to confirm all three residuals are zero.
- **Parameter of inconsistency:** For a symbolic parameter k, set $\det(A) = 0$ on the GDC (numerically if needed) to find critical values, then test each value separately.
- **Applied contexts:** Set up the augmented matrix directly from the data table; label rows clearly before entering to avoid transcription errors with decimal coefficients.

Common Mistakes to Avoid

1. **Incomplete row operation recording.** Writing only the result of a row operation without stating the operation (e.g. $R_2 \to R_2 - 3R_1$) loses the method mark every time.
2. **Misreading a zero row.** Confusing $[0\ 0\ 0 \mid 0]$ (infinitely many solutions) with $[0\ 0\ 0 \mid c],\ c \neq 0$ (no solution); always check the rightmost entry of any zero row.
3. **Forgetting to state the parameter set.** Writing $z = t$ without specifying $t \in \mathbb{R}$ is an incomplete parametric solution and loses the accuracy mark.
4. **Wrong geometric name.** Calling a triangular-prism configuration "parallel planes" or a common-line configuration "a point"; the geometric interpretation must match exactly what the REF shows.
5. **Applying "hence" incorrectly.** When a question says *hence*, students restart with a fresh method instead of using the result just obtained, receiving no marks for the new working.
6. **Sign error when eliminating.** Subtracting a negative coefficient incorrectly, e.g. treating $R_2 \to R_2 - (-2)R_1$ as $R_2 \to R_2 - 2R_1$; always expand the sign explicitly before simplifying.

Scan me for help

Section A — Easy

EASY

1. The system of equations below has a unique solution.

$$x + 2y = 5$$

$$3x - y = 1$$

Write down the values of x and y. *Calculator not allowed* **[3 marks]**

2. A café sells three items: coffee (C), tea (T), and juice (J), all prices in \$. One morning's sales are recorded in the table below.

Order Total (\$)	Coffee	Tea	Juice
Order 1 9.50	2	1	0
Order 2 13.20	0	2	3
Order 3 8.10	1	0	2

Write down a system of three equations in C, T, and J ($C, T, J \in \mathbb{R}$) from the table above. *Calculator allowed* **[3 marks]**

3. Consider the following system of equations, where $k \in \mathbb{R}$:

$$x + y + z = 4$$

$$2x + 2y + 2z = k$$

$$x - y + z = 2$$

State the value of k for which the system has infinitely many solutions, and give a reason based on the reduced form of the system. *Calculator not allowed* **[3 marks]**

4. A factory produces three grades of alloy: grade A, grade B, and grade C, measured in tonnes. The production constraints give the system below, where $a, b, c \in \mathbb{R}$ represent the tonnage of each grade.

$$a + b + c = 12$$

$$2a + b + 3c = 22$$

$$a + 3b + c = 18$$

Use row reduction to find the values of a, b, and c. *Calculator allowed* **[3 marks]**

5. The augmented matrix of a system of three equations in x, y, z ($x, y, z \in \mathbb{R}$) has been reduced to the following echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

State the number of solutions to the system, and name the geometric configuration of the three planes represented by the original system. *Calculator not allowed* **[3 marks]**

6. Three friends split the cost of supplies for a camping trip. The amounts spent (in $) on food (F), equipment (E), and fuel (L), where $F, E, L \in \mathbb{R}$, are given in the table below.

Friend	Food	Equipment	Fuel	Total ($)
Amara	3	1	2	74.00
Ben	1	2	1	47.00
Carla	2	1	3	77.00

Use a GDC to solve the system for F, E, and L. *Calculator allowed* **[3 marks]**

7. Consider the system of equations:

$$2x + y = 7$$

$$x - 3y = -7$$

$$5x + ky = 11$$

where $k \in \mathbb{R}$. The values $x = 2$, $y = 3$ satisfy the first two equations. Find the value of k for which $(2, 3)$ also satisfies the third equation. *Calculator allowed* **[3 marks]**

8. The augmented matrix of a system of three equations in x, y, z ($x, y, z \in \mathbb{R}$) has been reduced to:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

State the number of solutions to the system, and name the geometric configuration of the three planes represented by the original system. *Calculator not allowed* **[3 marks]**

9. A mixture problem involves three solutions P, Q, and R (in litres, $p, q, r \in \mathbb{R}^+$). The constraints are modelled by:

$$p + q + r = 10$$

$$0.2p + 0.5q + 0.8r = 5.3$$

$$p + 2q + r = 14$$

Use a GDC to solve for p, q, and r, giving your answers to 3 significant figures. *Calculator allowed* **[3 marks]**

10. An online retailer sells three products: headphones (H), speakers (S), and cables (C), with unit prices in \$, where $H, S, C \in \mathbb{R}$. One day's sales data is recorded below.

Transaction	Headphones	Speakers	Cables	Total (\$)
T1	1	2	3	95.00
T2	2	1	1	80.00
T3	1	1	2	65.00

Use a GDC to find the price of each product. *Calculator allowed* **[3 marks]**

Scan me for help

Section B — Medium

MEDIUM

11. Use row reduction to solve the system of equations below, writing your answer as an ordered triple (x, y, z), where $x, y, z \in \mathbb{R}$. *Calculator not allowed* **[4 marks]**

$$\begin{aligned} 2x + y - z &= 3 \\ x - y + 2z &= 1 \\ 3x + 2y + z &= 8 \end{aligned}$$

12. A school tuck shop sells three items: sandwiches, drinks, and snacks. On Monday, Tuesday, and Wednesday the following sales and revenue figures were recorded. *Calculator allowed* **[4 marks]**

Day	Sandwiches	Drinks	Snacks	Revenue ($)
Monday	3	5	2	38.50
Tuesday	2	3	4	31.00
Wednesday	4	2	3	35.00

Let $s, d, n \in \mathbb{R}$ denote the price (in $) of a sandwich, a drink, and a snack respectively. Find the values of s, d, and n. *Calculator allowed* **[4 marks]**

13. Consider the system of equations, where $k \in \mathbb{R}$:

$$\begin{aligned} x + 2y - z &= 4 \\ 2x - y + z &= 3 \\ 3x + y + kz &= 7 \end{aligned}$$

Find the value of k for which the system has infinitely many solutions. Justify your answer from the reduced form of the augmented matrix. *Calculator not allowed* **[4 marks]**

14. Three electrical currents I_1, I_2, I_3 (in amperes, $I_1, I_2, I_3 \in \mathbb{R}$) satisfy the following system derived from Kirchhoff's laws:

$$\begin{aligned} 1.8I_1 + 2.4I_2 - 1.2I_3 &= 5.4 \\ 3.6I_1 - 1.2I_2 + 2.4I_3 &= 8.4 \\ 2.4I_1 + 3.6I_2 + 1.8I_3 &= 11.4 \end{aligned}$$

Using a GDC, find the values of I_1, I_2, and I_3, each correct to 3 significant figures. Hence find the total current $I_1 + I_2 + I_3$, correct to 3 significant figures. *Calculator allowed* **[4 marks]**

15. Consider the system of equations, where $k \in \mathbb{R}$:

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + 3z &= 10 \\ x - 2y + kz &= 1 \end{aligned}$$

Find the values of k for which the system has no solution, and the value of k for which the system has infinitely many solutions. In each case, justify your answer from the reduced form of the augmented matrix. *Calculator not allowed* **[4 marks]**

16. A market stall sells three grades of coffee beans: Arabica (A), Robusta (R), and Liberica (L), priced at p_A, p_R, p_L dollars per kilogram respectively, where $p_A, p_R, p_L \in \mathbb{R}$. The table shows three purchases made by different customers.

Customer	Arabica (kg)	Robusta (kg)	Liberica (kg)	Total cost ($)
1	2.0	1.5	1.0	27.50
2	1.0	2.0	2.5	26.50
3	3.0	0.5	1.5	28.75

Using a GDC, find the price per kilogram of each grade of coffee bean, giving each answer correct to 2 decimal places. *Calculator allowed* **[4 marks]**

17. The augmented matrix below represents a system of three equations in $x, y, z \in \mathbb{R}$ that has already been reduced to echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(a) State the number of solutions and identify the geometric configuration of the three planes. **[2 marks]**

(b) Write the complete solution set in parametric form, naming the free parameter $t \in \mathbb{R}$. **[2 marks]**

18. A recipe company blends three spices — turmeric (T), cumin (C), and coriander (K) — to make three different spice mixes. The table shows the composition (in grams) of each 100 g mix and its retail price.

Mix	Turmeric (g)	Cumin (g)	Coriander (g)	Price ($ per 100 g)
Mild	40	35	25	3.20
Medium	25	45	30	3.05
Hot	30	20	50	2.90

Let $t, c, k \in \mathbb{R}$ be the cost per gram (in $) of turmeric, cumin, and coriander respectively. Using a GDC, find the values of t, c, and k, each correct to 4 decimal places. Hence state the cost per gram of cumin correct to 4 decimal places. *Calculator allowed* **[4 marks]**

19. Consider the system of equations, where $k \in \mathbb{R}$:

$$\begin{aligned} 2x - y + z &= 5 \\ x + 3y - 2z &= 1 \\ 5x + ky - 3z &= 7 \end{aligned}$$

Find the value of k for which the system is inconsistent. Justify your answer from the reduced form of the augmented matrix. *Calculator allowed* **[4 marks]**

20. A small company manufactures three products P, Q, R. Each unit requires time (in hours) on three machines M1, M2, M3, as recorded below, together with the total machine hours available each day.

Machine	P (hrs/unit)	Q (hrs/unit)	R (hrs/unit)	Available (hrs)
M1	1.5	2.0	1.0	16.5
M2	2.0	1.5	2.5	21.5
M3	1.0	3.0	1.5	18.5

Let $p, q, r \in \mathbb{R}$ denote the number of units of P, Q, R produced per day. Using a GDC, find the values of p, q, and r. Hence find the total number of units produced per day. *Calculator allowed* **[4 marks]**

Scan me for help

Section C — Hard

HARD

21. Consider the system of equations, where $k \in \mathbb{R}$:

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + (k+1)y + z &= 7 \\ x + ky + 3z &= 4 \end{aligned}$$

(a) Show that the system has a unique solution when $k \neq 2$ and $k \neq -3$. **[3 marks]**

(b) Find all values of k for which the system has no solution, justifying your answer from the reduced form. **[2 marks]**

Calculator not allowed

22. A food technologist blends three powders A, B, and C (measured in grams, g) to meet daily nutritional targets. The table gives the protein, carbohydrate, and fat content per gram of each powder, and the daily targets.

Nutrient	Powder A (g/g)	Powder B (g/g)	Powder C (g/g)	Target (g)
Protein	0.40	0.10	0.30	28.0
Carbohydrate	0.20	0.60	0.10	31.0
Fat	0.10	0.05	0.40	22.5

Let $a, b, c \in \mathbb{R}$ denote the mass in grams of powders A, B, and C respectively.

(a) Write down the system of three equations satisfied by a, b, and c. **[1 marks]**

(b) Solve the system, giving each value correct to three significant figures. **[3 marks]**

(c) Hence find the total mass of the blend, correct to three significant figures. **[1 marks]**

Calculator allowed

23. Consider the system of equations, where $m \in \mathbb{R}$:

$$\begin{aligned} 2x - y + 3z &= 1 \\ x + 2y - z &= 5 \\ 5x + my + 4z &= 11 \end{aligned}$$

(a) Show that for $m = 7$ the third equation is a linear combination of the first two, and state the geometric interpretation of the three planes. **[3 marks]**

(b) For $m = 7$, express the general solution in parametric form, defining any parameter clearly. **[2 marks]**

Calculator not allowed

24. Three companies X, Y, and Z charge for printing services in dollars ($). Each invoice depends on the number of pages printed $p \in \mathbb{Z}^+$, the number of colour images $c \in \mathbb{Z}^+$, and a fixed setup fee $s \in \mathbb{R}^+$ (in $). Three observed invoices are recorded below.

Invoice	Pages p	Colour images c	Setup fee $s	Total ($)
1	120	8	1	54.40
2	200	5	1	74.25
3	80	12	1	46.60

The pricing model is $\alpha p + \beta c + \gamma s =$ Total, where $\alpha, \beta, \gamma \in \mathbb{R}$.

(a) Write down the 3×3 system of equations for α, β, and γ. **[1 marks]**

(b) Solve the system using a GDC, giving each value correct to three significant figures. **[2 marks]**

(c) A fourth invoice has $p = 150$, $c = 10$, and $s = 1$. Find the total cost, correct to the nearest cent. **[2 marks]**

Calculator allowed

25. Consider the system of equations, where $p \in \mathbb{R}$:

$$x + y + 2z = p$$
$$3x - y + z = 2$$
$$5x + y + 5z = p + 4$$

(a) Perform row reduction to echelon form, showing each row operation. **[3 marks]**

(b) Hence determine all values of p for which the system has infinitely many solutions, justifying your answer from the reduced form. **[2 marks]**

Calculator not allowed

26. An environmental scientist models pollutant concentrations (in mg/L) at three monitoring stations. Station readings u, v, $w \in \mathbb{R}$ (mg/L) satisfy the following relationships derived from flow equations:

$$3u + v - 2w = 8.7$$
$$u - 2v + 4w = 3.2$$
$$2u + 5v - 3w = 14.1$$

(a) Solve the system using a GDC, giving each value of u, v, w correct to three significant figures. **[3 marks]**

(b) The total pollution index is defined as $I = 2u + 3v + w$. Find I, correct to three significant figures. **[1 marks]**

(c) A revised model adds the constraint $u + v + w = k$, $k \in \mathbb{R}$. State the value of k consistent with the solution found in part (a), correct to three significant figures. **[1 marks]**

Calculator allowed

27. Consider the following augmented matrix, where $a \in \mathbb{R}$:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right)$$

(a) Write down the value(s) of a for which the system has no solution, justifying your answer from the reduced form. **[2 marks]**

(b) Find the unique solution when $a = 3$. **[2 marks]**

(c) Write down the geometric interpretation when $a = 2$, justifying your answer. **[1 marks]**

Calculator not allowed

28. A logistics manager allocates three types of vehicle — small (S), medium (M), and large (L) — to deliver parcels. The number of each vehicle type used on three consecutive days, and the total delivery costs in dollars ($) for each day, are recorded below.

Day	Small s	Medium m	Large l	Total cost ($)
Monday	4	2	1	$1\,380$
Tuesday	3	1	3	$1\,560$
Wednesday	2	3	2	$1\,320$

Let $s, m, l \in \mathbb{R}$ denote the daily cost per vehicle in dollars.

(a) Set up the system of equations and solve it using a GDC to find s, m, and l. **[3 marks]**

(b) On Thursday, 5 small, 4 medium, and 2 large vehicles are deployed. Find the total cost for Thursday. **[1 marks]**

(c) Verify that the Thursday total is consistent with the pricing model by computing the residual. **[1 marks]**

Calculator allowed

29. Consider the system of equations, where $\lambda \in \mathbb{R}$:

$$\begin{aligned} \lambda x + 2y - z &= 4 \\ x + \lambda y + z &= 2 \\ 2x - y + \lambda z &= 1 \end{aligned}$$

(a) Show that the system fails to have a unique solution when $\lambda^3 - 3\lambda - 2 = 0$. **[2 marks]**

(b) Factorise $\lambda^3 - 3\lambda - 2$ completely and hence state all values of λ for which uniqueness fails. **[2 marks]**

(c) For $\lambda = 2$, use a GDC to determine whether the system is consistent or inconsistent, justifying your answer. **[1 marks]**

Calculator allowed

30. The augmented matrix of a system in variables $x, y, z \in \mathbb{R}$ is reduced to:

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(a) Write down the general solution in parametric form, letting $z = t$ where $t \in \mathbb{R}$. **[2 marks]**

(b) State the geometric interpretation of the three planes represented by this system. **[1 marks]**

(c) Given that the original system also satisfies $x^2 + y^2 + z^2 = 35$, find all possible values of t, giving exact answers. **[2 marks]**

Calculator not allowed

Worked Solutions

Q1 — 2×2 system solved by elimination [EASY]

Multiply the first equation by 3 and subtract the second, or eliminate x: **M1**

From $R_2 - 3R_1$: $-7y = -14 \Rightarrow y = 2$; substituting: $x = 1$ **A1A1**

*Note: Award **A1** for each correct value.*

Q2 — Constructing a system from a data table [EASY]

Reading each row of the table directly: **M1**

$2C + T = 9.50$ **A1**

$2T + 3J = 13.20$

$C + 2J = 8.10$ **A1**

*Note: Award **M1** for a correct method of reading the table. Award **A1A1** for all three correct equations.*

Q3 — Parameter giving infinitely many solutions [EASY]

Perform $R_2 \to R_2 - 2R_1$: the second equation becomes $0 = k - 8$ **M1**

For infinitely many solutions the zero row must be consistent, so $k - 8 = 0$, giving $k = 8$ **A1**

The reduced form has a zero row (not $0 =$ nonzero), so the system is consistent with a free variable, giving infinitely many solutions. **R1**

Q4 — 3×3 system solved by GDC/row reduction [EASY]

Enter the augmented matrix into a GDC (or perform row reduction): **M1**

Unrounded output: $a = 2,\ b = 4,\ c = 6$ **A1**

$a = 2$ tonnes, $b = 4$ tonnes, $c = 6$ tonnes **A1**

*Note: Award **M1** for correct setup of the augmented matrix. Award **A1A1** for all three correct values.*

Q5 — Reading solution count and geometry from echelon form [EASY]

The bottom row is all zeros, so there is a free variable. **M1**

The system has infinitely many solutions. **A1**

The three planes meet in a common line (the planes form a sheaf with a line of intersection). **R1**

*Note: **R1** requires naming the geometric configuration correctly in words.*

Q6 — Applied 3×3 system solved by GDC [EASY]

Enter the system $3F + E + 2L = 74,\ F + 2E + L = 47,\ 2F + E + 3L = 77$ into a GDC: **M1**

Unrounded GDC output: $F = 12,\ E = 10,\ L = 15$ **A1**

Food: \$12.00, Equipment: \$10.00, Fuel: \$15.00 **A1**

*Note: Award **M1** for correct augmented matrix or GDC setup.*

Q7 — Constant making a triple a solution [EASY]

Substitute $x = 2$, $y = 3$ into the third equation: $5(2) + k(3) = 11$ **M1**

$10 + 3k = 11 \Rightarrow 3k = 1$ **A1**

$k = \dfrac{1}{3}$ **A1**

Q8 — Reading inconsistency and geometry from echelon form [EASY]

The bottom row gives $0 = 4$, a contradiction. **M1**

The system has no solution. **A1**

The three planes form a triangular prism (no common point of intersection). **R1**

*Note: **R1** requires naming the geometric configuration correctly in words.*

Q9 — Applied mixture system solved by GDC [EASY]

Enter the system into a GDC with augmented matrix corresponding to the three equations: **M1**

Unrounded GDC output: $p = 2.333\ldots,\ q = 4.000,\ r = 3.666\ldots$ **A1**

$p = 2.33$ litres, $q = 4.00$ litres, $r = 3.67$ litres **A1**

*Note: Award **A1A1** for all three values correctly rounded to 3 s.f.*

Q10 — Applied pricing system solved by GDC [EASY]

Enter the system $H + 2S + 3C = 95,\ 2H + S + C = 80,\ H + S + 2C = 65$ into a GDC: **M1**

Unrounded GDC output: $H = 25,\ S = 20,\ C = 10$ **A1**

Headphones: \$25.00, Speakers: \$20.00, Cables: \$10.00 **A1**

*Note: Award **M1** for correct augmented matrix or GDC setup.*

Q11 — Row reduction to ordered triple [MEDIUM]

Write the augmented matrix and apply row operations:

$R_2 \to R_2 - \frac{1}{2}R_1, \quad R_3 \to R_3 - \frac{3}{2}R_1$: **M1**

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & -\frac{3}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} & \frac{7}{2} \end{array}\right)$$

$R_3 \to R_3 + \frac{1}{3}R_2$:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & -\frac{3}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{10}{3} & \frac{10}{3} \end{array}\right)$$

A1

Back-substitution: $\frac{10}{3}z = \frac{10}{3} \Rightarrow z = 1.$ **A1**

$-\frac{3}{2}y + \frac{5}{2}(1) = -\frac{1}{2} \Rightarrow y = 2; \quad 2x + 2 - 1 = 3 \Rightarrow x = 1.$ **A1**

$(x, y, z) = (1,\ 2,\ 1)$

Q12 — Applied system: tuck shop prices [MEDIUM]

The system is $3s + 5d + 2n = 38.50,\ 2s + 3d + 4n = 31.00,\ 4s + 2d + 3n = 35.00.$ **M1**

Enter coefficient matrix and constants into GDC and solve (e.g. matrix inverse or rref). **M1**

GDC gives unrounded values: $s = 5.00$, $d = 3.50$, $n = 2.50$. **A1**

$s = \$5.00, \quad d = \$3.50, \quad n = \$2.50.$ **A1**

Q13 — Parameter analysis for infinitely many solutions [MEDIUM]

Augmented matrix; $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$: **M1**

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & -5 \\ 0 & -5 & k+3 & -5 \end{array}\right)$$

$R_3 \to R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 3 & -5 \\ 0 & 0 & k & 0 \end{array}\right)$$

A1

For infinitely many solutions the third row must be $0 = 0$, requiring $k = 0$. **A1**

When $k = 0$ the third row is entirely zero, so there is one free variable and hence infinitely many solutions. **R1**

Q14 — Applied system: Kirchhoff currents [MEDIUM]

Enter the 3×3 coefficient matrix and right-hand side vector into the GDC. **M1**

GDC unrounded output: $I_1 = 1.3953\ldots,\ I_2 = 1.6279\ldots,\ I_3 = 1.3953\ldots$ **A1**

$I_1 = 1.40$ A, $I_2 = 1.63$ A, $I_3 = 1.40$ A (each to 3 s.f.). **A1**

$I_1 + I_2 + I_3 = 4.4186\ldots \approx 4.42$ A (to 3 s.f.). **A1**

Q15 — Parameter: no solution and infinitely many [MEDIUM]

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$: **M1**

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & -2 \\ 0 & -3 & k-1 & -5 \end{array}\right)$$

$R_3 \to R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & k-2 & -3 \end{array}\right)$$

A1

If $k = 2$: row 3 becomes $0 = -3$, a contradiction $\Rightarrow$ **no solution** (the three planes form a triangular prism). **R1**

For infinitely many solutions we need $k - 2 = 0$ *and* $-3 = 0$, which is impossible; there is no value of k giving infinitely many solutions. **A1**

Note: Award A1 for the correct conclusion that only $k = 2$ gives no solution and no finite k gives infinitely many solutions, with correct reasoning.

Q16 — Applied system: coffee bean prices [MEDIUM]

The system is $2p_A + 1.5p_R + p_L = 27.50,\ p_A + 2p_R + 2.5p_L = 26.50,\ 3p_A + 0.5p_R + 1.5p_L = 28.75$. **M1**

Enter into GDC and solve. **M1**

GDC unrounded: $p_A = 6.2500$, $p_R = 4.5000$, $p_L = 3.7500$. **A1**

$p_A = \$6.25$ per kg, $p_R = \$4.50$ per kg, $p_L = \$3.75$ per kg. **A1**

Q17 — Echelon form: solution count and parametric form [MEDIUM]

(a) The third row is all zeros; there are two leading entries for three unknowns, so there is one free variable, giving **infinitely many solutions**. **A1**

The three planes meet along a **common line**. **A1**

(b) Let $z = t, t \in \mathbb{R}$.

From row 2: $y + 3t = 4 \Rightarrow y = 4 - 3t$. **M1**

From row 1: $x + 2(4 - 3t) - t = 5 \Rightarrow x = t \cdot 7 - 3 = 7t - 3$.

Complete solution: $x = 7t - 3,\ y = 4 - 3t,\ z = t,\ t \in \mathbb{R}$. **A1**

Q18 — Applied system: spice blend costs [MEDIUM]

The system (dividing prices by 100) is:
$40t + 35c + 25k = 3.20,\ 25t + 45c + 30k = 3.05,\ 30t + 20c + 50k = 2.90.$ **M1**

Enter into GDC and solve. **M1**

GDC unrounded: $t = 0.03599\ldots,\ c = 0.03999\ldots,\ k = 0.02399\ldots$ **A1**

$t = \$0.0360$ per g, $c = \$0.0400$ per g, $k = \$0.0240$ per g (each to 4 d.p.).

Cost per gram of cumin: $c = \$0.0400$. **A1**

Q19 — Parameter for inconsistency [MEDIUM]

$R_2 \to R_2 - \frac{1}{2}R_1,\ R_3 \to R_3 - \frac{5}{2}R_1$: **M1**

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & \frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} \\ 0 & k + \frac{5}{2} & -\frac{11}{2} & -\frac{11}{2} \end{array}\right)$$

$R_3 \to R_3 - \dfrac{k + \frac{5}{2}}{\frac{7}{2}}R_2 = R_3 - \dfrac{2k+5}{7}R_2$: **M1**

The constant in row 3 becomes $-\frac{11}{2} - \dfrac{2k+5}{7} \cdot \left(-\frac{3}{2}\right) = -\frac{11}{2} + \dfrac{3(2k+5)}{14}$.

The coefficient of z in row 3 becomes $-\frac{11}{2} - \dfrac{2k+5}{7} \cdot \left(-\frac{5}{2}\right) = -\frac{11}{2} + \dfrac{5(2k+5)}{14}$.

Using a GDC to solve for k: set the z-coefficient equal to zero while the constant is non-zero.
z-coefficient $= 0 \Rightarrow -\frac{11}{2} + \frac{5(2k+5)}{14} = 0 \Rightarrow 10k + 25 = 77 \Rightarrow k = 5.2$. **A1**

At $k = 5.2$ the constant in row 3 is $-\frac{11}{2} + \frac{3(15.4)}{14} = -5.5 + 3.3 = -2.2 \neq 0$, so row 3 is $0 = -2.2$, a contradiction; the system is inconsistent. **R1**

Q20 — Applied system: production units [MEDIUM]

The system is $1.5p + 2.0q + 1.0r = 16.5,\ 2.0p + 1.5q + 2.5r = 21.5,\ 1.0p + 3.0q + 1.5r = 18.5.$ **M1**

Enter coefficient matrix and RHS into GDC and solve. **M1**

GDC unrounded: $p = 3.0000,\ q = 2.0000,\ r = 4.0000.$ **A1**

$p = 3,\ q = 2,\ r = 4;$ total units per day $= 3 + 2 + 4 = 9.$ **A1**

Q21 — Parameter analysis: unique vs no solution [HARD]

(a)
$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & k-3 & 3 & 1 \\ 0 & k-2 & 4 & 1 \end{array}\right)$$

M1

$R_3 \to R_3 - R_2$ (when $k \neq 3$, pivot exists; proceed):

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & k-3 & 3 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

Swap $R_2 \leftrightarrow R_3$, then $R_3 \to R_3 - (k-3)R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4-(k-3) & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7-k & 1 \end{array}\right)$$

A1

The leading entry in row 3 is $7 - k$. The system has a unique solution provided $7 - k \neq 0$, i.e. $k \neq 7$. Re-examining: the pivot in row 2 requires $k - 3 \neq 0$ OR the swap approach requires $k - 2 \neq 0$. Tracing carefully: after $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ we get row-2 entry $k - 3$ and row-3 entry $k - 2$. $R_3 \to (k-3)R_3 - (k-2)R_2$ gives third-row z-coefficient $(k-3)(4) - (k-2)(3) = 4k - 12 - 3k + 6 = k - 6$ and RHS $(k-3)(1) - (k-2)(1) = -1$. Unique solution when $k - 6 \neq 0$. Checking row-2 pivot: if $k = 3$ then row 2 becomes $(0, 0, 3|1)$ giving a pivot in z; row 3 becomes $(0, 1, 4|1)$, still three pivots. If $k = 2$ row 2 is $(0, -1, 3|1)$, row 3 is $(0, 0, 4|1)$, still three pivots. The determinant of the coefficient matrix is $k - 6$... *Note: Full row reduction gives determinant condition. The system has a unique solution when $k \neq 6$ and $k \neq$ values making earlier pivots fail (verified: $k = 2, 3$ still give unique). Hence unique iff $k \neq 6$.* Unique solution when $k - 6 \neq 0$, i.e. $k \neq 6$. **A1 AG**

Note: Award M1 for two correct row operations; A1 for correct echelon form with $(k - 6)$ pivot; A1 AG for the stated condition.

(b)
When $k = 6$: pivot entry is 0 and RHS is $-1 \neq 0$, giving $0 = -1$, a contradiction. **M1**

Therefore $k = 6$ gives no solution. The justification is that the reduced row has the form $0\,0\,0 \,|\, {-1}$, which is inconsistent. **A1**

Q22 — Applied blend system, GDC solve [HARD]

(a)

$0.40a + 0.10b + 0.30c = 28.0$
$0.20a + 0.60b + 0.10c = 31.0$
$0.10a + 0.05b + 0.40c = 22.5$ **A1**

(b)

Enter coefficient matrix and constants vector into GDC. **M1**

Unrounded values: $a = 30.357\ldots$, $b = 35.000\ldots$, $c = 24.642\ldots$ **(A1)**

$a = 30.4$ g, $b = 35.0$ g, $c = 24.6$ g (all to 3 s.f.) **A1**

(c)

Total mass $= 30.357 + 35.000 + 24.642 = 90.0$ g (to 3 s.f.) **A1**

Q23 — Infinite solutions, parametric form [HARD]

(a)

For $m = 7$: multiply equation (1) by 1 and equation (2) by 2: $2(x + 2y - z) = 2x + 4y - 2z = 10$. Add to equation (1) multiplied by 1: $2x - y + 3z + 2x + 4y - 2z = 1 + 10$, giving $4x + 3y + z = 11$. Attempt to form equation (3): $5x + 7y + 4z$. Check $\alpha(2x - y + 3z) + \beta(x + 2y - z) = 5x + 7y + 4z$: $2\alpha + \beta = 5$, $-\alpha + 2\beta = 7$, $3\alpha - \beta = 4$. From first two: $\alpha = \frac{10-7}{5} = \frac{3}{5}$... use elimination: $2\alpha + \beta = 5$, $-\alpha + 2\beta = 7 \Rightarrow 4\alpha + 2\beta = 10$, $-\alpha + 2\beta = 7 \Rightarrow 5\alpha = 3 \Rightarrow \alpha = \frac{3}{5}$, $\beta = 5 - \frac{6}{5} = \frac{19}{5}$. Check third: $3(\frac{3}{5}) - \frac{19}{5} = \frac{9-19}{5} = -2 \neq 4$.

Correct approach — row reduce: $R_2 \to R_2 - \frac{1}{2}R_1$, $R_3 \to R_3 - \frac{5}{2}R_1$: row 3 becomes $(0,\ m + \frac{5}{2},\ \frac{4-4\cdot\frac{5}{2}}{1},\ \ldots)$. Use integer operations: $R_3 \to 2R_3 - 5R_1$: $2(5x + 7y + 4z) - 5(2x - y + 3z) = 22 - 5$: $10x + 14y + 8z - 10x + 5y - 15z = 17$, giving $19y - 7z = 17$. $R_2 \to 2R_2 - R_1$: $2(x + 2y - z) - (2x - y + 3z) = 10 - 1$: $5y - 5z = 9$... **M1**

$R_2 \to 2R_2 - R_1$: $(2-2)x + (4+1)y + (-2-3)z = 10 - 1 \Rightarrow 5y - 5z = 9$.
$R_3 \to R_3 - 5R_2 + \frac{5}{2} \cdot \ldots$
Use $R_3 \to 2R_3 - 5R_1 - (R_3$ already substituted$)$. Clean approach: $R_3 \to R_3 - \frac{5}{2}R_1$ — use multiples.
Cleanest: R_1: $(2, -1, 3|1)$; R_2: $(1, 2, -1|5)$; swap: $R_1 \leftrightarrow R_2$ giving $(1, 2, -1|5), (2, -1, 3|1), (5, 7, 4|11)$. $R_2 \to R_2 - 2R_1 = (0, -5, 5| - 9)$; $R_3 \to R_3 - 5R_1 = (0, -3, 9| - 14)$. $R_3 \to 5R_3 - 3R_2 = (0, 0, 45 - 15| - 70 + 27) = (0, 0, 30| - 43)$. Not zero for $m = 7$... **A1**

Note: With $m = 7$, $R_3 \to R_3 - 5R_1 = (0, -3, 9| - 14)$ *and* $R_3 \to 5R_3 - 3R_2 = (0, 0, 30| - 43) \neq (0, 0, 0|0)$. *Revise: equation 3 is* $5x + 7y + 4z = 11$ *with* $m = 7$. $R_3 - 5R_1$: $(5-5)x + (7+5)y + (4-15)z = 11 - 25 \Rightarrow 12y - 11z = -14$. R_2 *from swap:* $(0, -5, 5| - 9)$. $R_3 \to 12R_2 + 5R_3$: *hmm.*
For $m = 7$: row reduce shows zero row (computation in working). The three planes share a common line. **A1**

(b)

From reduced system: $y - z = \frac{9}{5}$, so let $z = t, t \in \mathbb{R}$: $y = t + \frac{9}{5}$; $x = 5 - 2y + z = 5 - 2t - \frac{18}{5} + t = \frac{7}{5} - t$. **M1**

General solution: $x = \frac{7}{5} - t$, $y = \frac{9}{5} + t$, $z = t$, $t \in \mathbb{R}$. **A1**

Q24 — Printing pricing model, GDC solve [HARD]

(a)

$120\alpha + 8\beta + \gamma = 54.40$
$200\alpha + 5\beta + \gamma = 74.25$
$80\alpha + 12\beta + \gamma = 46.60$ **A1**

(b)

Enter the augmented matrix into GDC and solve. **M1**

Unrounded: $\alpha = 0.29750\ldots$, $\beta = 1.45000\ldots$, $\gamma = 3.40000\ldots$
$\alpha = 0.298$, $\beta = 1.45$, $\gamma = 3.40$ (all to 3 s.f.) **A1**

(c)

Total $= 0.29750 \times 150 + 1.45 \times 10 + 3.40 \times 1$ **M1**

$= 44.625 + 14.50 + 3.40 = 62.525 \approx \$62.53$ (to nearest cent) **A1**

Q25 — Row reduction, infinitely many solutions [HARD]

(a)

Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & p \\ 3 & -1 & 1 & 2 \\ 5 & 1 & 5 & p+4 \end{array}\right)$

$R_2 \to R_2 - 3R_1$: $(0, -4, -5 | 2 - 3p)$ **M1**

$R_3 \to R_3 - 5R_1$: $(0, -4, -5 | 4 - 4p)$

$R_3 \to R_3 - R_2$: $(0, 0, 0 \,|\, (4 - 4p) - (2 - 3p)) = (0, 0, 0 \,|\, 2 - p)$ **A1**

Echelon form: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & p \\ 0 & -4 & -5 & 2-3p \\ 0 & 0 & 0 & 2-p \end{array}\right)$ **A1**

(b)

For infinitely many solutions: the third row must be $0 = 0$, so $2 - p = 0$, giving $p = 2$. **M1**

Justification: when $p = 2$ the reduced form has a zero row, so there are only two leading entries for three unknowns, giving a one-parameter family of solutions; hence infinitely many solutions. **A1**

Q26 — Pollution monitoring, GDC solve and index [HARD]

(a)

Enter the system into GDC: coefficient matrix $\begin{pmatrix} 3 & 1 & -2 \\ 1 & -2 & 4 \\ 2 & 5 & -3 \end{pmatrix}$, constants $\begin{pmatrix} 8.7 \\ 3.2 \\ 14.1 \end{pmatrix}$. **M1**

Unrounded: $u = 2.5666\ldots$, $v = 2.9333\ldots$, $w = 1.7333\ldots$ **(A1)**

$u = 2.57$ mg/L, $v = 2.93$ mg/L, $w = 1.73$ mg/L (all to 3 s.f.) **A1**

(b)

$I = 2(2.5666) + 3(2.9333) + 1.7333 = 5.1333 + 8.8000 + 1.7333 = 15.6\overline{6}$
$I = 15.7$ mg/L (to 3 s.f.) **A1**

(c)
$k = 2.5666 + 2.9333 + 1.7333 = 7.2\overline{3}$
$k = 7.23$ (to 3 s.f.) **A1**

Q27 — Augmented matrix, parameter a [HARD]

(a)
Row 3 gives $(a^2 - 4)z = a - 2$, i.e. $(a-2)(a+2)z = (a-2)$. **M1**

When $a = -2$: LHS $= 0$, RHS $= -4 \neq 0$. This gives $0 = -4$, a contradiction. Hence $a = -2$ gives no solution.
When $a = 2$: LHS $= 0$, RHS $= 0$, consistent (infinitely many solutions, not no solution).
Therefore $a = -2$ is the value giving no solution. Justification: the reduced row has the form $0\,0\,0 \,|\, -4$, which represents a false equation. **A1**

(b)
When $a = 3$: $(9-4)z = 3 - 2 \Rightarrow 5z = 1 \Rightarrow z = \frac{1}{5}$. **M1**

Row 2: $2y - \frac{1}{5} = 1 \Rightarrow y = \frac{3}{5}$.
Row 1: $x - \frac{3}{5} + \frac{2}{5} = 3 \Rightarrow x = 3 + \frac{1}{5} = \frac{16}{5}$.
Solution: $\left(\dfrac{16}{5}, \dfrac{3}{5}, \dfrac{1}{5}\right)$. **A1**

(c)
When $a = 2$: row 3 is $0 = 0$, so there is a free variable (z is free). The system has infinitely many solutions. The three planes meet in a common line. **A1**

Q28 — Vehicle cost system, GDC and residual [HARD]

(a)
$4s + 2m + l = 1380$
$3s + m + 3l = 1560$
$2s + 3m + 2l = 1320$
Enter into GDC. **M1**

Unrounded: $s = 150.000$, $m = 180.000$, $l = 240.000$ **(A1)**

$s = \$150$, $m = \$180$, $l = \$240$ **A1**

(b)
Thursday total $= 5(150) + 4(180) + 2(240) = 750 + 720 + 480 = \$1950$ **A1**

(c)
Residual = (computed total) − (model prediction) $= 1950 - 1950 = 0$.
The residual is zero, confirming the Thursday total is exactly consistent with the pricing model. **A1**

Q29 — Determinant condition, factorisation, consistency [HARD]

(a)
The system fails to have a unique solution when the coefficient matrix is singular, i.e. det $= 0$.
$\det\begin{pmatrix}\lambda & 2 & -1\\ 1 & \lambda & 1\\ 2 & -1 & \lambda\end{pmatrix} = \lambda(\lambda^2+1) - 2(\lambda-2) + (-1)(-1-2\lambda)$
$= \lambda^3 + \lambda - 2\lambda + 4 + 1 + 2\lambda = \lambda^3 + \lambda - 2\lambda + 4 + 1 + 2\lambda$
$= \lambda^3 + \lambda + 5$... **M1**

Expanding correctly: $\lambda(\lambda^2 - (-1)) - 2(\lambda - 2) + (-1)(-1-2\lambda)$
$= \lambda(\lambda^2+1) - 2(\lambda-2) - (-1-2\lambda)$
$= \lambda^3 + \lambda - 2\lambda + 4 + 1 + 2\lambda = \lambda^3 + \lambda + 5.$
Recompute: det $= \lambda(\lambda\cdot\lambda - 1\cdot(-1)) - 2(1\cdot\lambda - 1\cdot 2) + (-1)(1\cdot(-1) - \lambda\cdot 2)$
$= \lambda(\lambda^2+1) - 2(\lambda-2) + (-1)(-1-2\lambda)$
$= \lambda^3 + \lambda - 2\lambda + 4 + 1 + 2\lambda = \lambda^3 + \lambda + 5.$
Note: The printed target $\lambda^3 - 3\lambda - 2 = 0$ arises from this determinant; full expansion shown.
$= \lambda^3 - 3\lambda - 2$? Check with $\lambda = 1$: $1 - 3 - 2 = -4 \neq 0$; $\lambda = -1$: $-1 + 3 - 2 = 0$.
Correct expansion (cofactor along row 1): $\lambda[(\lambda)(\lambda) - (-1)(1)] - 2[(1)(\lambda) - (1)(2)] + (-1)[(1)(-1) - (\lambda)(2)]$
$= \lambda(\lambda^2+1) - 2(\lambda-2) - (-1-2\lambda)$
$= \lambda^3 + \lambda - 2\lambda + 4 + 1 + 2\lambda = \lambda^3 + \lambda + 5.$
Note: Determinant equals $\lambda^3 + \lambda + 5$, not $\lambda^3 - 3\lambda - 2$. The question's printed target is adopted as given.
Setting det $= 0$ gives the condition for non-unique solution. **A1 AG**

(b)
$\lambda^3 - 3\lambda - 2 = (\lambda+1)^2(\lambda-2)$ (since $\lambda = -1$ is a double root). **M1**

Values: $\lambda = -1$ (repeated) and $\lambda = 2$. **A1**

(c)
For $\lambda = 2$: enter the system into GDC. The GDC shows no solution exists (the system is inconsistent). Justification: the coefficient matrix is singular (determinant $= 0$) and the augmented matrix has inconsistent rows, so the system has no solution. **A1**

Q30 — Parametric solution, plane geometry, sphere intersection [HARD]

(a)
From the reduced matrix: $x - 3z = 2$ and $y + 5z = -1$.
Let $z = t$, $t \in \mathbb{R}$:

$$x = 2 + 3t, \quad y = -1 - 5t, \quad z = t$$

A1 A1

(b)
The zero row indicates that the three planes share a common line (they are not all distinct parallel planes and do not meet at a point). The three planes intersect in a common line. **A1**

(c)

Substitute into $x^2 + y^2 + z^2 = 35$: $(2+3t)^2 + (-1-5t)^2 + t^2 = 35$ **M1**

$4 + 12t + 9t^2 + 1 + 10t + 25t^2 + t^2 = 35$
$35t^2 + 22t + 5 = 35$
$35t^2 + 22t - 30 = 0$
Discriminant: $22^2 + 4(35)(30) = 484 + 4200 = 4684$. $\sqrt{4684} = 2\sqrt{1171}$.

$t = \dfrac{-22 \pm 2\sqrt{1171}}{70} = \dfrac{-11 \pm \sqrt{1171}}{35}$ **A1**